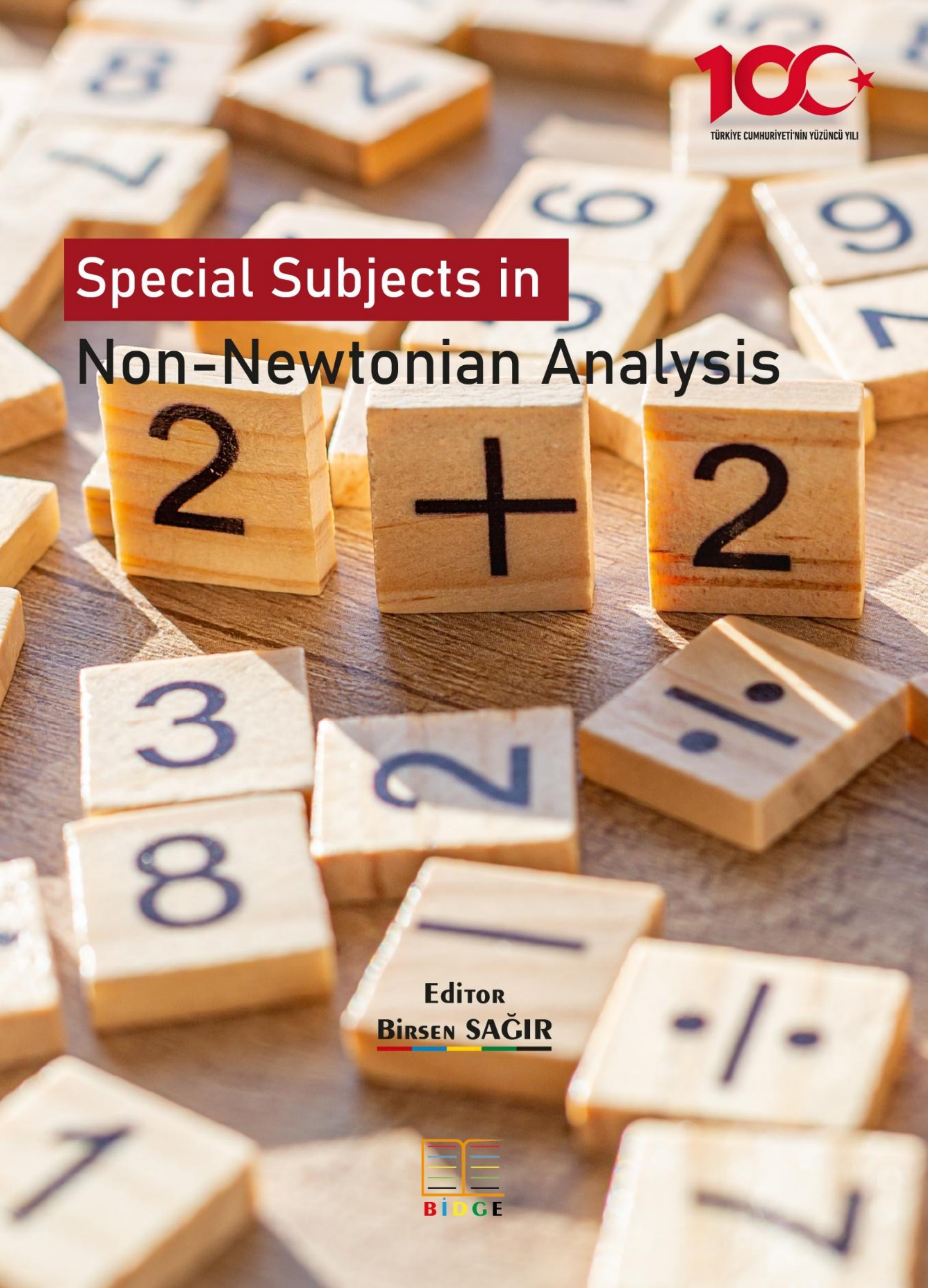

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Content

CHAPTER I

Remarks on Non-Newtonian 2-Normed Spaces

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Introduction

The theory of 2-normed spaces was first developed by Gahler in 1965. Between 1973 and 1977, Diminnie, Gahler and White made detailed studies on 2-inner product spaces and introduced the concept of strictly convexity on 2-normed spaces. Especially in 2001, Gunawan and Mashadi discussed the concepts of convergence and Cauchy sequence in 2-normed spaces. However, in their study in the same year, they generalized 2-normed spaces to n-normed spaces (Gunawan & Mashadi, 2001). In recent years, the new studies have been carried out on the equivalence of n-normed spaces

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(Diminnie et al., 1973; Dutta, 2010; Duyar et al., 2017; Oğur, 2018; Oğur, 2022).

Now let us summarize the mentioned concepts by using their sources (Dutta, 2010; Gunawan & Mashadi, 2001; White, 1969)

Definition 1.1. If a real-valued non-negative function $d: E \times E \times E \to \mathbb{R}^+ \cup \{0\}$ satisfies the following conditions, then this function is called a 2-metric and the pair (E, d) is called a 2-metric space:

(2M1) There is a point s in E such that $d(u, v, s) \neq 0$ for different points u and v of E.

(2M2) When $d(u, v, s) = 0$, at least two of the elements u, v, s must be equal.

(2M3) $d(u, v, s) = d(u, s, v) = d(v, s, u)$.

(2M4) $d(u, v, s) \leq d(u, v, w) + d(u, w, s) + d(w, s, v)$.

Definition 1.2. (Gähler, 1965) Let E be a real linear space with dimension greater than one. A real-valued function $\|.\|$ on $E \times E$ that satisfies the following conditions is called a 2-norm on E, and the pair $(E, \|.\|)$ is called a 2-norm space:

(2N1) $\|u, v\| \geq 0$ for each $u, v \in E$ and $\|u, v\| = 0$ if and only if the set $\{u, v\}$ is linearly dependent,

(2N2) $\|u, v\| = \|v, u\|$ for each $u, v \in E$,

(2N3) $\|au, v\| = |a|\|u, v\|$ for each $u, v \in E$ and $a \in \mathbb{R}$,

(2N4) $\|u + v, s\| \leq \|u, s\| + \|v, s\|$ for each $u, v, s \in E$.

There is the following relation between 2-normed space and 2-metric space:

$$d(u, v, s) = \|u - s, v - s\|.$$

According to the property (2M2) here, it can easily be seen that $\|.\|$ is a non-negative function. Also, for each $a \in \mathbb{R}$, the equality $\|u, v\| = \|u, v + au\|$ holds. 2-normed linear spaces are a special case of 2-metric spaces.

Now let us state the definition of a convergent sequence and Cauchy sequence in the 2-normed space (White, 1969; Malceski & Anevsk, 2014; Dutta, 2010).

Definition 1.3. Let $\{u_n\}$ be a sequence in the 2-normed space E. If there is an element $u \in E$ such that $\lim_{n\to\infty} \|u_n - u, v\| = 0$ for all $v \in E$, then the sequence $\{u_n\}$ is said to be convergent to the point u and, ${}^2\lim_{n\to\infty} u_n = u$ is written to indicate this convergence.

Definition 1.4. Let E be a 2-normed space and let $\{u_n\}$ be a sequence in E. If

$${}^2\lim_{n\to\infty} \|u_n - u_m, v\| = 0$$

for a $v \in E$, then $\{u_n\}$ is called a Cauchy sequence in E.

Non-Newtonian calculus was developed by Grossmann and Katz between 1967 and 1970. They first defined classical, geometric, quadratic and harmonic calculus, then bigeometric, biquadratic and biharmonic calculus. In 1972, they completed the book that formed the basic framework of non-Newtonian calculus. The expression *-calculus is also used instead of non-Newtonian calculus. It has many applications such as science, mathematics and engineering. Çakmak and Başar (2012) obtained some new results on sequence spaces and Değirmen (2021) obtained some new results for non-Newtonian approach to C_*-algebras.

In the study conducted by Duyar, Sağır and Oğur (2015), some basic topological properties on the non-Newtonian real axis were introduced and investigated. Also, one can take a look at the recent studies such as those by Işık and Eryılmaz (2023) on the properties of linear spaces defined over non-Newtonian fields and by Rohman and Eryılmaz (2023) on fundamental results in ν-normed spaces, and by Sager and Sağır (2021) on quasi-Banach algebra of non-Newtonian bicomplex numbers. Now, by making use of these works, and especially those of Grossmann and Katz (1972), the definitions and theorems to be used in this study will be given and the theory of non-Newtonian calculus will be briefly summarized.

Definition 1.5 A generator is an injective function whose domain is $\mathbb{R}$ and whose range is a subset A of $\mathbb{R}$. Arithmetic operations with respect to a generator $\varphi\colon \mathbb{R} \to A$, where u and v are elements of A, are defined as follows and are called as φ-arithmetic:

φ- summation $\quad u \oplus v = \varphi\{\varphi^{-1}(u) + \varphi^{-1}(v)\}$

φ- substraction $\quad u \ominus v = \varphi\{\varphi^{-1}(u) - \varphi^{-1}(v)\}$

φ- product $\quad u \odot v = \varphi\{\varphi^{-1}(u) \cdot \varphi^{-1}(v)\}$

φ- division $\quad u \oslash v = \varphi\{\varphi^{-1}(u) \,/\, \varphi^{-1}(v)\}$

φ- ordering $\quad u <_\varphi v \iff \varphi^{-1}(u) < \varphi^{-1}(v).$

φ-zero number is denoted by $\dot{0} = \varphi(0)$ and φ-one number is denoted by $\dot{1} = \varphi(1)$. In general, φ-integers are denoted as follows:

$$\ldots, \varphi(-2), \varphi(-1), \varphi(0), \varphi(1), \varphi(2), \ldots .$$

Numbers that satisfy the condition $\dot{0} <_{\varphi} u$ are called φ-positive numbers and those that satisfy the condition $u <_{\varphi} \dot{0}$ are called φ-negative numbers. Thus, according to φ-arithmetic, each integer n is denoted by $\dot{n} = \varphi(n)$. The set

$$\mathbb{R}(N)_{\varphi} = \mathbb{R}(N) = \mathbb{R}_{\varphi} = \{\varphi(u) : u \in \mathbb{R}\}$$

is called non-Newtonian real numbers set. Summation and Product operations and ordering relaion is defined as follows:

$$\oplus : \mathbb{R}_{\varphi} \times \mathbb{R}_{\varphi} \to \mathbb{R}_{\varphi}, \ \ (u,v) \to \ u \oplus v = \varphi\{\varphi^{-1}(u) + \varphi^{-1}(v)\}$$

$$\otimes : \mathbb{R}_{\varphi} \times \mathbb{R}_{\varphi} \to \mathbb{R}_{\varphi}, (u,v) \to \ u \otimes v = \varphi\{\varphi^{-1}(u) \cdot \varphi^{-1}(v)\}$$

$$\leq_{\varphi} : \ u \leq_{\varphi} v \Leftrightarrow \varphi^{-1}(u) < \varphi^{-1}(v), u \in \mathbb{R}_{\varphi}, \ v \in \mathbb{R}_{\varphi}.$$

$(\mathbb{R}_{\varphi}, \oplus, \otimes, \leq_{\varphi})$ is a complete ordered field (Çakmak & Başar, 2012). The φ-square of a number $u \in \mathbb{R}_{\varphi}$ denoted by $u^{2_{\varphi}}$ is defined as $u^{2_{\varphi}} = u \otimes u = \varphi\{[\varphi^{-1}(u)]^2\}$. Also, we define the p-non-Newtonian power with $u^{p_{\varphi}} = \varphi\{[\varphi^{-1}(u)]^p\}$ and q-non-Newtonian root with $\sqrt[q_{\varphi}]{u} = \varphi\left\{\sqrt[q]{\varphi^{-1}(u)}\right\}$ of a number $u \in \mathbb{R}_{\varphi}$. The φ-absolute value of $u \in \mathbb{R}_{\varphi}$ is denoted by

$$|u|_{\varphi} = \begin{cases} u \ \ , \ \dot{0} <_{\varphi} u \\ 0 \ \ , \ u = \dot{0} \\ \dot{0} \ominus u, \ u <_{\varphi} \dot{0} \end{cases} .$$

The φ- absolute value has the following peoperties with $u, v \in \mathbb{R}_{\varphi}$:

(i) $|u \otimes v|_{\varphi} = |u|_{\varphi} \otimes |v|_{\varphi}$

(ii) $|u \oplus v|_{\varphi} = |u|_{\varphi} \oplus |v|_{\varphi}$

(iii) $\left||u|_{\varphi} \ominus |v|_{\varphi}\right|_{\varphi} \leq_{\varphi} |u \ominus v|_{\varphi}$.

Definition 1.6. If a function $d_{\varphi} : E \times E \to \mathbb{R}_{\varphi}$ satisfies the following non-Newtonian metric axioms, where E is a non-empty set, then the function d_φ is called a φ-metric on E and the pair (E, d_φ) is called a non-Newtonian metric space: For $u, v, z \in E$,

(φ_{M1}) $d_\varphi(u, v) = \dot{0} \Leftrightarrow u = v$,

(φ_{M2}) $d_\varphi(u, v) = d_\varphi(v, u)$,

(φ_{M3}) $d_\varphi(u, v) \leq_\varphi d_\varphi(u, z) \oplus d_\varphi(z, v)$.

The function $d_\varphi: \mathbb{R}_\varphi \times \mathbb{R}_\varphi \to \mathbb{R}_\varphi,\ (u, v) \to d_\varphi(u, v) = |u \ominus v|_\varphi$ is a non-Newtonian metric on $\mathbb{R}_\varphi$.

Definition 1.7. (Değirmen, 2021; Çakmak & Başar, 2012) Let E be a non-empty set and let the φ-addition and φ-product operations on E be defined as follows:

$$\oplus_E: E \times E \to E, (u, v) \to u \oplus_E v,$$

$$\odot_E: \mathbb{R}_\varphi \times E \to E, (a, v) \to a \odot_E v.$$

In this case, if the following conditions are satisfied, then the set E is called a linear space over the field $\mathbb{R}_\varphi$:

(1) $(E, \oplus_E)$ is a abelian group by φ- arithmetic.

(2) $a \odot_E (u \oplus_E v) = (a \odot_E u) \oplus_E (a \odot_E v)$

for all $u, v \in E$ and $a \in \mathbb{R}_\varphi$.

(3) $(a \oplus_E b) \odot_E u = (a \odot_E u) \oplus_E (a \odot_E v)$

for all $u, v \in E$ and $a, b \in \mathbb{R}_\varphi$.

(4) $(a \odot_E b) \odot_E u = a \odot_E (b \odot_E u)$

for all $u \in E$ and $a, b \in \mathbb{R}_\varphi$.

(5) $u \odot_E 1_{\mathbb{R}_\varphi} = u$, where $1_{\mathbb{R}_\varphi}$ is the unity of $\mathbb{R}_\varphi$.

The establishment of the $*$-calculus is accomplished by employing two generators, namely $\gamma: \mathbb{R} \to A = \mathbb{R}_\gamma$ and $\varphi: \mathbb{R} \to B = \mathbb{R}_\varphi$, which are chosen arbitrarily. Let γ and φ defined an isomorphism ι as follows:

$$\iota: \mathbb{R}_\gamma \to \mathbb{R}_\varphi, \iota(u) = \varphi\left(\gamma^{-1}(u)\right).$$

$$\begin{array}{ccc} \mathbb{R} & & \mathbb{R} \\ \gamma \downarrow & & \downarrow \varphi \\ \mathbb{R}_\gamma & \underset{\iota}{\to} & \mathbb{R}_\varphi \end{array}$$

Then

$$\iota\left(u \oplus_\gamma v\right) = \iota(u) \oplus_\varphi \iota(v)$$

$$\iota\left(u \ominus_\gamma v\right) = \iota(u) \ominus_\varphi \iota(v)$$

$$\iota\left(u \otimes_\gamma v\right) = \iota(u) \otimes_\varphi \iota(v)$$

$$\iota\left(u \oslash_\gamma v\right) = \iota(u) \oslash_\varphi \iota(v) \ (v \neq \dot{0})$$

$$u <_\gamma v \Leftrightarrow \iota(u) <_\varphi \iota(v),$$

for any $u, v \in \mathbb{R}_\gamma$.

Definition 1.8. (Çakmak & Başar, 2012; Rohman & Eryılmaz, 2023) Let E be a linear space over $\mathbb{R}_\varphi$. The function $\|.\|_\varphi: E \to \mathbb{R}_\varphi$ is called φ-norm on E, if it is satisfies

(φN_1) $\|u\|_\varphi \geq_\varphi \dot{0} = \varphi(0)$ and $\|u\|_\varphi = \dot{0} \Leftrightarrow u = 0$,

(φN_2) $\|\mu \odot_E u\|_\varphi = |\mu|_\varphi \otimes \|u\|_\varphi$

(φN_3) $\|u \oplus_E v\|_\varphi \leq \|u\|_\varphi \oplus \|v\|_\varphi$

for all $u, v \in E$ and $\mu \in \mathbb{R}_\varphi$. The ordered pair $\left(E, \|.\|_\varphi\right)$ is called non-Newtonian normed space or φ-normed space. Here $\|u\|_\varphi = \iota(\|u\|)$. It is clear that

$$\left|\|u\|_\varphi - \|v\|_\varphi\right|_\varphi \leq_\varphi \|u \ominus_E v\|_\varphi.$$

The inequality shows that the function $u \to \|u\|_\varphi$ is a φ-continuous function. Furthermore, for any $u, u_0, v, v_0 \in E$ and $\mu, \mu_0 \in \mathbb{R}_\varphi$,

$$\|(u \oplus_E v) \ominus_E (u_0 \oplus_E v_0)\|_\varphi \leq_\varphi \|u \ominus_E u_0\|_\varphi \oplus \|v \ominus_E v_0\|_\varphi$$

and

$$\|(\mu \odot_E u) \ominus_E (\mu_0 \odot_E u_0)\|_\varphi \leq_\varphi \left(|\mu|_\varphi \odot \|u \ominus_E u_0\|_\varphi\right)$$
$$\oplus\left(|u_0|_\varphi \odot \|\mu \ominus_E \mu_0\|_\varphi\right).$$

The continuity of φ-norm function implies that the mappings $(u, v) \to u \oplus_E v$ and $(\mu, v) \to \mu \odot_E v$ are φ-continuous from E to E.

In this study, for the first time, a non-Newtonian 2-normed space (in short, a 2φ-normed space) will be introduced and the non-Newtonian counterparts of the concepts of convergence of sequences and Cauchy sequence known for 2-normed spaces and and some related basic properties will be investigated.

Main Results

Definition 2.1. Let E be a nonempty set. If a function d_φ on $E \times E \times E$ to $R_\varphi^+ \cup \{\dot{0}\}$ satisfies the following properties, then d_φ is called a non-Newtonian 2-metric (or 2_φ-metric) on E and the pair (E, d) is called a non-Newtonian 2-metric space (or 2_φ-metric space):

(i) There is an element z in E such that $d_\varphi(u, v, z) \neq \dot{0} = \varphi(0)$ for different points u and v of E.

(ii) When $d_\varphi(u, v, z) = \dot{0} = \varphi(0)$, at least two of the elements u, v, z must be equal.

(iii) $d_\varphi(u, v, z) = d_\varphi(u, z, v) = d_\varphi(v, z, u)$.

(iv) $d_\varphi(u, v, z) \leq_\varphi d_\varphi(u, v, w) \oplus d_\varphi(u, w, z) \oplus d_\varphi(w, v, z)$.

Definition 2.2. Let E be a φ-linear space of dimension greater than $\dot{1} = \varphi(1)$ on the field $\mathbb{R}_\varphi$. If a function $\|.,.\|_\varphi$ on

$E \times E$ to R_φ^+ satisfies the following properties, then this function is called a non-Newtonian 2-norm (or 2_φ-norm) on E and the pair $(E, \|.,.\|_\varphi)$ is called a non-Newtonian 2-normed space (or 2_φ-normed space):

(N1) $\|u, v\|_\varphi = \dot{0} = \varphi(0) \Leftrightarrow$ u and v are linearly dependent,

(N2) $\|u, v\|_\varphi = \|v, u\|_\varphi$ for all $u, v \in E$,

(N3) $\|a \odot_E u, v\|_\varphi = |a|_\varphi \odot \|u, v\|_\varphi$

for all $u, v \in E$ and $a \in \mathbb{R}_\varphi$,

(N4) $\|u \oplus_E v, z\|_\varphi \leq_\varphi \|u, z\|_\varphi \oplus \|v, z\|_\varphi$ for all $u, v, z \in E$.

In parallel with the definition of φ-normed space,

$$\|u, v\|_\varphi = \iota(\|u, v\|)$$

is written for all $u, v \in E$. Obviously, there is equality

$$d_\varphi(u, v, z) = \|u \ominus_E z, v \ominus_E z\|_\varphi$$

between 2_φ-metric and 2_φ-norm. From the definitions of the non-Newtonian 2-metric space and the 2_φ-normed space, it is seen that the φ-real valued function $\|.,.\|_\varphi$ does not take φ-negative values. Also, for all $u, v \in E$ and all $a \in \mathbb{R}_\varphi$, the equality $\|u, v\|_\varphi = \|u, v \oplus_E a \odot_E u\|_\varphi$ holds.

The definitions and basic properties of the concepts of convergence of a sequence and Cauchy sequence in non-Newtonian normed spaces can be found in the work of Rohman and İlker (2023). In this section, the definitions of convergence of a sequence in 2_φ-

normed space and Cauchy sequence will be given and some related properties will be given.

Definition 2.3. Let $\{u_n\}$ be a sequence in a non-Newtonian 2-normed linear space E. If ,for every $v \in E$, we can find a $u \in E$ such that

$$\lim_{n\to\infty} \|u_n \ominus_E u, v\|_\varphi = \dot{0},$$

then the sequence $\{u_n\}$ is said to 2_φ -converge to u and ${}^{2_\varphi}\lim_{n\to\infty} u_n = u$ or $u_n \overset{2_\varphi}{\to} u$ is written.

Definition 2.4. Let E be a 2_φ-normed linear space and let $\{u_n\}$ be a sequence in this space. If there are linearly independent $v, z \in E$ such that

$$\lim_{n,m\to\infty} \|u_n \ominus_E u_m, v\|_\varphi = \dot{0} \text{ and } \lim_{n,m\to\infty} \|u_n \ominus_E u_m, z\|_\varphi = \dot{0}$$

then the sequence $\{u_n\}$ is said to 2_φ-Cauchy sequence.

Theorem 2.5. Let E be a 2_φ-normed linear space.

(i) If $\{u_n\}$ is a 2_φ-Cauchy sequence in E with respect to u and v, then $\{\|u_n, u\|_\varphi\}$ and $\{\|u_n, v\|_\varphi\}$ are $\mathbb{R}_\varphi$- Cauchy sequences.

(ii) If $\{u_n\}$ and $\{v_n\}$ are 2_φ-Cauchy sequences in E with respect to u and v, and also $\{\mu_n\}$ is a $\mathbb{R}_\varphi$- Cauchy sequence, then $\{u_n \oplus_E v_n\}$ and $\{\mu_n \odot_E v_n\}$ are 2_φ-Cauchy sequences.

Proof. (i) Since

$$\|u_n, u\|_\varphi = \iota(\|u_n, u\|) = \iota(\|(u_n \ominus_E u_m) \oplus_E u_m, u\|)$$

$$\leq_\varphi \iota(\|(u_n \ominus_E u_m), u\| \oplus \|u_m, u\|)$$

$$= \|u_n \ominus_E u_m, u\|_\varphi \oplus \|u_m, u\|_\varphi,$$

we have

$$\|u_n, u\|_\varphi \ominus \|u_m, u\|_\varphi \leq_\varphi \|u_n \ominus_E u_m, u\|_\varphi.$$

Similarly, one gets

$$\|u_m, u\|_\varphi \ominus \|u_n, u\|_\varphi \leq_\varphi \|u_n \ominus_E u_m, u\|_\varphi.$$

Therefore

$$\left| \|u_n, u\|_\varphi \ominus \|u_m, u\|_\varphi \right|_\varphi \leq_\varphi \|u_n \ominus_E u_m, u\|_\varphi.$$

Thus $\{\|u_n, u\|_\varphi\}$ is a $\mathbb{R}_\varphi$ -Cauchy sequence, since $\lim_{n,m\to\infty} \|u_n \ominus_E u_m, u\|_\varphi = \dot{0}$. Similarly, it can be seen that $\{\|u_n, v\|_\varphi\}$ is an $\mathbb{R}_\varphi$-Cauchy sequence.

(ii) We have

$$\|(u_n \oplus_E v_n) \ominus_E (u_m \oplus_E v_m), u\|_\varphi$$

$$= \|(u_n \ominus_E v_n) \oplus_E (u_m \ominus_E v_m), u\|_\varphi$$

$$\leq_\varphi \|u_n \ominus_E u_m, u\|_\varphi \oplus \|v_n \ominus_E v_m, u\|_\varphi$$

and similarly

$$\|(u_n \oplus_E v_n) \ominus_E (u_m \oplus_E v_m), v\|_\varphi \leq_\varphi \|u_n \ominus_E u_m, u\|_\varphi$$

$$\oplus \|v_n \ominus_E v_m, v\|_\varphi.$$

Then, according to the hypothesis, $\{u_n \oplus_E v_n\}$ is a 2_φ-Cauchy sequence in E. Since

$$\|(\mu_n \odot_E u_n) \ominus_E (\mu_m \odot_E u_m), u\|_\varphi$$

$$= \iota(\|(\mu_n \odot_E u_n) \ominus_E (\mu_m \odot_E u_m), u\|)$$

$$\leq_\varphi \iota(\|(\mu_n \odot_E u_n) \ominus_E (\mu_n \odot_E u_m), u\| \oplus \|(\mu_n \odot_E u_m) \ominus_E (\mu_m \odot_E u_m), u\|)$$

$$= \iota\big(|\mu_n|_\varphi \odot_E \|u_n \ominus_E u_m, u\| \oplus |\mu_n - \mu_m|_\varphi \odot_E \|u_m, u\|\big)$$

$$= \iota\big(|\mu_n|_\varphi \odot_E \|u_n \ominus_E u_m, u\|\big) \oplus \iota\big(|\mu_n - \mu_m|_\varphi \odot_E \|u_m, u\|\big)$$

$$= |\mu_n|_\varphi \odot_E \|u_n \ominus_E u_m, u\|_\varphi \oplus |\mu_n - \mu_m|_\varphi \odot_E \|u_m, u\|_\varphi,$$

Using the fact that $\{u_n\}$ is a 2_φ-Cauchy sequences in E and $\{\mu_n\}$ is a $\mathbb{R}_\varphi$- Cauchy sequence, we have

$$\|(\mu_n \odot_E u_n) \ominus_E (\mu_m \odot_E u_m), u\|_\varphi \to 0.$$

Similarly

$$\|(\mu_n \odot_E u_n) \ominus_E (\mu_m \odot_E u_m), v\|_\varphi \to 0.$$

Hence $\{\mu_n \odot_E u_n\}$ is a 2_φ-Cauchy sequence in E.

Theorem 2.6. In any 2_φ-normed linear space, The following properties hold:

(i) If $u_n \overset{2_\varphi}{\to} u$ and $v_n \overset{2_\varphi}{\to} v$, then $u_n \oplus_E v_n \overset{2_\varphi}{\to} u \oplus_E v$.

(ii) If $u_n \overset{2_\varphi}{\to} u$ and $\mu_n \overset{|.|_\varphi}{\longrightarrow} \mu$, then $\mu_n \odot_E u_n \overset{2_\varphi}{\to} \mu \odot_E u$.

(iii) If $\dim E \geq 2$, $u_n \overset{2_\varphi}{\to} u$ and $u_n \overset{2_\varphi}{\to} v$, then $u = v$.

Proof. (i) By Definition 2.2 (N4), we have

$$\|(u_n \oplus_E v_n) \ominus_E (u \oplus_E v), z\|_\varphi$$

$$= \|(u_n \ominus_E u) \oplus_E (v_n \ominus_E v), z\|_\varphi$$

$$\leq_\varphi \|u_n \ominus_E u, z\|_\varphi \oplus \|v_n \ominus_E v, z\|_\varphi \overset{2_\varphi}{\to} 0_\varphi$$

for each $z \in E$ and so $u_n \oplus_E v_n \overset{2\varphi}{\rightarrow} u \oplus_E v$.

(ii) By Definition 2.2 (N3) and (N4), we have

$$\|\mu_n \odot_E u_n \ominus_E \mu \odot_E u, z\|_\varphi$$

$$= \|(\mu_n \odot_E u_n \ominus_E \mu_n \odot_E u) \oplus_E (\mu_n \odot_E u \ominus_E \mu \odot_E u), z\|_\varphi$$

$$\leq_\varphi \|\mu_n \odot_E u_n \ominus_E \mu_n \odot_E u, z\|_\varphi \oplus \|\mu_n \odot_E u \ominus_E \mu \odot_E u, z\|_\varphi$$

$$= \left(|\mu_n|_\varphi \odot \|u_n \ominus_E u, z\|_\varphi\right) \oplus \left(|\mu_n - \mu|_\varphi \odot \|u, z\|_\varphi\right)$$

$$\leq_\varphi M\|u_n \ominus_E u, z\|_\varphi \oplus \left(|\mu_n - \mu|_\varphi \odot \|u, z\|_\varphi\right),$$

using the fact that a φ-convergent sequence is bounded. Hence we find $\mu_n \odot_E u_n \overset{2\varphi}{\rightarrow} \mu \odot_E u$, since $u_n \overset{2\varphi}{\rightarrow} u$ and $\mu_n \overset{|.|_\varphi}{\longrightarrow} \mu$.

(iii) We can write

$$\|u \ominus_E v, z\|_\varphi \leq_\varphi \|u_n \ominus_E v, z\|_\varphi \oplus \|u \ominus_E u_n, z\|_\varphi$$

and $\|u \ominus_E v, z\|_\varphi = 0_\varphi$ for for each $z \in E$, since $u_n \overset{2\varphi}{\rightarrow} u$ and $u_n \overset{2\varphi}{\rightarrow} v$. Therefore $u \ominus_E v$ is linearly dependent on z for all $z \in E$. Since dim$E$, It is only possible for $u \ominus_E v$ to be linearly dependent on each z if $u \ominus_E v = 0$.

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CHAPTER II

On Measurable Sets in Multiplicative Calculus

Oğuz OĞUR[1]

Introduction

Classical Newtonian analysis has long been a fundamental tool in mathematical analysis. However, the **Non-Newtonian Calculus**, developed by (Grossman & Katz 1972), aimed to overcome the limitations of traditional analysis and introduced a new perspective. This alternative type of calculus, which replaces addition and multiplication with multiplication and exponential operations, offers a distinctive approach to mathematical modeling and problem-solving (Grossman, 1979), (Stanley, 1999).

Non-Newtonian and Multiplicative Calculus have provided innovative solutions in areas such as differential and integral equations, sequence spaces, and calculus of variations. (Bashirov,

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Kurpinar & Özyapıcı, 2008) laid the theoretical foundations of Multiplicative Calculus and emphasized its importance in applications. This method has proven to be a significant tool, especially in complex analysis (Uzer, 2010) and the study of function spaces (Çakmak & Başar, 2014).

Studies on sequence spaces and matrix transformations (Çakmak & Başar, 2012, 2015; Türkmen & Başar, 2012) have highlighted the impact of non-Newtonian and Multiplicative Calculus in abstract mathematics. (Değirmen, 2021) conducted an in-depth analysis of C*-algebras using Non-Newtonian approaches, while (Değirmen & Duyar, 2023) explored Non-Newtonian interpretations of Fibonacci and Lucas numbers. (Torres, 2021) demonstrated the applicability of this type of calculus in the calculus of variations, opening new avenues for solving mathematical problems.

Additionally, studies on integral equations (Güngör, 2020a, 2020b, 2022) have shown how Non-Newtonian analysis diversifies solution methods. In topology, (Duyar, Sağır & Oğur, 2015), as well as (Duyar & Oğur, 2017), investigated fundamental topological properties of the Non-Newtonian real line. Also, recent studies, such as those by (Işık & Eryılmaz, 2023) on the properties of linear spaces defined over Non-Newtonian fields and by (Rohman & Eryılmaz, 2023) on fundamental results in ν-normed spaces, have contributed significantly to the understanding and development of alternative mathematical frameworks.

Finally, Non-Newtonian analysis has found a broad application area in measure theory. (Oğur & Demir, 2019, 2020) examined the Non-Newtonian Lebesgue measure, while (Duyar & Sağır, 2017) studied its implications on the Non-Newtonian interpretation of real numbers. (Oğur & Güneş, 2024) extended these findings by exploring Non-Newtonian measurable sets, further demonstrating the potential of this type of calculus in abstract mathematics.

Now, let's introduce some basic concepts in non-Newtonian analysis. Let ρ be a generator which is an injective function from $\mathbb{R}\ \ to\ A = \mathbb{R}(N)_\rho \subseteq \mathbb{R}$. Let's define the non-Newtonian algebraic operations as follows;

$$
\begin{array}{ll}
\rho - addition & s \dot{+} t = \rho(\rho^{-1}(s) + \rho^{-1}(t)) \\
\rho - subtraction & s \dot{-} t = \rho(\rho^{-1}(s) - \rho^{-1}(t)) \\
\rho - multiplication & s \dot{\times} t = \rho(\rho^{-1}(s) \times \rho^{-1}(t)) \\
\rho - division & s \dot{\div} t = \rho(\rho^{-1}(s) \div \rho^{-1}(t)) \\
\rho - order & s \dot{<} t \Leftrightarrow \rho^{-1}(s) < \rho^{-1}(t)
\end{array}
$$

for any $s, t \in \mathbb{R}(N)_\rho$ (Grossman & Katz, 1972). The non-Newtonian absolute value of any element of $t \in \mathbb{R}(N)_\rho$ defines as follows

$$
|t|_\rho = \begin{cases} t, & if\ t \dot{>} \dot{0} \\ \dot{0}, & if\ t \doteq \dot{0} \\ \dot{0} \dot{-} t & if\ t \dot{<} \dot{0} \end{cases}
$$

where $\rho(0) = \dot{0}$. Also, we have $\sqrt[n]{t}^{\rho} = \rho\left(\sqrt[n]{\rho^{-1}(t)}\right)$ and $t^{n_\rho} = \rho\left(\left(\rho^{-1}(t)\right)^n\right)$ (Grossman & Katz, 1972).

At this point, we can introduce *geometric analysis*, a specific case within the broader framework of non-Newtonian analysis. For this purpose, setting $\rho(s) = \exp\,(s)$ will be sufficient. Let

$$\rho: \mathbb{R} \to \mathbb{R}^+, \ s \to \rho(s) = exp\ (s)$$

and so

$$\rho^{-1}: \mathbb{R}^+ \to \mathbb{R}, \ \rho^{-1}(t) = \ln(t)\ .$$

Thus, we get

$$\mathbb{R}(N)_\rho = \mathbb{R}_{exp} = \{exp(s): s \in \mathbb{R}\} = \mathbb{R}^+,$$

$$\mathbb{R}^+_{exp} = \{exp(s): s \in \mathbb{R}^+\} = (1, +\infty)$$

and

$$\mathbb{R}^-_{exp} = \{exp(s): s \in \mathbb{R}^-\} = (0,1)$$

where $exp(0) = 1$. By the definition, we have the multiplicative sum of $s, t \in \mathbb{R}_{exp}$ as follows;

$$s \dot{+} t = exp(lns + lnt) = e^{\ln(st)} = st.$$

By using similar way, we get the multiplicative algebraic operations as follows;

$s \dot{+} t = st$
$s \dot{-} t = {}^{s}/_{t}$
$s \dot{\times} t = s^{lnt}$
$s \dot{\div} t = s^{\frac{1}{lnt}}$

In this section, we will use the symbols $(,)_{exp}$, λ_{exp}, ${}_{exp}\sum.$, ${}^{exp}inf$, ${}^{exp}sup$ to represent the open interval, Lebesgue measure, sum, infimum and supremum, respectively, in the context of multiplicative calculus.

Let us now present some definitions and results commonly known in real analysis; to this end, it will suffice to provide the fundamental concepts outlined in Natanson's book (Natanson, 1964).

Definition 1. The length of an open interval $(a, b), i.e., b - a,$ is called the measure of the interval (a, b). This number is written as

$$\lambda(a, b) = b - a$$

(Natanson, 1964).

Definition 2. The measure $\lambda(G)$ of a non-void bounded open set G is the sum of the lengths of all its component intervals ϱ_k;

$$\lambda(G) = \sum_k \lambda(\varrho_k)$$

(Natanson, 1964).

Theorem 1. Let G_1 and G_2 be two bounded open sets. If $G_1 \subset G_2$, then

$$\lambda(G_1) \leq \lambda(G_2)$$

(Natanson, 1964).

Theorem 2. If the bounded open set G is the union of a finite or denumerable family of pairwise disjoint open sets, i. e.

$$G = \bigcup_k G_k \ , \quad G_k \cap G_l = \emptyset \ \ for \ \ k \neq l,$$

then

$$\lambda(G) = \sum_k \lambda(G_k)$$

(Natanson, 1964).

Definition 3. The measure of a non-void bounded closed set F is the number

$$\lambda(F) = D - C - \lambda(T - F)$$

where $T = [C, D]$ is the smallest closed interval containing the set F (Natanson, 1964).

Definition 4. The outer measure $\lambda^o(E)$ of a bounded set E is defined as

$$\lambda^o(E) = inf\{\lambda(T): E \subset T,\ \ T\ is\ bounded\ open\ set\}$$

(Natanson, 1964).

Definition 5. The inner measure $\lambda^i(E)$ of a bounded set E is defined as

$$\lambda^i(E) = sup\{\lambda(U): U \subset E,\ \ U\ is\ bounded\ closed\ set\}$$

(Natanson, 1964).

It is clear that by the definition of inner and outer measure, the inner $\lambda^i(E)$ and outer measure $\lambda^o(E)$ are well-defined for every bounded set E.

The definition of measure of open intervals, as known in real analysis, has been extended to Non-Newtonian analysis by Duyar and Sağır as follows;

Definition 6. The measure $\lambda_N(s, t)$ of an open interval (s, t) in $\mathbb{R}(N)_\rho$ is defined by

$$\lambda_N(s, t) = \rho\big(\lambda(\rho^{-1}(s), \rho^{-1}(t))\big) = t \dot{-} s$$

(Duyar&Sağır, 2017).

Additionally, in this study, fundamental definitions and theorems regarding the measurements of non-Newtonian bounded open sets are presented. Considering the definition above, the measure of closed sets in Non-Newtonian analysis was provided by Oğuz and Demir as follows;

Definition 7. The measure of a non-void bounded closed set F in $\mathbb{R}(N)_\rho$ is defined as follows;

$$\lambda_N \mathrm{F} = \rho\left(\lambda(\rho^{-1}(C), \rho^{-1}(D)) - \lambda\big(\rho^{-1}(S - F)\big)\right)$$

where $S = [C, D]$ is the smallest closed interval containing the set F in $\mathbb{R}(N)_\rho$ (Oğur & Demir, 2020).

In this section, these concepts, which have been extended to Non-Newtonian analysis, will be further adapted to geometric analysis by taking the special case $\rho(x) = exp(x)$. The fundamental definitions and properties of measurable sets in geometric analysis will be examined. This special case will provide convenience for applications of multiplicative measurable sets.

Following this general information, we can now give the definition of the Lebesgue measure of open sets in geometric analysis by taking $\rho(x) = exp(x)$ in (Duyar & Sağır, 2017).

Definition 8. The measure of a multiplicative open interval $(s, t)_{exp}$ is defined by

$$\begin{aligned} \lambda_{exp}(s, t)_{exp} &= exp\big(\lambda(lns, lnt)\big) \\ &= exp(lnt - lns) \\ &= exp\left(ln\frac{t}{s}\right) \\ &= \frac{t}{s} \end{aligned}$$

where λ is Lebesgue measure in real line.

Example 1. The measure of the interval $(1,4)_{exp}$ can be found as follows

$$\lambda_{exp}(1,4)_{exp} = exp\big(\lambda(ln1, ln4)\big)$$
$$= exp\big((ln4 - ln1)\big)$$
$$=4.$$

Example 2. $\lambda_{exp}(e^1, e^3)_{exp} = exp\big(\lambda(lne^1, lne^3)\big)$
$$= e^2.$$

Definition 9. Let (γ_k) be a family of component of intervals of bounded open set $T\ in\ \mathbb{R}_{exp}$. Then, the multiplicative measure of G is defined as follows

$$\lambda_{exp}(T) = {}_{exp}\sum_k \lambda_{exp}(s_k, t_k)_{exp}$$
$$= {}_{exp}\sum_k (t_k \dot{-} s_k)$$
$$= {}_{exp}\sum_k (t_k \div s_k)$$
$$=\exp(\sum_k ln(t_k \div s_k))$$
$$=\exp\big(ln(\prod_k (t_k \div s_k))\big)$$
$$=\prod_k (t_k \div s_k)$$

where $\gamma_k = (s_k, t_k)_{exp}$.

Example 3. Let $T = (1,3)_{exp} \cup (4,7)_{exp} \cup (16, e^6)_{exp}$. Then, the multiplicative measure of T is

$$\lambda_{exp}(T) = \frac{3}{1}\frac{7}{4}\frac{e^6}{16} = \frac{21e^6}{64}.$$

Having defined the Lebesgue measure for open sets in the multiplicative sense, we can now proceed to define the measure for bounded, closed sets within this framework.

Definition 10. Let U be a bounded, closed set in the sense of multiplicative sense. Then, the multiplicative measure of U is defined as

$$\lambda_{exp}(U) = exp\{\lambda(lnK, lnL) - \lambda\big(ln(S - U)\big)\}.$$

Here, the set $S = [K, L]_{exp}$ is smallest interval containing the set U. It is note that the set $S - U$ is an open set in sense of multiplicative calculus.

Example 4. Let $U = [e^1, e^2]_{exp} \cup [e^3, e^4]_{exp}$. Then, the multiplicative measure of U is

$$\lambda_{exp}(U) = exp\{\lambda(lne^1, lne^4) - \lambda(lne^2, lne^3)\}$$

$$=exp\{(lne^4 - lne^1) - (lne^3 - lne^2)\}$$

$$=exp\left\{ln\left(\frac{e^4e^2}{e^1e^3}\right)\right\}$$

$$= e^2.$$

Theorem 3. If the set U can be write as union of finite pairwise disjoint sets in multiplicative sense, i. e. $U = \bigcup_{l=1}^{r} U_l$, $U_l \cap U_s = \emptyset \; for \; l \neq s$. Then, we have

$$\lambda_{exp}(U) = \prod_{l=1}^{r}\Big(\lambda_{exp}(U_l)\Big).$$

Proof. It is clear that $ln(U) = \bigcup_{l=1}^{r} ln(U_l)$. Then, by the measure properties in real case, we have

$$\lambda_{exp}\big(ln(U)\big) = exp\Big(\sum_{l=1}^{r} \lambda\big(ln(U_l)\big)\Big)$$

$$= exp\left(\sum_{l=1}^{r} ln\left(exp\left(\lambda\big(ln(U_l)\big)\right)\right)\right)$$

$$= exp\left(\sum_{l=1}^{r} ln\Big(\lambda_{exp}(U_l)\Big)\right)$$

$$= exp\left(ln\left(\prod_{l=1}^{r} \lambda_{exp}(U_l)\right)\right)$$

$$= \prod_{l=1}^{r}\left(\lambda_{exp}(U_l)\right).$$

Definition 11. Let E be a non-empty, bounded set in $\mathbb{R}_{exp}$. The multiplicative outer and inner measure of E are defined as follows, respectively,

$$\lambda_{exp}^{o}(E) = {}^{exp}inf\left\{ \lambda_{exp}(G): E \subset G, G \; is \; open \; bounded \; set \; in \; \mathbb{R}_{exp}\right\}$$

and

$$\lambda_{exp}^{i}(E) = {}^{exp}sup\left\{ \lambda_{exp}(F): F \subset E, G \; is \; closed \; set \; in \; \mathbb{R}_{exp}\right\}.$$

Theorem 4. If $G \; is \; an \; open, bounded \; set \; in \; \mathbb{R}_{exp}$, then

$$\lambda_{exp}(G) = \lambda_{exp}^{o}(G) = \lambda_{exp}^{i}(G).$$

Proof. Let E be an open set with $G \subset E$ and F be a closed set with $F \subset G \; in \; \mathbb{R}_{exp}$. Thus, we have

$$\lambda_{exp}^{o}(G) = {}^{exp}inf_{G\subset E}\left\{ \lambda_{exp}(E)\right\}$$

$$= exp\left\{inf_{lnG\subset lnE} ln\left(\lambda_{exp}(E)\right)\right\}$$

$$= exp\left\{inf_{lnG\subset lnE} ln\left(exp\left(\lambda\left(ln(E)\right)\right)\right)\right\}$$

$$= exp\left\{inf_{lnG\subset lnE} \lambda\left(ln(E)\right)\right\}$$

$$= exp\left\{\lambda\left(ln(G)\right)\right\}$$

$$= \lambda_{exp}(G)$$

and

$$\lambda_{exp}^{i}(G) = {}^{exp}sup_{F\subset G}\left\{ \lambda_{exp}(F)\right\}$$

$$= exp\left\{sup_{lnF\subset lnG} ln\left(\ \lambda_{exp}(F)\right)\right\}$$
$$= exp\left\{sup_{lnF\subset lnG} ln\left(exp\left(\lambda\left(ln(F)\right)\right)\right)\right\}$$
$$= exp\{sup_{lnF\subset lnG}\lambda\left(ln(F)\right)\}$$
$$= exp\{\lambda\left(ln(F)\right)\}$$
$$= \ \lambda_{exp}(G)$$

which gives the proof.

Theorem 5. If F is a bounded, closed set $in\ \mathbb{R}_{exp}$, then we have

$$\lambda_{exp}(F) = \lambda^{o}_{exp}(F) = \lambda^{i}_{exp}(F).$$

Proof. The proof is derived in a manner similar to the one above.

Theorem 6. For every bounded set $M\ in\ \mathbb{R}_{exp}$ we have the following inequality;

$$\lambda^{i}_{exp}(M) \dot{\leq} \lambda^{o}_{exp}(M).$$

Proof. Let G be an open set with $M \subset G$ and F be a closed set with $F \subset M\ in\ \mathbb{R}_{exp}$. Then, we get

$$\lambda^{i}_{exp}(M) = {}^{exp}sup_{F\subset M}\{\ \lambda_{exp}(F)\}$$
$$= exp\left\{sup_{lnF\subset lnM} ln\left(\ \lambda_{exp}(F)\right)\right\}$$
$$= exp\left\{sup_{lnF\subset lnM} ln\left(exp\left(\lambda\left(ln(F)\right)\right)\right)\right\}$$
$$= exp\{sup_{lnF\subset lnM}\lambda\left(ln(F)\right)\}$$
$$\dot{\leq} exp\{inf_{lnM\dot{\subset} lnG}\lambda\left(ln(G)\right)\}$$

$$= exp\left\{inf_{lnM\subset lnG} ln\left(exp\left(\lambda\big(ln(G)\big)\right)\right)\right\}$$

$$= exp\left\{inf_{lnM\subset lnG} ln\left(\ \lambda_{exp}(G)\right)\right\}$$

$$= {}^{exp}inf_{M\subset G}\{\ \lambda_{exp}(G)\}$$

$$= \lambda^{o}_{exp}(M).$$

Theorem 7. Let C and D be two bounded sets $in\ \mathbb{R}_{exp}$. If $C \subset D$, then we have

$$\lambda^{i}_{exp}(C) \dot{\leq} \lambda^{i}_{exp}(D)$$

and

$$\lambda^{o}_{exp}(C) \dot{\leq} \lambda^{o}_{exp}(D).$$

Proof. Let E be an open set with $C \subset E$ and G be a closed set with $G \subset M\ in\ \mathbb{R}_{exp}$. Then, we have

$$\lambda^{i}_{exp}(C) = {}^{exp}sup_{G\subset C}\{\ \lambda_{exp}(G)\}$$

$$= exp\left\{sup_{lnG\subset lnC} ln\left(\ \lambda_{exp}(G)\right)\right\}$$

$$= exp\left\{sup_{lnG\subset lnC} ln\left(exp\left(\lambda\big(ln(G)\big)\right)\right)\right\}$$

$$= exp\left\{sup_{lnG\subset lnC} \lambda\big(ln(G)\big)\right\}$$

$$\dot{\leq} exp\left\{sup_{lnL\subset lnD} \lambda\big(ln(L)\big)\right\}$$

$$= exp\left\{sup_{lnL\subset lnD} ln\left(exp\left(\lambda\big(ln(L)\big)\right)\right)\right\}$$

$$= exp\left\{sup_{lnL\subset lnD} ln\left(\ \lambda_{exp}(L)\right)\right\}$$

$$= {}^{exp}sup_{L\subset D}\{\ \lambda_{exp}(L)\}$$

$$= \lambda^{i}_{exp}(D).$$

The second inequality can be obtained by similar way.

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CHAPTER III

Non-Newtonian Approach to Fixed Point Problem in C^*- Algebra Valued Metric Spaces

Nilay DEĞİRMEN [1]

Introduction and Preliminaries

The Banach fixed point theorem (Banach, 1922) claims that any contraction mapping defined on a complete metric space has exactly one fixed point. Since then, the theorem has been generalized to various other types of metric spaces, leading to new findings and results.

If Λ is a complex Banach algebra with involution $y \to y^*$ and $\|y^* y\| = \|y\|^2$ for all $y \in \Lambda$, then we say that Λ is a C^*- algebra (Murphy, 2014). In 2014, Ma et al. (Ma, Jiang & Sun, 2014) defined the notion of C^*- algebra valued metric space. Since then, there

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have been many results and applications related to such spaces, we refer to (Ma and Jiang, 2015; Chandok, Kumar & Park, 2019; Asim and Imdad, 2020; Maheswari et al., 2022).

Calculus, a branch of analysis and an important area of mathematics is the mathematical study of change and motion. The history of calculus has been shaped by many contributions of mathematicians and thinkers, culminating in the 17th century when Gottfried Wilhelm Leibniz and Isaac Newton independently and simultaneously defined and developed the fundamental principles of differential and integral calculus. Michael Grosman and Robert Katz (Grossman and Katz, 1972) created non-Newtonian calculus between 1967-1970 years. This new approach uses functions called generators to reshape arithmetic operations and introduce new mathematical structures, especially multiplicative arithmetic. Non-Newtonian calculus includes various unique and infinite types of calculus, such as harmonic, bigeometric, geometric, and anageometric calculus. The field has advanced rapidly in recent years due to its wide-ranging applications in areas like engineering, economics, biology, probability theory, approximation theory and weighted calculus, which has led to substantial interest from many scholars. Particularly noteworthy are the important applications of multiplicative calculus in statistics (Carr and Cirillo, 2024), economics (Filip and Piatecki, 2014), finance (Bashirov et al., 2011), biomedical image analysis (Florack and van Assen, 2012), logistic growth models (Pinto et al., 2020), contour detection in noisy images (Mora, Córdova-Lepe & Del-Valle, 2012), linear and nonlinear signal representation (Bilgehan, 2015), physics (Czachor, 2021), geometric magnetic energy (Ekinci et al., 2024), exponential signal processing (Özyapıcı and Bilgehan, 2016), integral equations (Güngör & Durmaz, 2020) and cancer treatment (Momenzadeh, Obi & Hincal, 2022), etc. Also, Non-Newtonian calculus has been applied to topological, algebraic and analysis problems in the area of theoretical mathematics (Işık & Eryılmaz, 2023; Rohman & Eryılmaz, 2023; Duyar & Oğur, 2017; Oğur & Güneş, 2024; Güngör, 2022; Değirmen & Duyar, 2023).

Arithmetic is a term usually associated with positive integers, but here the term "arithmetic" refers to a complete ordered field whose universe is a subset of R. The generator of an arithmetic system generates classical arithmetic if I is the identity function and geometric arithmetic if $\exp$ is the function. Let $\alpha : R \rightarrow U \subset R$ be a bijection. Then, it is said to be a generator with range U and defines an arithmetic. The range of generator α is denoted by R_α.

Assume that $\alpha : R \rightarrow U$ and $\beta : R \rightarrow V$ be arbitrarily selected generators and additionally $*-$ calculus be represented the ordered pair of arithmetics $(\alpha - \text{arithmetic}, \beta - \text{arithmetic})$.

$\alpha -$arithmetic $\beta -$arithmetic

Realm	$A\left(=R_\alpha\right)$	$B\left(=R_\beta\right)$
Addition	$r\dot{+}s=\alpha\left\{\alpha^{-1}(r)+\alpha^{-1}(s)\right\}$	$\ddot{+}$
Subtraction	$r\dot{-}s=\alpha\left\{\alpha^{-1}(r)-\alpha^{-1}(s)\right\}$	$\ddot{-}$
Multiplication	$r\dot{\times}s=\alpha\left\{\alpha^{-1}(r)\times\alpha^{-1}(s)\right\}$	$\ddot{\times}$
Division	$r\dot{/}s=\frac{r}{s}\alpha=\alpha\left\{\frac{\alpha^{-1}(r)}{\alpha^{-1}(s)}\right\}\left(s\neq\dot{0}\right)$	$\ddot{/}$
Ordering	$r\dot{\leq}s\Leftrightarrow\alpha^{-1}(r)\leq\alpha^{-1}(s).$	$\ddot{\leq}$

The unique function ι that serves as the isomorphism from $\alpha-$arithmetic to $\beta-$arithmetic satisfies

1. ι is injective.
2. $\iota:U\rightarrow V$ is surjective.
3. For all $\tau,\upsilon\in U$,

$$\iota\left(\tau\dot{\mp}\upsilon\right)=\iota(\tau)\ddot{\mp}\iota(\upsilon),$$

$$\iota(\tau\dot{\times}\upsilon)=\iota(\tau)\ddot{\times}\iota(\upsilon),\ \iota\left(\tau\dot{/}\upsilon\right)=\iota(\tau)\ddot{/}\iota(\upsilon),\upsilon\neq\dot{0}$$

$$\tau\dot{<}\upsilon\Leftrightarrow\iota(\tau)\ddot{<}\iota(\upsilon),$$

$$\iota(\tau)=\beta\left\{\alpha^{-1}(\tau)\right\},$$

Also, for $n\in Z$, we note that $\iota(\dot{n})=\ddot{n}$.

If $u \in R_\alpha$ and $\dot{0} \dot{<} u$ (or $u \dot{<} \dot{0}$), then we say that it is a $\alpha-$ positive number (or $\alpha-$ negative number). Additionally, R_α^+ denotes the set of $\alpha-$ positive numbers. Also, $\alpha(-u) = \alpha\left\{-\alpha^{-1}\left(\dot{u}\right)\right\} = \dot{-}\dot{u}$ for all $u \in R$. Besides, the number

$u \dot{\times} u$ is called the $\alpha-$ square of u, denoted by $u^{\dot{2}}$. If $u \in R_\alpha^+ \cup \left\{\dot{0}\right\}$, then we say that $\alpha\left[\sqrt{\alpha^{-1}(u)}\right]$ is the $\alpha-$square root

of u, denoted by $\dot{\sqrt{u}}$ (Grossman and Katz, 1922; Çakmak and Başar, 2012).

Let $\dot{u} \in \left(U, \dot{+}, \dot{-}, \dot{\times}, \dot{/}, \dot{\leq}\right)$ and $\ddot{v} \in \left(V, \ddot{+}, \ddot{-}, \ddot{\times}, \ddot{/}, \ddot{\leq}\right)$. The collection of

all $*-$points $\left(\dot{u}, \ddot{v}\right)$ is said to be the set of non-Newtonian complex

numbers and is denoted by $C(N)$, that is,

$$C(N) = \left\{\left(\dot{u}, \ddot{v}\right) \; : \; \dot{u} \in U \subseteq R, \ddot{v} \in V \subseteq R\right\}.$$

We will use the abbreviation “w.r.t.” for “with respect to”.

The $*-$ norm $\ddot{\|}.\ddot{\|}_1 \; : C(N) \to \ddot{[}\ddot{0}, \infty\ddot{)}$ of z^* is defined by

$$\ddot{\|} z^* \ddot{\|}_1 = \ddot{\sqrt{\left[\imath\left(\dot{u} \dot{-} \dot{0}\right)\right]^{\ddot{2}} \ddot{+} \left(\ddot{v} \ddot{-} \ddot{0}\right)^{\ddot{2}}}} = \beta\left(\sqrt{u^2 + v^2}\right)$$

where $z^{*}=\left(\dot{u},\ddot{v}\right)$ (Tekin and Başar, 2013).

Non-Newtonian calculus offers a new perspective in mathematical analysis by overcoming the limitations of classical calculus. In (Değirmen, 2022), the author defined the notions of a $C^{*_N}-$ algebra, a Banach $C(N)-$ algebra and a $C(N)-$ algebra as follows:

Definition 1. (Değirmen, 2022) A $C(N)-$ algebra is a $C(N)-$ vector space Λ such that

$$\alpha\overset{\frown}{\times}\left(\beta\overset{\frown}{\times}\gamma\right)=\left(\alpha\overset{\frown}{\times}\beta\right)\overset{\frown}{\times}\gamma,$$

$$\left(\alpha\dot{+}\beta\right)\overset{\frown}{\times}\gamma=\alpha\overset{\frown}{\times}\gamma\dot{+}\beta\overset{\frown}{\times}\gamma,$$

$$\alpha\overset{\frown}{\times}\left(\beta\dot{+}\gamma\right)=\alpha\overset{\frown}{\times}\beta\dot{+}\alpha\overset{\frown}{\times}\gamma,$$

$$\lambda\overset{\frown}{\cdot}\left(\alpha\overset{\frown}{\times}\chi\right)=\alpha\overset{\frown}{\times}\left(\lambda\overset{\frown}{\cdot}\beta\right)=\left(\lambda\overset{\frown}{\cdot}\alpha\right)\overset{\frown}{\times}\beta$$

for all $\alpha,\beta,\gamma\in\Lambda$ and $\lambda\in C(N)$.

If Λ is a $C(N)-$ normed space w.r.t. a $*-$ norm $\ddot{\|}.\ddot{\|}$ and $\ddot{\|}\alpha\overset{\frown}{\times}\beta\ddot{\|}\ddot{\leq}\ddot{\|}\alpha\ddot{\|}\ddot{\times}\ddot{\|}\beta\ddot{\|}$ for all $\alpha,\beta\in\Lambda$, then Λ is said to be a normed $C(N)-$ algebra. If Λ is also a Banach space with the $*-$ norm $\ddot{\|}.\ddot{\|}$, then Λ is called a Banach $C(N)-$ algebra.

Definition 2. (Değirmen, 2022) Let Λ be a Banach $C(N)-$ algebra with an $N-$ involution ${}^{*_N}:\Lambda\rightarrow\Lambda,\quad \alpha\rightarrow\alpha^{*_N}$ and

$\ddot{\|}\alpha^{*_N} \widehat{\times} \alpha \ddot{\|} = \ddot{\|}\alpha\ddot{\|}^{\ddot{2}}$ for all $\alpha \in \Lambda$, then Λ is said to be a $C^{*_N}-$ algebra (non-Newtonian $C^{*}-$algebra).

We will use the abbreviations "CA", "CAVM", "CAVMS", "CCAVMS", "CAVCM", "C.S.", "f.p." and "u.f.p." for "$C^{*_N}-$ algebra", "$C^{*_N}-$ algebra valued metric", "$C^{*_N}-$ algebra valued metric space", "complete $C^{*_N}-$ algebra valued metric space", "$C^{*_N}-$ algebra valued contractive mapping", "Cauchy sequence", "fixed point" and "unique fixed point", respectively.

Motivated by the preceding discussion and extensive applications of fixed point theory, this study discuss new types of contractive mappings in a novel $C^{*}-$algebra. We then prove several new some f.p. theorems based on the concept of CA and its properties. The main objectives of this study are to understand the place of this new space in the mathematical world, to establish a general mathematical structure, and to explore how they can be used in specific areas of application.

Main Results

The following definition presents a new type of spectrum.

Definition 3. Let Λ be a CA and $\alpha \in \Lambda$. Then, the $N-$ spectrum of α is defined as

$$\sigma_N(\alpha) = \left\{ \lambda \in C(N) \ : \ \lambda \widehat{\cdot} I_{\Lambda} \widehat{-} \alpha \notin \Lambda^{-1_N} \right\}.$$

Here, Λ^{-1_N} is the set of $N-$ invertible elements in Λ (see (Değirmen, 2022)).

We now introduce one of the fundamental definitions of the study.

Definition 4. Let Λ be a CA with a unit I_Λ and $\alpha \in \Lambda.$ If α is $N-$ hermitian (see (Değirmen, 2022)) and $\sigma_N(\alpha) \subset \ddot{[}\ddot{0},\ddot{\infty}\ddot{)},$ then α is said to be an $N-$ positive element in $\Lambda,$ denoted by $\alpha \succsim_N 0_\Lambda,$ where 0_Λ mean the zero element in $\Lambda.$ The set $\{\alpha \in \Lambda \,:\, \alpha \succsim_N 0_\Lambda\}$ is denoted by Λ_+^N. For $\alpha, \beta \in \Lambda_h^N;$ if their difference $\beta \overset{\frown}{-} \alpha \in \Lambda_+^N$ (or $\beta \overset{\frown}{-} \alpha \in \Lambda_+^N - \{0_\Lambda\}$), then we write $\alpha \precsim_N \beta$ (or $\alpha \prec_N \beta$). This relation $\precsim_N$ is reflexive, anti - symmetric, transitive and so defines a partial ordering on Λ_h^N.

Remark 1. When Λ is a CA with a unit $I_\Lambda,$ then, for any $\alpha \in \Lambda_+^N,$ we have $\alpha \precsim_N I_\Lambda \Leftrightarrow \ddot{\|}\alpha\ddot{\|} \ddot{\leq} \ddot{1}.$

We are ready to enunciate the following:

Definition 5. Let $\Psi \neq \varnothing$ and $d_{C^*N} \,:\, \Psi \times \Psi \rightarrow \Lambda$ be a function such that

i) $0_\Lambda \precsim_N d_{C^*N}(\mu, \nu)$ for all $\mu, \nu \in \Psi.$

ii) $d_{C^*N}(\mu, \nu) = 0_\Lambda$ if and only if $\mu = \nu.$

iii) $d_{C^*N}(\mu, \nu) = d_{C^*N}(\nu, \mu)$ for all $\mu, \nu \in \Psi.$

iv) $d_{C^*N}(\mu, \nu) \precsim_N d_{C^*N}(\mu, \eta) \ddot{+} d_{C^*N}(\eta, \nu)$ for all $\mu, \nu, \eta \in \Psi.$

Then, d_{C^*N} and $(\Psi, \Lambda, d_{C^*N})$ are called a CAVM and CAVMS, respectively.

This definition generalizes the concept of CAVMS and so the classical concept of metric space. If $\alpha = \beta = I$ (identity function), then the concepts of CA and CAVM are identical with those of classical $C^* -$ algebra and $C^* -$ algebra valued metric.

In the following, we state some noteworthy concepts:

Definition 6. Let $\left(\Psi, \Lambda, d_{C^{*N}}\right)$ be a CAVMS. Then, the elementary concepts can be given as follows:

i) $(\mu_n) \subset \Psi$ is called convergent w.r.t. $d_{C^{*N}}$ if for $\varepsilon \succsim_N 0_\Lambda$ there is an $n_0 = n_0(\varepsilon) \in N$ and $\mu \in \Psi$ such that $d_{C^{*N}}(\mu_n, \mu) \precsim_N \varepsilon$, $n \geq n_0$ and is denoted by $\mu_n \xrightarrow{d_{C^{*N}}} \mu$ or $\lim\limits_{n \to \infty}^{N} \mu_n = \mu$.

ii) $(\mu_n) \subset \Psi$ is called a C.S. w.r.t. $d_{C^{*N}}$ if for every given $\varepsilon \succsim_N 0_\Lambda$ there is an $n_0 = n_0(\varepsilon) \in N$ such that $d_{C^{*N}}(\mu_n, \mu_m) \precsim_N \varepsilon$, $n, m \geq n_0$.

iii) If every C.S. w.r.t. $d_{C^{*N}}$ is convergent w.r.t. $d_{C^{*N}}$, then $\left(\Psi, \Lambda, d_{C^{*N}}\right)$ is called a CCAVMS.

We proceed in this section by introducing a new definition that generalizes the concept of a contractive mapping on a metric space.

Definition 7. Let $\left(\Psi, \Lambda, d_{C^{*N}}\right)$ be a CAVMS. We call a mapping $T_N : \Psi \to \Psi$ is a CAVCM on Ψ, if there is an $A \in \Lambda$ with $\|A\| < 1$ such that

$$d_{C^{*N}}(T_N \tau, T_N \nu) \precsim_N A^{*_N} \hat{\times} d_{C^{*N}}(\tau, \nu) \hat{\times} A$$

for all $\tau, \nu \in \Psi$.

Now, we present our main result which is analogous to the Banach f.p. theorem (Banach, 1922).

Theorem 1. If $\left(\Psi, \Lambda, d_{C^{*N}}\right)$ is a CCAVMS and T_N is a CAVCM, then there is a u.f.p. in Ψ.

Proof: Let $\tau_* \in \Psi$ and $\tau_{j+1} = T_N \tau_j = T_N^{j+1} \tau_*, \quad j = 1, 2, ...$ So,

$$\begin{aligned} d_{C^{*N}}\left(\tau_{j+1}, \tau_j\right) &= d_{C^{*N}}\left(T_N \tau_j, T_N \tau_{j-1}\right) \\ &\precsim_N A^{*_N} \widehat{\times} d_{C^{*N}}\left(\tau_j, \tau_{j-1}\right) \widehat{\times} A \\ &\precsim_N \left(A^{*_N}\right)^2 \widehat{\times} d_{C^{*N}}\left(\tau_{j-1}, \tau_{j-2}\right) \widehat{\times} A^2 \\ &\precsim_N \cdots \precsim_N \left(A^{*_N}\right)^j \widehat{\times} d_{C^{*N}}\left(\tau_1, \tau_*\right) \widehat{\times} A^j. \end{aligned}$$

This implies that for $j + 1 > m$

$$
\begin{aligned}
d_{C^{*N}}\left(\tau_{j+1},\tau_m\right) &\precsim_N d_{C^{*N}}\left(\tau_{j+1},\tau_n\right)+d_{C^{*N}}\left(\tau_j,\tau_{j-1}\right)+\ldots+d_{C^{*N}}\left(\tau_{m+1},\tau_m\right)\\
&\precsim_N \left(A^{*N}\right)^j \times d_{C^{*N}}\left(\tau_1,\tau_*\right)\times A^p+\ldots+\left(A^{*N}\right)^m \times d_{C^{*N}}\left(\tau_1,\tau_*\right)\times A^m\\
&=\left(A^{*N}\right)^j \times d_{C^{*N}}\left(\tau_1,\tau_*\right)\times A^p+\ldots+\left(A^{*N}\right)^m \times d_{C^{*N}}\left(\tau_1,\tau_*\right)\times A^m\\
&=\left(d_{C^{*N}}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\times A^j\right)^{*N}\times\left(d_{C^{*N}}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\times A^j\right)+\ldots\\
&\quad+\left(d_{C^{*N}}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\times A^m\right)^{*N}\times\left(d_{C^{*N}}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\times A^m\right)\\
&=\left|d_{C^{*N}}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\times A^j\right|^2+\ldots+\left|d_{C^{*N}}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\times A^m\right|^2\\
&\precsim_N \left\|\left|d_{C^{*N}}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\times A^j\right|^2+\ldots+\left|d_{C^{*N}}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\times A^m\right|^2\right\| I_\Lambda\\
&\precsim_N \left[\left\|d_{C^{*N}}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\right\|^2\times\left\|A^j\right\|^2+\left\|d_{C^{*N}}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\right\|^2\times\left\|A^m\right\|^2\right] I_\Lambda\\
&\precsim_N \left\|d_{C^{*N}}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\right\|^2\times\left[\left\|A\right\|^{2m}+\ldots+\left\|A\right\|^{2j}\right] I_\Lambda\\
&\precsim_N \left\|d_{C^{*N}}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\right\|^2\times\frac{\left\|A\right\|^{2m}}{1-\left\|A\right\|}\beta I_\Lambda \xrightarrow{m\to\infty} 0_\Lambda.
\end{aligned}
$$

Thus, $\left(\tau_j\right)$ is a C.S. w.r.t. $d_{C^{*N}}$. Since $\left(\Psi,\Lambda,d_{C^{*N}}\right)$ is complete, there is an element $\tau\in\Psi$ such that $\tau_j=T_N\tau_{j-1}\xrightarrow{j\to\infty}\tau$. Also, we have

$$
\begin{aligned}
0_\Lambda &\precsim_N d_{C^{*N}}\left(T_N\tau,\tau\right)\\
&\precsim_N d_{C^{*N}}\left(T_N\tau,T_N\tau_j\right)+d_{C^{*N}}\left(T\tau_j,\tau\right)\\
&\precsim_N A^{*N}\times d_{C^{*N}}\left(\tau,\tau_j\right)\times A+d_{C^{*N}}\left(\tau_{j+1},\tau\right)\xrightarrow{j\to\infty} 0_\Lambda,
\end{aligned}
$$

therefore $T_N\tau = \tau$ and so τ is a f.p. of T_N.

For uniqueness, let $\nu \in \Psi$ be another f.p. of T_N . Then it follows that

$$0_\Lambda \precsim_N d_{C^{*N}}(\tau,\nu) = d_{C^{*N}}(T_N\tau, T_N\nu) \precsim_N A^{*_N} \widehat{\times} d_{C^{*N}}(\tau,\nu) \widehat{\times} A$$

and so

$$\begin{aligned} \ddot{0} \ddot{\leq} \ddot{\|} d_{C^{*N}}(\tau,\nu) \ddot{\|} = \ddot{\|} d_{C^{*N}}(T_N\tau, T_N\nu) \ddot{\|} \\ \ddot{\leq} \ddot{\|} A^{*_N} \widehat{\times} d_{C^{*N}}(\tau,\nu) \widehat{\times} A \ddot{\|} \\ \ddot{\leq} \ddot{\|} A^{*_N} \ddot{\|} \ddot{\times} \ddot{\|} d_{C^{*N}}(\tau,\nu) \ddot{\|} \ddot{\times} \ddot{\|} A \ddot{\|} \\ = \ddot{\|} A \ddot{\|}^{\ddot{2}} \ddot{\times} \ddot{\|} d_{C^{*N}}(\tau,\nu) \ddot{\|} \\ \ddot{<} \ddot{\|} d_{C^{*N}}(\tau,\nu) \ddot{\|}. \end{aligned}$$

This implies that $d_{C^{*N}}(\tau,\nu) = 0_\Lambda$. So, we conclude that $\tau = \nu$ and τ is unique. The proof is completed.

We will use the symbol Λ' for $\{a \in \Lambda : a \widehat{\times} b = b \widehat{\times} a, \forall b \in \Lambda\}$.

The following definition is a generalization of the Chatterjea's contractive condition (Chatterjea, 1972).

Definition 8. Let $(\Psi, \Lambda, d_{C^{*N}})$ be a CAVMS. We call a mapping $T_N : \Psi \to \Psi$ is a Chatterjea type CAVCM on Ψ, if there is an $A \in (\Lambda_+^N)'$ with $\ddot{\|} A \ddot{\|} \ddot{<} \frac{\ddot{1}}{\ddot{2}} \beta$ such that

$$d_{C^{*N}}\left(T_N\tau, T_N\nu\right) \precsim_N A\widehat{\times}\left(d_{C^{*N}}\left(T_N\tau,\nu\right)\widehat{+}d_{C^{*N}}\left(T_N\nu,\tau\right)\right)$$

for all $\tau,\nu\in\Psi$.

Theorem 2. If $\left(\Psi,\Lambda,d_{C^{*N}}\right)$ is a CCAVMS and T_N is a Chatterjea type CAVCM, then there is a u.f.p. in Ψ.

Proof: Let $\tau_*\in\Psi$ and $\tau_{j+1}=T_N\tau_j=T_N^{n+1}\tau_*,\quad j=1,2,....$ So,

$$\begin{aligned}
d_{C^{*N}}\left(\tau_{j+1},\tau_j\right) &= d_{C^{*N}}\left(T_N\tau_j, T_N\tau_{j-1}\right)\\
&\precsim_N A\widehat{\times}\left(d_{C^{*N}}\left(T_N\tau_j,\tau_{j-1}\right)\widehat{+}d_{C^{*N}}\left(T_N\tau_{j-1},\tau_j\right)\right)\\
&= A\widehat{\times}\left(d_{C^{*N}}\left(T_N\tau_j,T_N\tau_{j-2}\right)\widehat{+}d_{C^{*N}}\left(T_N\tau_{j-1},T_N\tau_{j-1}\right)\right)\\
&\precsim_N A\widehat{\times}\left(d_{C^{*N}}\left(T_N\tau_j,T_N\tau_{j-1}\right)\widehat{+}d_{C^{*N}}\left(T_N\tau_{j-1},T_N\tau_{j-2}\right)\right)\\
&= A\widehat{\times}d_{C^{*N}}\left(T_N\tau_j,T_N\tau_{j-1}\right)\widehat{+}A\widehat{\times}d_{C^{*N}}\left(T_N\tau_{j-1},T_N\tau_{j-2}\right)\\
&= A\widehat{\times}d_{C^{*N}}\left(\tau_{j+1},\tau_j\right)\widehat{+}A\widehat{\times}d_{C^{*N}}\left(\tau_j,\tau_{j-1}\right)
\end{aligned}$$

and hence $\left(I_\Lambda\widehat{-}A\right)\widehat{\times}d_{C^{*N}}\left(\tau_{j+1},\tau_j\right)\precsim_N A\widehat{\times}d_{C^{*N}}\left(\tau_j,\tau_{j-1}\right)$. Since $A\in\left(\Lambda_+^N\right)'$ with $\ddot{\|}A\ddot{\|}\ddot{<}\frac{\ddot{1}}{\ddot{2}}\beta,$ we can write $\left(I_\Lambda\widehat{-}A\right)^{-1_N}\in\left(\Lambda_+^N\right)'$ and $A\widehat{\times}\left(I_\Lambda\widehat{-}A\right)^{-1_N}\in\left(\Lambda_+^N\right)'$ with $\ddot{\|}A\widehat{\times}\left(I_\Lambda\widehat{-}A\right)^{-1_N}\ddot{\|}\ddot{<}\ddot{1}.$ Therefore we get $d_{C^{*N}}\left(\tau_{j+1},\tau_j\right)\precsim_N A\widehat{\times}\left(I_\Lambda\widehat{-}A\right)^{-1_N}\widehat{\times}d_{C^{*N}}\left(\tau_j,\tau_{j-1}\right).$

On the other hand, for $j+1>m,$ we have

$$d_{C^*N}\left(\tau_{j+1},\tau_m\right) \precsim_N d_{C^*N}\left(\tau_{j+1},\tau_j\right)+d_{C^*N}\left(\tau_j,\tau_{j-1}\right)+\ldots+d_{C^*N}\left(\tau_{m+1},\tau_m\right)$$

$$\precsim_N \left[\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^j+\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^{j-1}+\ldots+\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^m\right]\times d_{C^*N}\left(\tau_1,\tau_*\right)$$

$$=\left(d_{C^*N}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\times\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^{\frac{m}{2}}\right)^{*_N}\times\left(d_{C^*N}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\times\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^{\frac{m}{2}}\right)+\ldots$$

$$+\left(d_{C^*N}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\times\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^{\frac{j}{2}}\right)^{*_N}\times\left(d_{C^*N}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\times\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^{\frac{j}{2}}\right)$$

$$=\left|d_{C^*N}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\times\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^{\frac{m}{2}}\right|^2+\ldots+\left|d_{C^*N}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\times\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^{\frac{j}{2}}\right|^2$$

$$\precsim_N \left\|\left|d_{C^*N}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\times\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^{\frac{m}{2}}\right|^2+\ldots+\left|d_{C^*N}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\times\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^{\frac{j}{2}}\right|^2\right\| I_\Lambda$$

$$\precsim_N \left[\begin{array}{l}\left\|d_{C^*N}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\right\|^2\times\left\|\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^{\frac{m}{2}}\right\|^2+\ldots\\ +\left\|d_{C^*N}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\right\|^2\times\left\|\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^{\frac{j}{2}}\right\|^2\end{array}\right] I_\Lambda$$

$$\precsim_N \left\|d_{C^*N}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\right\|^2\times\left[\left\|A\times\left(I_\Lambda-A\right)^{-1_N}\right\|^m+\ldots+\left\|A\times\left(I_\Lambda-A\right)^{-1_N}\right\|^j\right] I_\Lambda$$

$$\precsim_N \left\|d_{C^*N}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\right\|^2\times\frac{\left\|A\times\left(I_\Lambda-A\right)^{-1_N}\right\|^m}{1-\left\|A\times\left(I_\Lambda-A\right)^{-1_N}\right\|}\beta I_\Lambda \overset{m\to\infty}{\to} 0_\Lambda.$$

Thus, $\left(\tau_j\right)$ is a C.S. w.r.t. $d_{C^{*N}}$. Since $\left(\Psi,\Lambda,d_{C^{*N}}\right)$ is complete, there is an element $\tau\in\Psi$ such that $\tau_j=T_N\tau_{j-1}\overset{j\to\infty}{\to}\tau.$ Also, we have

$$d_{C^{*N}}\left(T_N\tau,\tau\right) \precsim_N d_{C^{*N}}\left(T_N\tau,T_N\tau_j\right) + d_{C^{*N}}\left(T\tau_j,\tau\right)$$

$$\precsim_N A\widehat{\times}\left(d_{C^{*N}}\left(T_N\tau,\tau_j\right)+d_{C^{*N}}\left(T_N\tau_j,\tau\right)\right)+d_{C^{*N}}\left(\tau_{j+1},\tau\right)$$

$$\precsim_N A\widehat{\times}\left(d_{C^{*N}}\left(T_N\tau,\tau\right)+d_{C^{*N}}\left(\tau,\tau_j\right)+d_{C^{*N}}\left(\tau_{j+1},\tau\right)\right)+d_{C^{*N}}\left(\tau_{j+1},\tau\right)$$

and equivalently

$$\left(I_\Lambda \widehat{-} A\right)\widehat{\times} d_{C^{*N}}\left(T_N\tau,\tau\right) \precsim_N A\widehat{\times}\left(d_{C^{*N}}\left(\tau,\tau_j\right)+d_{C^{*N}}\left(\tau_{j+1},\tau\right)\right)+d_{C^{*N}}\left(\tau_{j+1},\tau\right).$$

Therefore, we get

$$\ddot{\|} d_{C^{*N}}\left(T_N\tau,\tau\right)\ddot{\|}\ddot{\leq}\ddot{\|} A\widehat{\times}\left(I_\Lambda \widehat{-} A\right)^{-1_N}\ddot{\|}\ddot{\times}\left[\ddot{\|} d_{C^{*N}}\left(\tau,\tau_j\right)\ddot{\|}\ddot{+}\ddot{\|} d_{C^{*N}}\left(\tau_{j+1},\tau\right)\ddot{\|}\right]$$

$$\ddot{+}\ddot{\|}\left(I_\Lambda \widehat{-} A\right)^{-1_N}\ddot{\|}\ddot{\times}\ddot{\|} d_{C^{*N}}\left(\tau_{j+1},\tau\right)\ddot{\|}\xrightarrow{j\to\infty}\ddot{0}.$$

This means that $T_N\tau = \tau$ and so τ is a f.p. of T_N.

For uniqueness, let $\nu \in \Psi$ be another f.p. of T_N . Then it follows that

$$0_\Lambda \precsim_N d_{C^{*N}}\left(\tau,\nu\right) = d_{C^{*N}}\left(T_N\tau,T_N\nu\right)$$

$$\precsim_N A\widehat{\times}\left(d_{C^{*N}}\left(T_N\tau,\nu\right)+d_{C^{*N}}\left(T_N\nu,\tau\right)\right)$$

$$= A\widehat{\times}\left(d_{C^{*N}}\left(\tau,\nu\right)+d_{C^{*N}}\left(\nu,\tau\right)\right)$$

and so $d_{C^{*N}}\left(\tau,\nu\right) \precsim_N A\widehat{\times}\left(I_\Lambda \widehat{-} A\right)^{-1_N}\widehat{\times} d_{C^{*N}}\left(\tau,\nu\right).$ Using the inequality $\ddot{\|} A\widehat{\times}\left(I_\Lambda \widehat{-} A\right)^{-1_N}\ddot{\|}\ddot{<}\ddot{1},$ we obtain

$$
\begin{aligned}
\ddot{0}\,\ddot{\le}\,\ddot{\|} d_{C^{*N}}(\tau,\nu)\ddot{\|}\,\ddot{\le}\,\ddot{\|} A\widehat{\times}\left(I_\Lambda\widehat{-}A\right)^{-1_N}\widehat{\times} d_{C^{*N}}(\tau,\nu)\ddot{\|} \\
\ddot{\le}\,\ddot{\|} A\widehat{\times}\left(I_\Lambda\widehat{-}A\right)^{-1_N}\ddot{\|}\,\ddot{\times}\,\ddot{\|} d_{C^{*N}}(\tau,\nu)\ddot{\|} \\
\ddot{<}\,\ddot{\|} d_{C^{*N}}(\tau,\nu)\ddot{\|}.
\end{aligned}
$$

This implies that $d_{C^{*N}}(\tau,\nu)=0_\Lambda$. So, we conclude that $\tau=\nu$ and τ is unique. The proof is completed.

Now, we give another of generalizations of the Kannan's contractive condition (Kannan, 1968).

Definition 9. Let $\left(\Psi,\Lambda,d_{C^{*N}}\right)$ be a CAVMS. We call a mapping $T_N : \Psi\to\Psi$ is a Kannan type CAVCM on Ψ, if there is an $A\in\left(\Lambda_+^N\right)'$ with $\ddot{\|}A\ddot{\|}\,\ddot{<}\,\frac{1}{\ddot{2}}\beta$ such that

$$
d_{C^{*N}}(T_N\tau,T_N\nu)\precsim_N A\widehat{\times}\left(d_{C^{*N}}(T_N\tau,\tau)+d_{C^{*N}}(T_N\nu,\nu)\right)
$$

for all $\tau,\nu\in\Psi$.

Theorem 3. If $\left(\Psi,\Lambda,d_{C^{*N}}\right)$ is a CCAVMS and T_N is a Kannan type CAVCM, then there is a u.f.p. in Ψ.

Proof: Let $\tau_*\in\Psi$ and $\tau_{j+1}=T_N\tau_j=T_N^{j+1}\tau_*,\quad n=1,2,....$ So,

$$
\begin{aligned}
d_{C^{*N}}\left(\tau_{j+1},\tau_j\right) &= d_{C^{*N}}\left(T_N\tau_j,T_N\tau_{j-1}\right) \\
&\precsim_N A\widehat{\times}\left(d_{C^{*N}}\left(T_N\tau_j,\tau_j\right)+d_{C^{*N}}\left(T_N\tau_{j-1},\tau_{j-1}\right)\right) \\
&= A\widehat{\times}\left(d_{C^{*N}}\left(T_N\tau_{j+1},T_N\tau_j\right)+d_{C^{*N}}\left(T_N\tau_j,T_N\tau_{j-1}\right)\right)
\end{aligned}
$$

and hence $d_{C^{*N}}\left(\tau_{j+1},\tau_j\right)\precsim_N \widehat{A\times}\left(I_\Lambda \widehat{-}A\right)^{-1_N}\widehat{\times}d_{C^{*N}}\left(\tau_j,\tau_{j-1}\right).$

In addition, for $l\geq 1, p\geq 1,$ we get

$$\begin{aligned}
d_{C^{*N}}\left(\tau_{l+p},\mu_l\right) &\precsim_N d_{C^{*N}}\left(\tau_{l+p},\tau_{l+p-1}\right)+d_{C^{*N}}\left(\tau_{l+p-1},\tau_l\right)\\
&\precsim_N d_{C^{*N}}\left(\tau_{l+p},\tau_{l+p-1}\right)+\left[d_{C^{*N}}\left(\tau_{l+p-1},\tau_{l+p-2}\right)+d_{C^{*N}}\left(\tau_{l+p-2},\tau_l\right)\right]\\
&\precsim_N d_{C^{*N}}\left(\tau_{l+p},\tau_{l+p-1}\right)+d_{C^{*N}}\left(\tau_{l+p-1},\tau_{l+p-2}\right)+d_{C^{*N}}\left(\tau_{l+p-2},\tau_{l+p-3}\right)+\ldots+\\
&\quad d_{C^{*N}}\left(\tau_{l+2},\tau_{l+1}\right)+d_{C^{*N}}\left(\tau_{l+1},\tau_l\right)
\end{aligned}$$

$$\begin{aligned}
&=\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^{l+p-1}\times d_{C^{*N}}\left(\tau_1,\tau_*\right)+\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^{l+p-2}\times d_{C^{*N}}\left(\tau_1,\tau_*\right)\\
&\quad+\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^{l+p-3}\times d_{C^{*N}}\left(\tau_1,\tau_*\right)+\ldots+\\
&\quad+\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^{l+1}\times d_{C^{*N}}\left(\tau_1,\tau_*\right)+\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^{l}\times d_{C^{*N}}\left(\tau_1,\tau_*\right)\\
&=\left|d_{C^{*N}}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\times\left(A\times\left(I_\Lambda-A\right)^{-1_N}\right)^{\frac{l+p-1}{2}}\right|^2+\ldots\\
&\quad+\left|d_{C^{*N}}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\widehat{\times}\left(\widehat{A\times}\left(\widehat{I_\Lambda}\widehat{-A}\right)^{-1_N}\right)^{\frac{l+1}{2}}\right|^2\\
&\widehat{+\left|d_{C^{*N}}\right.}\left(\tau_1,\tau_*\right)^{\frac{1}{2}}\widehat{\times}\left(\widehat{A\times}\left(\widehat{I_\Lambda}\widehat{-A}\right)^{-1_N}\right)^{\frac{l}{2}}\Big|^2\\
&\precsim_N \ddot{\|}d_{C^{*N}}\left(\tau_1,\tau_*\right)\ddot{\|}\ddot{\times}\left[\ddot{\|}\widehat{A\times}\left(I_\Lambda\widehat{-}A\right)^{-1_N}\ddot{\|}^{l+p-1}\ddot{+}\ldots\ddot{+}\ddot{\|}\widehat{A\times}\left(I_\Lambda\widehat{-}A\right)^{-1_N}\ddot{\|}^{l+1}\right]I_\Lambda\\
&\quad\ddot{+}\ddot{\|}d_{C^{*N}}\left(\tau_1,\tau_*\right)\ddot{\|}\ddot{\times}\ddot{\|}\widehat{A\times}\left(I_\Lambda\widehat{-}A\right)^{-1_N}\ddot{\|}^{l}I_\Lambda\\
&\precsim_N \ddot{\|}d_{C^{*N}}\left(\tau_1,\tau_*\right)\ddot{\|}\ddot{\times}\frac{\ddot{\|}\widehat{A\times}\left(I_\Lambda\widehat{-}A\right)^{-1_N}\ddot{\|}^{l+1}}{\ddot{1}\ddot{-}\ddot{\|}\widehat{A\times}\left(I_\Lambda\widehat{-}A\right)^{-1_N}\ddot{\|}}\beta\ddot{+}\ddot{\|}d_{C^{*N}}\left(\tau_1,\tau_*\right)\ddot{\|}\ddot{\times}\ddot{\|}\widehat{A\times}\left(I_\Lambda\widehat{-}A\right)^{-1_N}\ddot{\|}^{l}I_\Lambda\\
&\overset{l\to\infty}{\to}0_\Lambda.
\end{aligned}$$

Thus, $\left(\tau_j\right)$ is a C.S. w.r.t. $d_{C^{*N}}$. Since $\left(\Psi,\Lambda,d_{C^{*N}}\right)$ is complete, there is an element $\tau\in\Psi$ such that $\tau_j=T_N\tau_{j-1}\overset{j\to\infty}{\to}\tau.$ Also, we have

$$\begin{aligned} d_{C^{*N}}\left(T_N\tau,\tau\right) &\precsim_N d_{C^{*N}}\left(T_N\tau,T_N\tau_j\right)\hat{+}d_{C^{*N}}\left(T_N\tau_j,\tau\right) \\ &\precsim_N A\hat{\times}\left(d_{C^{*N}}\left(T_N\tau,\tau\right)\hat{+}d_{C^{*N}}\left(T_N\tau_j,\tau_j\right)\right)\hat{+}d_{C^{*N}}\left(T_N\tau_j,\tau\right) \\ &\precsim_N A\hat{\times}\left(d_{C^{*N}}\left(T_N\tau,\tau\right)\hat{+}d_{C^{*N}}\left(T_N\tau_j,T_N\tau_{j-1}\right)\right)\hat{+}d_{C^{*N}}\left(T_N\tau_j,\tau\right) \end{aligned}$$

and equivalently

$$\begin{aligned} d_{C^{*N}}\left(T_N\tau,\tau\right) \precsim_N &\left(I_\Lambda\hat{-}A\right)^{-1_N}\hat{\times}A\hat{\times}d_{C^{*N}}\left(T_N\tau_j,T_N\tau_{j-1}\right) \\ &\hat{+}\left(I_\Lambda\hat{-}A\right)^{-1_N}\hat{\times}d_{C^{*N}}\left(T_N\tau_j,\tau\right). \end{aligned}$$

Therefore, we get

$$\begin{aligned} &\ddot{\|}d_{C^{*N}}\left(T_N\tau,\tau\right)\ddot{\|}\ddot{\le}\ddot{\|}A\hat{\times}\left(I_\Lambda\hat{-}A\right)^{-1_N}\ddot{\|}\ddot{\times}\ddot{\|}d_{C^{*N}}\left(T_N\tau_j,\tau_j\right)\ddot{\|} \\ &\ddot{+}\ddot{\|}A\hat{\times}\left(I_\Lambda\hat{-}A\right)^{-1_N}\ddot{\|}\ddot{\times}\ddot{\|}d_{C^{*N}}\left(T_N\tau_j,\tau\right)\ddot{\|}\overset{n\to\infty}{\to}\ddot{0}. \end{aligned}$$

This means that $T_N\tau=\tau$ and so τ is a f.p. of T_N.

For uniqueness, let $\nu\in\Psi$ be another f.p. of T_N . Then it follows that

$$\begin{aligned} 0_\Lambda \precsim_N d_{C^{*N}}\left(\tau,\nu\right) &= d_{C^{*N}}\left(T_N\tau,T_N\nu\right) \\ &\precsim_N A\hat{\times}\left(d_{C^{*N}}\left(T_N\tau,\tau\right)\hat{+}d_{C^{*N}}\left(T_N\nu,\nu\right)\right) \\ &= 0_\Lambda \end{aligned}$$

and so $\tau=\nu$ and τ is unique. The proof is completed.

Conclusion

In this article, we have introduced the concept of a CAVMS. Additionally, we have presented f.p. results for CCAVMS. Given that our findings extend several known results from the existing literature, we believe they will be valuable for future research and new applications.

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CHAPTER IV

Bigeometric Sumudu Transform

Nihan GÜNGÖR [1]

Introduction

Integral transforms can be applied to deal with many mathematically stated processes and phenomena in research, engineering, and everyday life. They are extensively utilized across fields of engineering and science to easily handle problems related to thermal science, heat conduction, control theory, electrical networks, exponential growth and decay problems, statistics, physics, mathematics, chemistry, economics, biology, medicine, telecommunications, nuclear reactors, quantum mechanics, deflection of beams, Brownian motion, and many more. A multitude of innovative integral transforms has been introduced owing to their extensive applicability. The integral transforms that are most utilized

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and recognized include the Laplace, Fourier, and Sumudu transforms. The examination of the Sumudu transforms has been conducted through various concepts, including fractional (Gupta et al., 2015), conformable fractional (Al-Zhour et al., 2019), fuzzy (Rahman & Ahmad, 2016), multiplicative calculus (Bhat et al., 2019) and non-Newtonian calculus (Gungor & Dinc, 2024).

Since the establishment of classical calculus by Newton and Leibnitz, numerous calculi have been developed, recognizing that a prominent and preferred approach to establishing a new mathematical system is to modify the axioms of an existing system. Furthermore, a mathematical problem that may be difficult or unresolvable with one calculus can be effectively addressed using an alternative calculus. Grossman and Katz (1972) developed a novel structure known as non-Newtonian calculus, which serves as an alternative to classical calculus. This framework encompasses various specialized calculi, including geometric, bigeometric, anageometric, harmonic and quadratic calculus. Modern derivatives and integrals have been introduced, which have transformed addition and subtraction into multiplication and division respectively. Non-Newtonian calculus, the innovative work carried out by Grossman and Katz, has attracted considerable interest in recent years, owing to its extensive applications across several disciplines including economics, biology, integral equations, probability theory, approximation theory, functional analysis, differential equations, computer science and more from many scholars (Rybaczuk & Stoppel, 2000; Córdova-Lepe, 2006; Florack & van Assen, 2012; Çakmak & Başar, 2014; Filip & Piatecki, 2014; Kadak & Özlük, 2014; Duyar & Oğur, 2017; Güngör, 2020; Boruah & Hazarika, 2021; Czachor, 2021; Sager & Sağır, 2021; Güngör, 2022; Değirmen & Duyar, 2023; Rohman & Eryılmaz, 2023; Ogur & Gunes, 2024a, 2024b) Bigeometric calculus is among the most widely recognized forms of non-Newtonian calculus. This calculus assesses the variations in function arguments and values through ratios.

A generator is one to one function whose domain is $\mathbb{R}$ and whose range is a subset of $\mathbb{R}$. Each generator produces precisely one arithmetic, and each arithmetic is produced by exactly one generator. Classical arithmetic is produced by the identity function I, whereas geometric arithmetic is produced by the exponential function. The range of the generator η is denoted by $\mathbb{R}_\eta \coloneqq \{\eta(u): u \in \mathbb{R}\}$ and η-arithmetic operations are represented by

η-addition	$\upsilon \dot{+} \nu = \eta\{\eta^{-1}(\upsilon) + \eta^{-1}(\nu)\}$,
η-subtraction	$\upsilon \dot{-} \nu = \eta\{\eta^{-1}(\upsilon) - \eta^{-1}(\nu)\}$,
η-multiplication	$\upsilon \dot{\times} \nu = \eta\{\eta^{-1}(\upsilon) \times \eta^{-1}(\nu)\}$,
η-division	$\upsilon \dot{/} \nu \ (\nu \neq \dot{0}) = \eta\{\eta^{-1}(\upsilon) / \eta^{-1}(\nu)\}$,
η-order	$\upsilon \dot{<} \nu \ \Leftrightarrow \eta^{-1}(\upsilon) < \eta^{-1}(\nu)$

for $\upsilon, \nu \in \mathbb{R}_\eta$. The establishment of the $*$-calculus is accomplished by employing two generators, namely η and β, which are chosen arbitrarily. Here are the specific calculuses that can be derive by using either the exponential function exp or the identity function I as the generators η and β:

Calculus	η	β
Classical	I	I
Geometric	I	exp
Anageometric	exp	I
Bigeometric	exp	exp.

This study will concentrate on bigeometric calculus, characterized as a $*$-calculus in which η and β are both equivalent to exp. In essence, when engaging with function arguments and values in bigeometric calculus, one utilizes geometric arithmetic. Initially, we will present the geometric arithmetic along with its fundamental characteristics. It is the responsibility of the exponential function to generate geometric arithmetic, and the following is a list of the definitions of operations:

geometric addition $\quad \upsilon \oplus \nu = e^{\{\ln \upsilon + \ln \nu\}} = \upsilon . \nu$,

geometric subtraction $\quad \upsilon \ominus \nu = e^{\{\ln \upsilon - \ln \nu\}} = \upsilon \div \nu, \nu \neq 0$,

geometric multiplication $\quad \upsilon \odot \nu = e^{\{\ln \upsilon \times \ln \nu\}} = \upsilon^{\ln \nu} = \nu^{\ln \upsilon}$,

geometric division $\quad \upsilon \oslash \nu = e^{\{\ln \upsilon \div \ln \nu\}} = \upsilon^{\frac{1}{\ln \nu}}, \nu \neq 1.$

It is evident that $\ln \upsilon < \ln \nu$ if $\upsilon < \nu$ for $\upsilon, \nu \in \mathbb{R}^+$. That is $\upsilon < \nu \Leftrightarrow \eta^{-1}(\upsilon) < \eta^{-1}(\nu)$. Thus, without loss of generality, we utilize $\upsilon < \nu$ in place of the geometric order $\upsilon <_{\exp} \nu$. The geometric factorial notation $!_{\exp}$ is defined as

$$n!_{\exp} = e^n \odot e^{n-1} \odot \ldots \odot e^2 \odot e = e^{n!}$$

for $n \in \mathbb{N}$. For $\upsilon \in \mathbb{R}_{\exp}$, $\upsilon^{p_{\exp}} = e^{(\ln \upsilon)^p} = \upsilon^{\ln^{p-1} \upsilon}$ and $\sqrt[p]{\upsilon}^{exp} =$

$e^{(\ln \upsilon)^{\frac{1}{p}}}$. The geometric absolute value of $\upsilon \in \mathbb{R}_{\exp}$ is denoted by

$$|\upsilon|_{\exp} = \begin{cases} \upsilon & ,\upsilon > 1 \\ 1 & ,\upsilon = 1. \\ 1/\upsilon & ,\upsilon < 1 \end{cases}$$

Hence, we can write $|\upsilon|_{\exp} = \mathrm{e}^{|\ln \upsilon|}$ (Boruah & Hazarika, 2018; 2021).

Definition 1. (Grossman & Katz, 1972; Sağır & Erdoğan, 2019; Güngör, 2020) Let the function $f: X \subset \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$ and $a \in X'^{\exp}$, $b \in \mathbb{R}_{\exp}$. If for every $\varepsilon > 1$ there is $\delta = \delta(\varepsilon) > 1$ such that $|f(x) \ominus b\,|_{\exp} < \varepsilon$ for all $x \in X$ whenever $1 < |x \ominus a\,|_{\exp} < \delta$, then it is said that the BG-limit of f at a is b and it is articulated as ${}_{BG}\lim_{x \to a} f(x) = b$.

Definition 2. (Grossman & Katz, 1972; Sağır & Erdoğan, 2019). Let the function $f: X \subset \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$ and $a \in X$. If for every $\varepsilon > 1$ there is $\delta = \delta(\varepsilon) > 1$ such that $|f(x) \ominus f(a)\,|_{\exp} < \varepsilon$ for all $x \in X$ whenever $1 < |x \ominus a\,|_{\exp} < \delta$, then it is called that f is BG-continuous at the point $a \in X$.

Remark 3. (Grossman & Katz, 1972; Güngör, 2020) The limits ${}_{BG}\lim_{x \to a} f(x)$ and $\lim_{t \to \ln a} \ln f(t)$ coexist and if they do exist, ${}_{BG}\lim_{x \to a} f(x) = \exp\left\{ \lim_{t \to \ln a} \ln f(e^t) \right\}$. Moreover, f is BG-continuous at a if and only if $\ln f$ is continuous at $\ln a$.

Definition 4. (Grossman & Katz, 1972; Boruah & Hazarika, 2021) Let the function $f: (r,s) \subset \mathbb{R}_{\exp} \to \mathbb{R}_{\exp}$ and $\in (r,s)$. If

${}_{BG}\lim_{x\to a}\frac{f(x)\ominus f(a)}{x\ominus a}\exp=\lim_{x\to a}\left[\frac{f(x)}{f(a)}\right]^{\frac{1}{\ln x-\ln a}}$ occurs, it is indicated by $(D^{BG}f)(a)=f^{BG}(a)$ is called the BG-derivative of f at a.

Remark 5. (Grossman & Katz, 1972) The derivatives $f^{BG}(a)$ and $(\ln f(\ln a))'$ coexist and if they do exist, $f^{BG}(a)=\exp\left[\left(\ln f\left(e^{\ln a}\right)\right)'\right]=e^{a\frac{f'(a)}{f(a)}}$.

Definition 6. (Grossman & Katz, 1972; Boruah & Hazarika, 2018) The BG-average of a BG-continuous positive function f on $[r,s]\subset\mathbb{R}_{\exp}$ is defined as the exp-limit of the exp-convergent sequence whose n-th term is geometric average of $f(a_1),f(a_2),\dots,f(a_n)$ where $a_1,a_2,\dots,a_n$ is the n-fold exp-partition of $[r,s]$ and denoted by ${M^{BG}}_r^s f$. The BG-integral of a BG-continuous function f on $[r,s]$ is denoted by ${}_{BG}\int_r^s f(x)d^{BG}x$, which is the number $\left[{M^{BG}}_r^s f\right]^{[\ln s-\ln r]}$.

Remark 7. (Grossman & Katz, 1972; Boruah & Hazarika, 2018) If f is BG-continuous $[r,s]\subset\mathbb{R}_{\exp}$, then ${}_{BG}\int_r^s f(x)d^{BG}x=\exp\left[\int_{\ln r}^{\ln s}\ln f(e^t)dt\right]$, i.e., the BG-integral of f is defined by ${}_{BG}\int_r^s f(x)d^{BG}x=\mathrm{e}^{\int_r^s\frac{\ln f(x)}{x}dx}$.

Definition 8. (Erdoğan & Duyar, 2018) Let the function $f:[a,+\infty)\subset\mathbb{R}_{\exp}\to\mathbb{R}_{\exp}$ be $*$-continuous on $[a,b]\subset\mathbb{R}_{\exp}$ for each $a\leq b$. The BG-limit ${}_{BG}\lim_{b\to+\infty}{}_{BG}\int_a^b f(t)d^{BG}t$ is called improper BG-integral of the function f on $[a,+\infty)$ and it is demonstrated by ${}_{BG}\int_a^{+\infty} f(t)d^{BG}t$. If the ${}_{BG}\lim_{b\to+\infty}{}_{BG}\int_a^b f(t)d^{BG}t$

exists and is equal to a number $L \in \mathbb{R}_{\exp}$, then it is said that the improper BG-integral ${}_{BG}\int_a^{+\infty} f(t)d^{BG}t$ is BG-convergent.

Definition 9. (Gungor & Dinc, 2024) If there are $M \in \mathbb{R}_{\exp}^{+}$ and $\gamma \in \mathbb{R}_{\exp}$ such that

$$|f(t)|_{\exp} \leq M \odot e^{(\ln(\gamma \odot t))_{\exp}} = M^{\gamma^{\ln t}}$$

for every $t \geq t_0$ with $t_0 \geq 1$, then it is called that f is a function of BG-exponential order γ.

Definition 10. (Gungor & Dinc, 2024) If ${}_{BG}\lim_{t \to t_0^+} f(t)$ and ${}_{BG}\lim_{t \to t_0^-} f(t)$ exist but are not equal, the function f is jump BG-discontinuity at a point t_0.

Definition 11. (Gungor & Dinc, 2024) If there is a finite subinterval $[a, t_1], [t_1, t_2], \ldots, [t_{n-1}, b]$ such that f is BG-continuous on each $(t_{i-1}, t_i) \subset \mathbb{R}_{\exp}$ with $t_0 = a, t_n = b$, $i = 1, \ldots, n$ and has the one-sided BG-limits ${}_{BG}\lim_{t \to t_{i-1}^+} f(t)$ and ${}_{BG}\lim_{t \to t_i^-} f(t)$, then the function f is sectionally (piecewise) BG-continuous on $[a, b] \subset \mathbb{R}_{\exp}$.

Motivated by the vast applications of both bigeometric calculus and integral transformations, this study discusses and analyzes the bigeometric Sumudu transform, a special case of the non-Newtonian Sumudu transform in Gungor & Dinc (2024).

Main Results

This section presents the Sumudu transform from a bigeometric perspective, offering a novel viewpoint on integral transforms and a fundamental explanation of the underlying theory of the bigeometric Sumudu transform.

Definition 12. The set of functions $\mathbb{A}$ is defined by

$$\mathbb{A} = \left\{ f(t) \colon \exists M > e, \tau_1, \tau_2 > 1, |f(t)|_{\exp} < M \odot e^{\left(\frac{\ln(|t|_{\exp})}{\ln(\tau_j)}\right)_{\exp}}, \right.$$

$$\left. t \in (1 \ominus e)^{j_{\exp}} \times [1, +\infty) \right\}$$

where $M \in \mathbb{R}_{\exp}$ and τ_1, τ_2 are finite exp-constants or infinite. The bigeometric Sumudu integral transform for a function in the set $\mathbb{A}$ is defined as

$$S_{BG}\{f(t)\} = F_{BG}(v)$$
$$= {}_{BG}\int_{1}^{+\infty} \frac{e}{v}\exp \odot e^{\left(\frac{-\ln t}{\ln v}\right)_{\exp}} \odot f(t) d^{BG}t \qquad (1)$$

for $v \in (\ominus \tau_1, \tau_2)$. The equation is also given as

$$S_{BG}\{f(t)\} = {}_{BG}\int_{1}^{+\infty} e^{(-\ln t)_{\exp}} \odot f(v \odot t) d^{BG}t. \qquad (2)$$

The relationship between classical calculus and bigeometric calculus indicates that equation (1) is equal to

$$S_{BG}\{f(t)\} = {}_{BG}\int_{1}^{+\infty} \frac{e}{v}\exp \odot e^{\left(\frac{-\ln t}{\ln v}\right)_{\exp}} \odot f(t) d^{BG}t$$

$$= {}_{BG}\lim_{c \to +\infty} {}_{BG}\int_{1}^{c} e^{\left(\frac{1}{\ln v} e^{\left(\frac{-\ln t}{\ln v}\right)} \cdot \ln(f(t))\right)} d^{BG}t$$

$$= {}_{BG}\lim_{c \to +\infty} \exp\left\{ \int_{1}^{c} \frac{1}{\ln v} \cdot e^{\left(\frac{-\ln t}{\ln v}\right)} \cdot \frac{\ln(f(t))}{t} dt \right\}$$

$$= \exp\left\{\lim_{c\to+\infty}\int_1^c \frac{1}{\ln\upsilon}\cdot e^{\left(\frac{-\ln t}{\ln\upsilon}\right)}\cdot\frac{\ln\big(f(t)\big)}{t}\,dt\right\}.$$

$$= \exp\left\{\int_1^{+\infty} \frac{1}{\ln\upsilon}\cdot e^{\left(\frac{-\ln t}{\ln\upsilon}\right)}\cdot\frac{\ln\big(f(t)\big)}{t}\,dt\right\}. \qquad (3)$$

If we change of variable as $t = e^z$ in equation (3), it is also written as

$$S_{BG}\{f(t)\} = \exp\left\{\int_0^{+\infty} \frac{1}{\ln\upsilon}\cdot e^{\left(\frac{-z}{\ln\upsilon}\right)}\cdot\ln\big(f(e^z)\big)\,dz\right\}. \qquad (4)$$

Remark 13. For $c, t \in \mathbb{R}_{\exp}$, let $\bar{c} = \ln c, \bar{\upsilon} = \ln\upsilon$. Let $\bar{f}(z) = \ln\big(f(e^z)\big)$, where f is a positive $\mathbb{R}_{\exp}$-valued function. By using the equation (4), how the classical Sumudu transform is related to the bigeometric Sumudu transform is demonstrated in the following way:

$$S_N\{f(t)\} = {}_{BG}\int_1^{+\infty} \frac{e}{\upsilon}\exp\odot\, e^{\left(\frac{-\ln t}{\ln\upsilon}\right)_{\exp}}\odot f(t)d^{BG}t$$

$$= \exp\left\{\int_0^{+\infty} \frac{1}{\ln\upsilon}\cdot e^{\left(\frac{-z}{\ln\upsilon}\right)}\cdot\ln\big(f(e^z)\big)\,dz\right\}$$

$$= \exp\left\{\lim_{c\to+\infty}\int_0^{\ln c} \frac{1}{\ln\upsilon}\cdot e^{\left(\frac{-z}{\ln\upsilon}\right)}\cdot\ln\big(f(e^z)\big)\,dz\right\}$$

$$= \exp\left\{\lim_{\bar{c}\to+\infty}\int_0^{\bar{c}} \frac{1}{\bar{\upsilon}}\cdot e^{\left(\frac{-z}{\bar{\upsilon}}\right)}\cdot\bar{f}(z)dz\right\}$$

$$= e^{S\left(\bar{f}(z)\right)}$$

$$= e^{S\left(\ln\left(f(e^z)\right)\right)}.$$

Hence we get the expression $S_{BG}\{f(t)\} = \; e^{S\left(\bar{f}(z)\right)} \; = e^{S\left(\ln\left(f(e^z)\right)\right)}$.

Example 14. The *BG*-Sumudu transform of the function $f(t) = t$ can be determined using the equation (4):

$$S_{BG}\{t\} = \; {}_{BG}\int_{1}^{+\infty} \frac{e}{v} \exp \odot e^{\left(\frac{-\ln t}{\ln v}\right)_{\exp}} \odot t \; d^{BG}t$$

$$= \exp\left\{\int_{0}^{+\infty} \frac{1}{\ln v} \cdot e^{\left(\frac{-z}{\ln v}\right)} \cdot \ln(e^z)\, dz\right\}$$

$$= \exp\left\{\int_{0}^{+\infty} \frac{1}{\ln v} \cdot e^{\left(\frac{-z}{\ln v}\right)} \cdot z\, dz\right\}$$

$$= \exp\left\{\lim_{a\to+\infty} \int_{0}^{a} \frac{1}{\ln v} \cdot e^{\left(\frac{-z}{\ln v}\right)} \cdot z\, dz\right\}$$

$$= \exp\left\{\lim_{a\to+\infty}\left(-e^{\frac{-z}{\ln v}}.z\Big|_{0}^{a} + \int_{0}^{a} e^{\frac{-z}{\ln v}}\, dz\right)\right\}$$

$$= \exp\left\{-\lim_{a\to+\infty}\left(a e^{\frac{-a}{\ln v}}\right) - \lim_{a\to+\infty} \ln v\, e^{\frac{-a}{\ln v}} + \ln v\right\}$$

$$= \exp\{\ln v\} = v = e \odot v = 1!_{\exp} \odot v.$$

This result can be generalized through the application of induction as follows:

$$S_{BG}\left\{t^{(m)_{\exp}}\right\} = m!_{\exp} \odot v^{m_{\exp}} = e^{m!} \odot v^{m_{\exp}} = e^{m!(\ln v)^m} (m \in \mathbb{N}).$$

The BG-Sumudu transforms of certain elementary functions are provided below.

$f(t)$	$S_{BG}[f(t)] = F_{BG}(v)$
e	e
t	v
$t^{(m)_{\exp}}, m \in \mathbb{N}$	$e^{m!} \odot v^{m_{\exp}} = e^{m!(\ln v)^m}$
$e^{(\ln k \ln t)_{\exp}}, e^{\frac{1}{\ln v}} > k, k \in \mathbb{R}_{\exp}$	$\frac{e}{e \ominus k \odot v} \exp$
$t \odot e^{(\ln k \ln t)_{\exp}}$	$\frac{v}{(e \ominus k \odot v)^{2_{\exp}}} \exp$
$\exp\{\sin(\ln k \ln t)\}$	$\frac{k \odot v}{e \oplus (k)^{2_{\exp}} \odot (v)^{2_{\exp}}} \exp, v > 1$
$\exp\{\cos(\ln k \ln t)\}$	$\frac{e}{e \oplus k^{2_{\exp}} \odot (v)^{2_{\exp}}} \exp, v > 1.$

Theorem 15 (Existence of BG-Sumudu transform). The BG-Sumudu transform $S_{BG}\{f(t)\}$ exists for $e^{\frac{1}{\ln v}} > \gamma$ and BG-converges exp-absolutely if f is sectionally BG-continuous on $[1, +\infty)$ and has BG-exponential order γ.

Proof. It can be written as

$$_{BG}\int_1^{+\infty} \frac{e}{v} \exp \odot e^{\left(\frac{-\ln t}{\ln v}\right)_{\exp}} \odot f(t) d^{BG} t$$

$$= \frac{e}{v} \exp \odot \ _{BG}\int_1^{t_0} e^{\left(\frac{-\ln t}{\ln v}\right)_{\exp}} \odot f(t) d^{BG} t \oplus$$

$$\oplus \frac{e}{v} \exp \odot \ _{BG}\int_{t_0}^{+\infty} e^{\left(\frac{-\ln t}{\ln v}\right)_{\exp}} \odot f(t) d^{BG} t \ . \qquad (5)$$

Since f is sectionally BG-continuous on $[e, t_0]$, the function f is BG-continuous on $(e, t_0) \subset \mathbb{R}_{\exp}$ with the exception of a finite number of points $t_1, t_2, \ldots, t_n$ in (e, t_0). For finite constants $M_i \in \mathbb{R}_{\exp}$, one gets

$$|f(t)|_{\exp} \leq M_i, \qquad t_i < t < t_{i+1} \; (i = 1,2, \ldots, n-1).$$

In order to integrate the sectionally BG-continuous function from 1 to t_0, the exp-sum of the BG-integrals over each of the exp-subintervals of f is computed, that is

$$_{BG}\int_1^{t_0} e^{\left(\frac{-\ln t}{\ln v}\right)_{\exp}} \odot f(t) d^{BG}t = {}_{BG}\int_1^{t_1} e^{\left(\frac{-\ln t}{\ln v}\right)_{\exp}} \odot f(t) d^{BG}t$$

$$\oplus \; {}_{BG}\int_{t_1}^{t_2} e^{\left(\frac{-\ln t}{\ln v}\right)_{\exp}} \odot f(t) d^{BG}t \oplus \ldots \oplus$$

$$\oplus \; {}_{BG}\int_{t_n}^{t_0} e^{\left(\frac{-\ln t}{\ln v}\right)_{\exp}} \odot f(t) d^{BG}t.$$

Since f is BG-continuous and exp-bounded on every exp-subinterval, it follows that each BG-integral is well-defined. Therefore, the first integral on the right side of (5) is accurate.

Given that f possesses BG-exponential order γ, there exist $M \in \mathbb{R}^+_{\exp}$ and $\gamma \in \mathbb{R}_{\exp}$ such that

$$|f(t)|_{\exp} \leq M \odot e^{(\ln(\gamma)\ln(t))_{\exp}}$$

for all $t > t_0$. Thus, we obtain

$$\left| {}_{BG}\int_{t_0}^{+\infty} e^{\left(\frac{-\ln t}{\ln v}\right)_{\exp}} \odot f(t) d^{BG}t \right|_{\exp}$$

$$\leq {}_{BG}\int_{t_0}^{+\infty} e^{\left(\frac{-\ln t}{\ln \upsilon}\right)_{\exp}} \odot |f(t)|_{\exp} d^{BG}t$$

$$\leq {}_{BG}\int_{t_0}^{+\infty} e^{\left(\frac{-\ln t}{\ln \upsilon}\right)_{\exp}} \odot M \odot e^{(\ln\gamma \ln t)_{\exp}} d^{BG}t$$

$$= M \odot {}_{BG}\int_{t_0}^{+\infty} e^{\left(-\ln t\left(\frac{1}{\ln \upsilon} - \ln \gamma\right)\right)_{\exp}} d^{BG}t$$

$$= M \odot {}_{BG}\lim_{c\to+\infty}\left[{}_{BG}\int_{t_0}^{c} e^{\left(-\ln t\left(\frac{1}{\ln \upsilon} - \ln \gamma\right)\right)_{\exp}} d^{BG}t\right]$$

$$= M \odot {}_{BG}\lim_{c\to+\infty}\left[{}_{BG}\int_{t_0}^{c} \exp\left\{e^{-\ln t\left(\frac{1}{\ln \upsilon} - \ln \gamma\right)}\right\} d^{BG}t\right]$$

$$= M \odot {}_{BG}\lim_{c\to+\infty}\left[\exp\left\{\int_{t_0}^{c} \frac{1}{t} e^{-\ln t\left(\frac{1}{\ln \upsilon} - \ln \gamma\right)} dt\right\}\right]$$

$$= M \odot {}_{BG}\lim_{c\to+\infty} \frac{1}{\left(\ln\gamma - \frac{1}{\ln \upsilon}\right)} \cdot \exp\left\{\left(\ln\gamma - \frac{1}{\ln\upsilon}\right)(\ln c - \ln t_0)\right\}$$

$$= M \odot \exp\left\{\lim_{c\to+\infty} \frac{\ln\upsilon}{(\ln\gamma \ln\upsilon - 1)} \cdot \exp\left\{\left(\ln\gamma - \frac{1}{\ln\upsilon}\right)(\ln c - \ln t_0)\right\}\right\}$$

$$= M \odot \exp\left\{\frac{\ln\upsilon}{(1 - \ln\gamma \ln\upsilon)} \cdot \exp\left\{\left(\ln\gamma - \frac{1}{\ln\upsilon}\right)\ln t_0\right\}\right\}.$$

The second integral on the right is also defined for $e^{\frac{1}{\ln\upsilon}} > \gamma$. Consequently, the argument is substantiated.

Theorem 16 (BG-linearity property). If f_1 and f_2 are two positive $\mathbb{R}_{\exp}$-valued functions with existing BG-Sumudu transforms, then

$$S_{BG}\{\lambda_1 \odot f_1(t) \oplus \lambda_2 \odot f_2(t)\} \;= \lambda_1 \odot S_{BG}\{f_1(t)\} \oplus \lambda_2 \odot S_{BG}\{f_2(t)\}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}_{\exp}$.

Proof. Suppose that

$$|f_1(t)|_{\exp} \leq M_1 \odot e^{(\ln\gamma \ln t)_{\exp}}$$

$$|f_2(t)|_{\exp} \leq M_2 \odot e^{(\ln\gamma \ln t)_{\exp}}.$$

Consequently, we can express

$$\begin{aligned} &|\lambda_1 \odot f_1(t) \oplus \lambda_2 \odot f_2(t)|_{\exp} \\ &\qquad \leq (\lambda_1 \odot M_1 \oplus \lambda_2 \odot M_2) \odot e^{(\ln\gamma \ln t)_{\exp}}, \end{aligned}$$

indicating that the BG-Sumudu transform of the function $\lambda_1 \odot f_1(t) \oplus \lambda_2 \odot f_2(t)$ exists. By properties of improper BG-integral, we find

$$\begin{aligned} &S_N\{\lambda_1 \odot f_1(t) \oplus \lambda_2 \odot f_2(t)\} \\ &= {}_{BG}\int_1^{+\infty} \frac{e}{v}\exp \odot e^{\left(\frac{-\ln t}{\ln v}\right)_{\exp}} \odot \big(\lambda_1 \odot f_1(t) \oplus \lambda_2 \odot f_2(t)\big) d^{BG}t \\ &= \lambda_1 \odot {}_{BG}\int_1^{+\infty} \frac{e}{v}\exp \odot e^{\left(\frac{-\ln t}{\ln v}\right)_{\exp}} \odot f_1(t) d^{BG}t \\ &\oplus \lambda_2 \odot {}_{BG}\int_1^{+\infty} \frac{e}{v}\exp \odot e^{\left(\frac{-\ln t}{\ln v}\right)_{\exp}} \odot f_2(t) d^{BG}t \\ &= \lambda_1 \odot S_{BG}\{f_1(t)\} \oplus \lambda_2 \odot S_{BG}\{f_2(t)\} \end{aligned}$$

which completes the proof.

Theorem 17 (BG-first translation theorem). If $S_{BG}\{f(t)\} = F_{BG}(v)$ exists for $e^{\frac{1}{\ln v}} > \gamma$, then

$$S_{BG}\left\{e^{(\ln k \ln t)_{\exp}} \odot f(t)\right\} = \frac{e}{e \ominus k \odot v}\exp \odot F_{BG}\left(\frac{v}{e \ominus k \odot v}\exp\right)$$

for any $k \in \mathbb{R}_{\exp}$.

Proof. Utilizing the definition of the BG-Sumudu transform as presented in equation (2) yields

$$S_N\left\{e^{(\ln k \ln t)_{\exp}} \odot f(t)\right\}$$

$$= {}_{BG}\int_1^{+\infty} e^{(-\ln t)_{\exp}} \odot e^{(\ln k \ln t \ln v)_{\exp}} \odot f(v \odot t) d^{BG}t$$

$$= {}_{BG}\int_1^{+\infty} e^{\left(-\ln t(1-\ln k \ln v)\right)_{\exp}} \odot f(v \odot t) d^{BG}t$$

$$= {}_{BG}\lim_{c\to+\infty}\left({}_{BG}\int_1^{c} \exp\left\{e^{-\ln t(1-\ln k \ln v)}\right\} \odot f(v \odot t) d^{BG}t\right)$$

$$= {}_{BG}\lim_{c\to+\infty}\exp\left\{\int_1^{c}\frac{1}{t}e^{-\ln t(1-\ln k \ln v)}\ln f((\ln t)^{v})\,dt\right\}$$

for $e^{\frac{1}{\ln v}} > \gamma$. If we consider it to be $\ln t\,(1 - \ln k \ln v) = w$, then we find that

$$S_N\left\{e^{(\ln k \ln t)_{\exp}} \odot f(t)\right\}$$

$$= {}_{BG}\lim_{c\to+\infty}\exp\left\{\int_1^{c}\frac{1}{t}e^{-\ln t(1-\ln k \ln v)}\ln f((\ln t)^{v})\,dt\right\}$$

$$= {}_{BG}\lim_{c\to+\infty} \exp\left\{\frac{1}{1-\ln k \ln v} \int_0^{\ln c\,(1-\ln k \ln v)} e^{-w} \ln f\left(\left(\frac{w}{1-\ln k \ln v}\right)^{v}\right) dw\right\}$$

$$= \exp\left\{\frac{1}{1-\ln k \ln v}\right\} {}_{BG}\lim_{c\to+\infty} \exp\left\{\int_0^{\ln c\,(1-\ln k \ln v)} e^{-w} \ln f\left(\left(\frac{w}{1-\ln k \ln v}\right)^{v}\right) dw\right\}$$

$$= \frac{e}{e \ominus k \odot v} \exp \odot$$

$${}_{BG}\lim_{c\to+\infty}\left({}_{BG}\int_1^{c\odot(e\ominus k\odot v)} e^{(-\ln w)_{\exp}} \odot f\left(\frac{v}{e\ominus k \odot v}\exp \odot w\right) d^{BG}w\right)$$

$$= \frac{e}{e \ominus k \odot v} \exp \odot F_{BG}\left(\frac{v}{e\ominus k \odot v}\exp\right).$$

Theorem 18 ($\boldsymbol{BG}$-second translation theorem). If $S_{BG}\{f(t)\} = F_{BG}(v)$ and $h(t) = \begin{cases} 1, & 1 < t < k \\ f(t \ominus k) & , t > k \end{cases}$, then

$$S_{BG}\{h(t)\} = e^{\left(\frac{-\ln k}{\ln v}\right)_{\exp}} \odot F_{BG}(v).$$

Proof. Utilizing equation (3), it is evident that

$$S_{BG}\{h(t)\} = {}_{BG}\int_1^{+\infty} \frac{e}{v}\exp \odot e^{\left(\frac{-\ln t}{\ln v}\right)_{\exp}} \odot h(t) d^{BG}t$$

$$= \exp\left\{\int_1^{+\infty} \frac{1}{\ln v} \cdot e^{\left(\frac{-\ln t}{\ln v}\right)} \cdot \frac{\ln(h(t))}{t} dt\right\}$$

$$= \exp\left\{\int_{1}^{k} \frac{1}{\ln v} \cdot e^{\left(\frac{-\ln t}{\ln v}\right)} \cdot \frac{\ln 1}{t} dt + \int_{k}^{+\infty} \frac{1}{\ln v} \cdot e^{\left(\frac{-\ln t}{\ln v}\right)} \cdot \frac{\ln\big(f(t \ominus k)\big)}{t} dt\right\}$$

$$= \exp\left\{\int_{k}^{+\infty} \frac{1}{\ln v} \cdot e^{\left(\frac{-\ln t}{\ln v}\right)} \cdot \frac{\ln\big(f(t/k)\big)}{t} dt\right\}.$$

Substituting $\frac{t}{k} = w$ gives

$$S_{BG}\{h(t)\} = \exp\left\{\int_{k}^{+\infty} \frac{1}{\ln v} \cdot e^{\left(\frac{-\ln k - \ln w}{\ln v}\right)} \cdot \frac{\ln\big(f(w)\big)}{w} dw\right\}$$

$$= \exp\left\{e^{\left(\frac{-\ln k}{\ln v}\right)}\right\} \cdot \exp\left\{\int_{1}^{+\infty} \frac{1}{\ln v} \cdot e^{\left(\frac{-\ln w}{\ln v}\right)} \cdot \frac{\ln\big(f(w)\big)}{w} dw\right\}$$

$$= e^{\left(\frac{-\ln k}{\ln v}\right)_{\exp}} \odot F_{BG}(v)$$

which completes the proof.

Theorem 19 (*BG*-derivative theorem). If $f(t)$ is BG-continuous on $[1, +\infty)$ and has BG-exponential order γ, and also $f^{BG}(t)$ is sectionally BG-continuous on $[1, +\infty)$, then

$$S_{BG}\{f^{BG}(t)\} = \frac{S_{BG}\{f(t)\} \ominus f(1)}{v} \text{ exp}$$

for $e^{\frac{1}{\ln v}} > \gamma$.

Proof. By using the equation (3), one gets

$$S_{BG}\{f^{BG}(t)\} = {}_{BG}\int_{1}^{+\infty} \frac{e}{v} \exp \odot e^{\left(\frac{-\ln t}{\ln v}\right)_{\exp}} \odot f^{BG}(t) d^{BG}t$$

$$= \exp\left\{\int_1^{+\infty} \frac{1}{\ln \upsilon} \cdot e^{\left(\frac{-\ln t}{\ln \upsilon}\right)} \cdot \frac{\ln\left(f^{BG}(t)\right)}{t} dt\right\}$$

$$= \exp\left\{\int_1^{+\infty} \frac{1}{\ln \upsilon} \cdot e^{\left(\frac{-\ln t}{\ln \upsilon}\right)} \cdot \frac{\ln\left(e^{\frac{tf'(t)}{f(t)}}\right)}{t} dt\right\}$$

$$= \exp\left\{\int_1^{+\infty} \frac{1}{\ln \upsilon} \cdot e^{\left(\frac{-\ln t}{\ln \upsilon}\right)} \cdot \frac{f'(t)}{f(t)} dt\right\}$$

$$= \exp\left\{\frac{1}{\ln \upsilon} \cdot \lim_{a\to+\infty} \int_1^{a} e^{\left(\frac{-\ln t}{\ln \upsilon}\right)} \cdot \frac{f'(t)}{f(t)} dt\right\}.$$

The following expression is derived utilizing the method of partial integration:

$$\int_1^{a} e^{\left(\frac{-\ln t}{\ln \upsilon}\right)} \cdot \frac{f'(t)}{f(t)} dt$$

$$= e^{\left(\frac{-\ln t}{\ln \upsilon}\right)} . \ln f(t)\Big|_1^{a} + \frac{1}{\ln \upsilon} . \int_1^{a} e^{\left(\frac{-\ln t}{\ln \upsilon}\right)} \cdot \frac{\ln f(t)}{t} dt$$

$$= e^{\left(\frac{-\ln a}{\ln \upsilon}\right)} . \ln f(a) - \ln f(1) + \frac{1}{\ln \upsilon} . \int_1^{a} e^{\left(\frac{-\ln t}{\ln \upsilon}\right)} \cdot \frac{\ln f(t)}{t} dt.$$

Consequently, we find

$$S_{BG}\{f^{BG}(t)\} = \exp\left\{\frac{1}{\ln \upsilon} \cdot \left(\lim_{a\to+\infty} e^{\left(\frac{-\ln a}{\ln \upsilon}\right)} \ln f(a) - \ln f(1)\right.\right.$$

$$\frac{1}{\ln v}.\int_1^{+\infty} e^{\left(\frac{-\ln t}{\ln v}\right)}\cdot\frac{\ln f(t)}{t}dt\Bigg)\Bigg\}.$$

Due to the fact that f is of BG-exponential order γ, it follows that there are exits $M \in \mathbb{R}_{\exp}^{+}$ and $\gamma \in \mathbb{R}_{\exp}$ such that

$$|f(t)|_{\exp} = \exp\{|\ln f(t)|\} \leq M \odot e^{(\ln(\gamma\odot t))_{\exp}} = \mu^{\gamma^{\ln t}}$$

$$|\ln f(t)| \leq \ln M\, \gamma^{\ln t}.$$

As a result,

$$\left|e^{\left(\frac{-\ln a}{\ln v}\right)}.\ln f(a)\right| \leq \ln M\, \gamma^{\ln a} e^{\left(\frac{-\ln a}{\ln v}\right)} = \ln M\, e^{\ln a \ln\gamma} e^{\left(\frac{-\ln a}{\ln v}\right)}$$

$$= \ln M\, e^{-\ln a\left(-\ln\gamma+\frac{1}{\ln v}\right)}$$

is found. As a result of the fact that $\lim_{a\to+\infty} \ln M\, e^{-\ln a\left(-\ln\gamma+\frac{1}{\ln v}\right)} = 0$ for $e^{\frac{1}{\ln v}} > \gamma$, we get $\lim_{a\to+\infty} e^{\left(\frac{-\ln a}{\ln v}\right)}.\ln f(a) = 0$ for $e^{\frac{1}{\ln v}} > \gamma$. Therefore, we achieve

$S_{BG}\{f^{BG}(t)\}$

$$= \exp\left\{\frac{1}{\ln v}\cdot\left(\int_1^{+\infty} \frac{1}{\ln v}.e^{\left(\frac{-\ln t}{\ln v}\right)}.\frac{\ln f(t)}{t}dt - \ln f(1)\right)\right\}$$

$$= \left(\frac{\exp\left\{\int_1^{+\infty}\frac{1}{\ln v}.e^{\left(\frac{-\ln t}{\ln v}\right)}.\frac{\ln f(t)}{t}dt\right\}}{\exp\{\ln f(1)\}}\right)^{\frac{1}{\ln v}}$$

$$= \frac{S_{BG}\{f(t)\}\ominus f(1)}{v}\ \exp$$

as intended.

Corollary 20. Assuming that $f(t), f^{BG}(t), \cdots, f^{BG(n-1)}(t)$ are BG-continuous functions on the $[1, +\infty)$ and exhibit BG-exponential order γ and further supposing that $f^{BG(n)}(t)$ is sectionally BG-continuous on $[1, +\infty)$, then follows that

$$S_{BG}\{f^{BG(n)}(t)\} = \frac{S_{BG}\{f(t)\}}{\upsilon^{n_{\exp}}} \exp \ominus \frac{f(1)}{\upsilon^{n_{\exp}}} \exp \ominus \frac{f^{BG}(1)}{\upsilon^{(n-1)_{\exp}}} \exp$$

$$\ominus \dots \ominus \frac{f^{BG(n-1)}(1)}{\upsilon} \exp$$

for $e^{\frac{1}{\ln \upsilon}} > \gamma$.

Definition 21. The inverse BG -Sumudu transform is defined $S_{BG}^{-1}\{F_N(\upsilon)\} = f(t)$, if $S_{BG}\{f(t)\} = F_{BG}(\upsilon)$.

Theorem 22. The inverse BG-Sumudu transform is linear.

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CHAPTER V

A Short Note on Measurable Sets in Multiplicative Calculus

Oğuz OĞUR[1]

Introduction

Non-Newtonian analysis, defined by (Grossman & Katz, 1972) as an alternative to the classical number system, has found applications across various fields, including physics, mechanics, mathematics and economics etc. (Grossman, 1979), (Stanley, 1999). The core concepts of non-Newtonian analysis-such as integrals, series, sequence spaces, convergence, metrics, and norms—have been thoroughly examined by numerous researchers (for details see references (Bashirov, Kurpinar & Özyapıcı, 2008), (Uzer, 2010), (Çakmak & Başar, 2014) (Değirmen, 2021), (Değirmen & Duyar, 2023), (Torres, 2021)). Additionally, studies on integral equations

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(Güngör, 2020a, 2020b, 2022) have shown how Non-Newtonian analysis diversifies solution methods. Recent studies, such as those by (Işık & Eryılmaz, 2023) on the properties of linear spaces defined over Non-Newtonian fields and by (Rohman & Eryılmaz, 2023) on fundamental results in ν-normed spaces, have contributed significantly to the understanding and development of alternative mathematical frameworks.

The groundwork for the concept of measurable sets in non-Newtonian analysis was first laid by (Duyar, Sağır & Ogur, 2015) and (Duyar & Ogur, 2017). Following this, (Duyar & Sağır, 2017) derived the Lebesgue measure for non-Newtonian open sets, laying the groundwork for a non-Newtonian measure concept and creating a need for defining measures on more general sets. To meet this need, (Ogur & Sezgin, 2019, 2020) as well as (Ogur & Zekiye, 2024), introduced the notion of non-Newtonian measurable sets and investigated some of their key properties.

In the second section of this book, the foundational concepts of multiplicative calculus were introduced, including the measures of multiplicative open and closed bounded sets, as well as the definitions and basic properties of multiplicative inner and outer measures. In this section, we will delve deeper into the key properties of these measures and define a Lebesgue measurable set using multiplicative inner and outer measures within the multiplicative framework.

Now, let’s introduce some basic concepts in non-Newtonian analysis. Let ρ be a generator which is an injective function from

$\mathbb{R}$ $to\ A = \mathbb{R}(N)_\rho \subseteq \mathbb{R}$. Let's define the non-Newtonian algebraic operations as follows;

$$\begin{array}{ll}
\rho - addition & s\dot{+}t = \rho(\rho^{-1}(s) + \rho^{-1}(t)) \\
\rho - subtraction & s\dot{-}t = \rho(\rho^{-1}(s) - \rho^{-1}(t)) \\
\rho - multiplication & s \dot{\times} t = \rho(\rho^{-1}(s) \times \rho^{-1}(t)) \\
\rho - division & s \dot{\div} t = \rho(\rho^{-1}(s) \div \rho^{-1}(t)) \\
\rho - order & s \dot{<} t \Leftrightarrow \rho^{-1}(s) < \rho^{-1}(t)
\end{array}$$

for any $s, t \in \mathbb{R}(N)_\rho$ (Grossman & Katz, 1972). The non-Newtonian absolute value of any element of $t \in \mathbb{R}(N)_\rho$ defines as follows

$$|t|_\rho = \begin{cases} t, & if\ t \dot{>} \dot{0} \\ \dot{0}, & if\ t \dot{=} \dot{0} \\ \dot{0}\dot{-}t & if\ t \dot{<} \dot{0} \end{cases}$$

where $\rho(0) = \dot{0}$. Also, we have $\sqrt[n]{t}^{\rho} = \rho\left(\sqrt[n]{\rho^{-1}(t)}\right)$ and $t^{n_\rho} = \rho\left(\left(\rho^{-1}(t)\right)^n\right)$ (Grossman & Katz, 1972).

At this point, we can introduce *geometric analysis*, a specific case within the broader framework of non-Newtonian analysis. For this purpose, setting $\rho(s) = \exp(s)$ will be sufficient. Let

$\rho: \mathbb{R} \to \mathbb{R}^+,\ s \to \rho(s) = exp\ (s)$

and so

$\rho^{-1}: \mathbb{R}^+ \to \mathbb{R},\ \rho^{-1}(t) = \ln(t)$.

Thus, we get

$$\mathbb{R}(N)_\rho = \mathbb{R}_{exp} = \{exp(s): s \in \mathbb{R}\} = \mathbb{R}^+,$$

$$\mathbb{R}^+_{exp} = \{exp(s): s \in \mathbb{R}^+\} = (1, +\infty)$$

and

$$\mathbb{R}_{exp}^{-} = \{exp(s): s \in \mathbb{R}^{-}\} = (0,1)$$

where $exp(0) = 1$. By the definition, we have the multiplicative sum of $s, t \in \mathbb{R}_{exp}$ as follows;

$$s \dot{+} t = exp(lns + lnt) = e^{\ln(st)} = st.$$

By using similar way, we get the multiplicative algebraic operations as follows;

$s \dot{+} t = st$
$s \dot{-} t = {}^{s}/_{t}$
$s \dot{\times} t = s^{lnt}$
$s \dot{\div} t = s^{\frac{1}{lnt}}$

In this section, as in the second section of this book we will use the symbols $(,)_{exp}$, λ_{exp}, ${}_{exp}\sum .$, ${}^{exp}inf$, ${}^{exp}sup$ to represent the open interval, Lebesgue measure, sum, infimum and supremum, respectively, in the context of multiplicative analysis.

Now, let us present the definitions of inner measure, outer measure, and Lebesgue measure, which are well-known in real analysis and will be used in this section. For more detailed information on these topics, the reader may refer to the book by Natanson (Natanson, 1964).

Definition 1. The outer measure $\lambda^{o}(E)$ of a bounded set E is defined as

$$\lambda^{o}(E) = inf\{\lambda(T): E \subset T,\ T\ is\ bounded\ open\ set\}$$

(Natanson, 1964).

Definition 2. The inner measure $\lambda^i(E)$ of a bounded set E is defined as

$$\lambda^i(E) = sup\{\lambda(U): U \subset E,\ \ U\ is\ bounded\ closed\ set\}$$

(Natanson, 1964).

Definition 3. A bounded set E is said to be measurable if its outer and inner measures are equal;

$$\lambda^i(E) = \lambda^o(E).$$

The common value of these two mesures is called the measure of the set E and is designated by $\lambda(E)$. This measure is sometimes referred to as a Lebesgue measurable set (Natanson, 1964).

Building on this general information, we can now present fundamental theorems on multiplicative inner and outer measures. As a special case, we will define the Lebesgue measurable set in geometric analysis using these measures, with $\rho(x) = exp(x)$, as discussed in (Oğur & Güneş, 2024).

Theorem 1. Let E be a bounded set with $E = \bigcup_{l=1}^{\infty} E_l$ in $\mathbb{R}_{exp}$. Then, we have the following inequality;

$$\lambda_{exp}^o(E) \dot{\leq} {}_{exp}\sum_{l=1}^{\infty} \lambda_{exp}^o(E_l).$$

Proof. If the sum is not finite in the right side, the proof is completed. Assume the sum is finite. Let G be an open set with $E \subset G$ and G_l be open sets with $E_l \subset G_l$ in $\mathbb{R}_{exp}$. Thus, we get

$$\lambda_{exp}^o(E) = {}^{exp}inf_{E \subset G}\{\ \lambda_{exp}(G)\}$$

$$= exp\left\{inf_{lnE \subset lnG} ln\left(\ \lambda_{exp}(G)\right)\right\}$$

$$= exp\left\{inf_{lnE\subset lnG} ln\left(exp\left(\lambda\big(ln(G)\big)\right)\right)\right\}$$

$$= exp\big\{inf_{lnE\subset lnG}\lambda\big(ln(G)\big)\big\}$$

$$\dot{\leq}\ exp\big\{\sum_{l=1}^{\infty} inf_{lnE_l\subset lnG_l}\lambda\big(ln(G_l)\big)\big\}$$

$$= exp\left\{\sum_{l=1}^{\infty} ln\left(exp\left(inf_{lnE_l\subset lnG_l}\lambda\big(ln(G_l)\big)\right)\right)\right\}$$

$$\underset{exp}{=}\sum_{l=1}^{\infty} exp\left(inf_{lnE_l\subset lnG_l}\lambda\big(ln(G_l)\big)\right)$$

$$\underset{exp}{=}\sum_{l=1}^{\infty} exp\left(inf_{lnE_l\subset lnG_l} ln\left(exp\left(\lambda\big(ln(G_l)\big)\right)\right)\right)$$

$$= {}_{exp}\sum_{l=1}^{\infty} {}^{exp}inf_{E_l\subset G_l}\big\{\lambda\big(ln(G_l)\big)\big\}$$

$$= {}_{exp}\sum_{l=1}^{\infty}\lambda_{exp}^{o}(E_l).$$

Theorem 2. Let E be a bounded set with $E = \bigcup_{l=1}^{\infty} E_l$, $E_l \cap E_k = \emptyset\ for\ l \neq k$ in $\mathbb{R}_{exp}$. Then, we have

$${}_{exp}\sum_{l=1}^{\infty}\lambda_{exp}^{i}(E_l)\ \dot{\leq}\ \lambda_{exp}^{i}(E).$$

Proof. Let (G_l) be a family of closed sets with $G_l \subset E_l$ and let $G = \bigcup_{l=1}^{\infty} G_l$. Thus, we get

$$\lambda_{exp}^{i}(E) = {}^{exp}sup_{G\subset E}\{\ \lambda_{exp}(G)\}$$

$$= exp\left\{sup_{lnG\subset lnE} ln\left(\ \lambda_{exp}(G)\right)\right\}$$

$$= exp\left\{sup_{lnG\subset lnE} ln\left(exp\left(\lambda\big(ln(G)\big)\right)\right)\right\}$$

$$= exp\big\{sup_{lnG\subset lnE}\lambda\big(ln(G)\big)\big\}$$

$$\dot{\geq}\ exp\{\sum_{l=1}^{\infty} sup_{lnG_l\subset lnE_l}\lambda(ln(G_l))\}$$

$$= exp\left\{\sum_{l=1}^{\infty} ln\left(exp\left(sup_{lnG_l\subset lnE_l}\lambda(ln(G_l))\right)\right)\right\}$$

$$= {}_{exp}\sum_{l=1}^{\infty} exp\left(sup_{lnG_l\subset lnE_l}\lambda(ln(G_l))\right)$$

$$= {}_{exp}\sum_{l=1}^{\infty} exp\left(sup_{lnG_l\subset lnE_l} ln\left(exp\left(\lambda(ln(G_l))\right)\right)\right)$$

$$= {}_{exp}\sum_{l=1}^{\infty}\left({}^{exp}sup_{G_l\subset E_l} exp\left(\lambda(ln(G_l))\right)\right)$$

$$= {}_{exp}\sum_{l=1}^{\infty}\ {}^{exp}sup_{G_l\subset E_l}\{\ \lambda_{exp}(G_l)\}$$

$$= {}_{exp}\sum_{l=1}^{\infty} \lambda_{exp}^{i}(E_l)$$

which gives the proof.

Theorem 3. Let E be a bounded set in $\mathbb{R}_{exp}$. If B is an open interval such that $E\subset B$, then

$$\lambda_{exp}(B) = \lambda_{exp}^{o}(E)\dot{+}\lambda_{exp}^{i}(B-E).$$

Proof. Let G be an open set with $E\subset G$ and let K be a closed set with $K\subset B-E$ in $\mathbb{R}_{exp}$. Thus, we get

$$\lambda_{exp}^{o}(E)\dot{+}\lambda_{exp}^{i}(B-E) = exp\left\{ln\left(\lambda_{exp}^{o}(E)\right) + ln\left(\lambda_{exp}^{i}(B-E)\right)\right\}$$

$$= exp\{ln({}^{exp}inf_{E\subset G}\{\ \lambda_{exp}(G)\})$$

$$+ln({}^{exp}sup_{K\subset B-E}\{\ \lambda_{exp}(K)\})\}$$

$$= exp\left\{ln\left(exp\left\{inf_{lnE\subset lnG} ln\left(\ \lambda_{exp}(G)\right)\right\}\right)\right.$$

$$\left.+ln\left(exp\left\{sup_{lnK\subset ln(B-E)} ln\left(\ \lambda_{exp}(K)\right)\right\}\right)\right\}$$

$$= exp\left\{inf_{lnE\subset lnG} ln\left(\ \lambda_{exp}(G)\right)\right.$$

$$\left.+sup_{lnK\subset ln(B-E)} ln\left(\ \lambda_{exp}(K)\right)\right\}$$

$$= exp\left\{inf_{lnE\subset lnG} ln\left(exp\left(\lambda\left(ln(G)\right)\right)\right)\right.$$

$$\left.+sup_{lnK\subset ln(B-E)} ln\left(exp\left(\lambda\left(ln(K)\right)\right)\right)\right\}$$

$$= exp\left\{inf_{lnE\subset lnG}\lambda\left(ln(G)\right)\right.$$

$$\left.+sup_{lnK\subset ln(B-E)}\lambda\left(ln(K)\right)\right\}$$

$$= exp\left\{\lambda^{o}\left(ln(E)\right) + \lambda^{i}\left(ln(B-E)\right)\right\}$$

$$= exp\left\{\lambda\left(ln(B)\right)\right\}$$

$$= \ \lambda_{exp}(B).$$

Here, λ^{o} and λ^{i} are Lebesgue outer and inner measure in real line, respectively.

Definition 4. Let E be a bounded set in $\mathbb{R}_{exp}$. If $\lambda_{exp}^{o}(E) = \lambda_{exp}^{i}(E)$, then the set E is called **multiplicative measurable set**.

Remark 1. It is easy to see that if E is multiplicative measurable set, then we have $\lambda^{o}(\ln(E)) = \lambda^{i}(\ln(E))$, where λ^{o} and λ^{i} are Lebesgue outer and inner measure in real line.

Theorem 4. If E is an open, bounded set in $\mathbb{R}_{exp}$, then E is a multiplicative measurable set.

Proof. The proof can be easily obtained by previously theorems.

Theorem 5. If G is a closed, bounded set in $\mathbb{R}_{exp}$, then G is a multiplicative measurable set.

Proof. The proof can be easily obtained by previously theorems.

Theorem 6. Let (E_k) be a family of pairwise disjoint multiplicative measurable set and let E be a bounded set with $E = \bigcup_{k=1}^{\infty} E_k$ in $\mathbb{R}_{exp}$. Then, the set E is a multiplicative measurable set and

$$\lambda_{exp}(E) = {}_{exp}\sum_{k=1}^{\infty} \lambda_{exp}(E_k).$$

Proof. If $E = \bigcup_{k=1}^{\infty} E_k$, then we have

$$\lambda_{exp}^{o}(E) \dot{\leq} {}_{exp}\sum_{k=1}^{\infty} \lambda_{exp}^{o}(E_k)$$

and

$${}_{exp}\sum_{k=1}^{\infty} \lambda_{exp}^{i}(E_k) \dot{\leq} \lambda_{exp}^{i}(E).$$

Also, if E is a bounded set in $\mathbb{R}_{exp}$, then we have

$$\lambda_{exp}^{i}(E) \dot{\leq} \lambda_{exp}^{o}(E).$$

Since E_k is a multiplicative measurable set for all natural numbers k, we get

$$\begin{aligned} {}_{exp}\sum_{k=1}^{\infty} \lambda_{exp}(E_k) &= {}_{exp}\sum_{k=1}^{\infty} \lambda_{exp}^{i}(E_k) \\ &\dot{\leq} \lambda_{exp}^{i}(E) \\ &\dot{\leq} \lambda_{exp}^{o}(E) \\ &\dot{\leq} {}_{exp}\sum_{k=1}^{\infty} \lambda_{exp}^{o}(E_k) \\ &\dot{\leq} {}_{exp}\sum_{k=1}^{\infty} \lambda_{exp}(E_k). \end{aligned}$$

This gives the proof.

Theorem 7. Let $E_1, E_2, \dots, E_n$ are multiplicative measurable sets in $\mathbb{R}_{exp}$. Then, $E = \bigcup_{k=1}^{n} E_k$ is a multiplicative measurable set.

Proof. Since E_k is a multiplicative measurable set for $k = 1,2,\dots,n$, then we have $\ln(E_k)$ is a measurable set in real line for $k = 1,2,\dots,n$. Thus, the set $\bigcup_{k=1}^{n} \ln(E_k)$ is a measurable set in real line. Since

$$\bigcup_{k=1}^{n} \ln(E_k) = ln(\bigcup_{k=1}^{n} E_k) = \ln(E),$$

the set $\ln(E)$ is measurable set. This shows that $E = \bigcup_{k=1}^{n} E_k$ is a multiplicative measurable set in $\mathbb{R}_{exp}$.

Theorem 8. Let $E_1, E_2, \dots, E_n$ are multiplicative measurable sets in $\mathbb{R}_{exp}$. Then, $E = \bigcap_{k=1}^{n} E_k$ is a multiplicative measurable set.

Proof. Since $E_1, E_2, \dots, E_n$ are multiplicative measurable sets in $\mathbb{R}_{exp}$, we have $\ln(E_1), \ln(E_2), \dots, \ln(E_n)$ are measurable sets in real line. Thus, we get

$$\bigcap_{k=1}^{n} ln(E_k) = ln(\bigcap_{k=1}^{n} E_k) = \ln(E)$$

is a measurable set in real line. This shows that E is a multiplicative measurable set in $\mathbb{R}_{exp}$.

Theorem 9. If E and F are two multiplicative measurable sets in $\mathbb{R}_{exp}$, then $E - F$ is a multiplicative measurable set.

Proof. Since E and F are two multiplicative measurable sets, then $\ln(E)$ and $\ln(F)$ are two measurable sets in real line. Thus, we have

$$\ln(E) - l\,n(F) = \ln(E - F)$$

is a measurable set in real line, which shows that $E - F$ is a multiplicative measurable set.

Theorem 10. Let E and F are two multiplicative measurable sets with $F \subset E$ and let $W = E - F$. Then, we have

$$\lambda_{exp}(W) = \lambda_{exp}(E) \dot{-} \lambda_{exp}(F).$$

Proof. By the Theorem 9, the set $E - F$ is a multiplicative measurable set. Since $E = W \cup F$, we get

$$\lambda_{exp}(E) = \lambda_{exp}(W) \dot{+} \lambda_{exp}(F)$$

which gives the proof.

Theorem 11. Let (E_k) be a family of multiplicative measurable sets in $\mathbb{R}_{exp}$. Then, $E = \bigcap_{k=1}^{\infty} E_k$ is a multiplicative measurable set in $\mathbb{R}_{exp}$.

Proof. Since E_k is a multiplicative measurable set for all k, we have $ln(E_k)$ is a measurable set for every k. Then, we have

$$\bigcap_{k=1}^{\infty} ln\,(E_k) = ln(\bigcap_{k=1}^{\infty} E_k) = \ln\,(E)$$

is a measurable set. Thus, we get E is a multiplicative measurable set.

Theorem 12. Let (E_k) be a family of multiplicative measurable sets with $E_1 \subset E_2 \subset \cdots$ in $\mathbb{R}_{exp}$. If $E = \bigcup_{k=1}^{\infty} E_k$ is a bounded set in $\mathbb{R}_{exp}$, then we have

$$\lambda_{exp}(E) = {}^{exp}lim_{n\to\infty} \lambda_{exp}(E_n).$$

Proof. Since E_k is a multiplicative measurable set for all k, we have $ln(E_k)$ is a measurable set for every k. Also, it is easy to see that $\ln(E_1) \subset ln(E_2) \subset \cdots$

and

$$\bigcup_{k=1}^{n} \ln(E_k) = ln(\bigcup_{k=1}^{n} E_k) = \ln(E).$$

Because of boundedness of E in $\mathbb{R}_{exp}$, we have $\ln(E)$ is bounded set in $\mathbb{R}$. Thus, we have

$$\lambda(ln\,(E)) = lim_{n\to\infty}\big(\lambda(ln\,(E_n))\big)$$
$$= lim_{n\to\infty}\Big(ln\Big(exp\big(\lambda(ln\,(E_n))\big)\Big)\Big)$$
$$= lim_{n\to\infty}\Big(ln\Big(\lambda_{exp}(E_n)\Big)\Big).$$

Therefore, we get

$$exp\big(\lambda(ln\,(E))\big) = exp\Big(lim_{n\to\infty}\Big(ln\Big(\lambda_{exp}(E_n)\Big)\Big)\Big)$$

which gives

$$\lambda_{exp}(E) = {}^{exp}lim_{n\to\infty}\lambda_{exp}(E_n).$$

Theorem 13. Let (E_k) be a family of multiplicative measurable sets with $E_1 \supset E_2 \supset \cdots$ in $\mathbb{R}_{exp}$ and let $E = \bigcap_{k=1}^{\infty} E_k$. Then, we have

$$\lambda_{exp}(E) = {}^{exp}lim_{n\to\infty}\lambda_{exp}(E_n).$$

Proof. Since E_k is a multiplicative measurable set for all k, we have $ln(E_k)$ is a measurable set for every k. Also, we have

$$\bigcap_{k=1}^{\infty} ln\,(E_k) = ln(\bigcap_{k=1}^{\infty} E_k) = \ln(E).$$

Thus, we have

$$\lambda(ln\,(E)) = lim_{n\to\infty}\big(\lambda(ln\,(E_n))\big)$$
$$= lim_{n\to\infty}\Big(ln\Big(exp\big(\lambda(ln\,(E_n))\big)\Big)\Big)$$

$$= lim_{n\to\infty}\left(ln\left(\lambda_{exp}(E_n)\right)\right).$$

Therefore, we get

$$exp\big(\lambda(ln\ (E))\big) = exp\left(lim_{n\to\infty}\left(ln\left(\lambda_{exp}(E_n)\right)\right)\right)$$

which gives

$$\lambda_{exp}(E) = {}^{exp}lim_{n\to\infty}\lambda_{exp}(E_n).$$

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