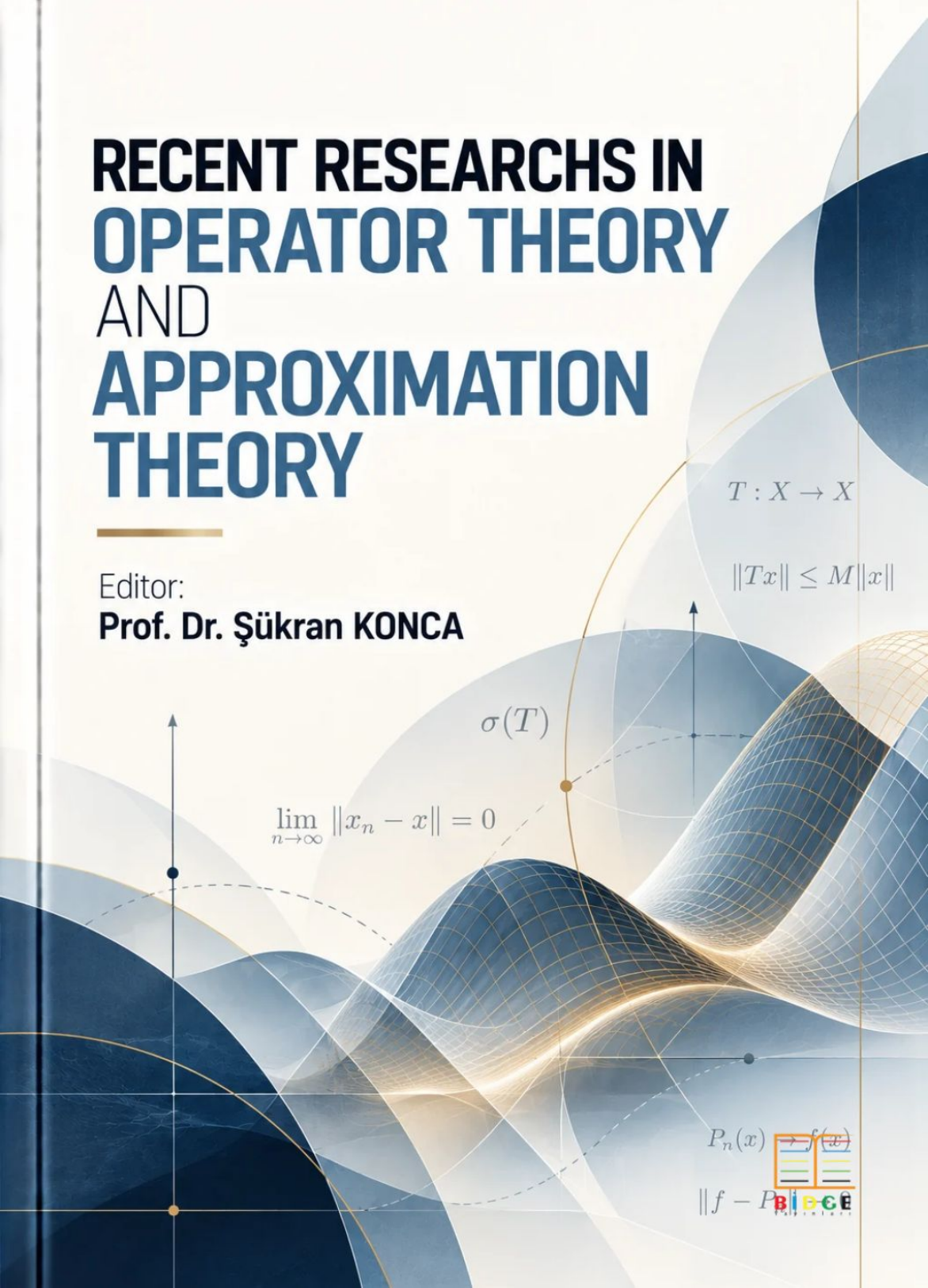
RECENT RESEARCHS IN OPERATOR THEORY AND APPROXIMATION THEORY

Editor:
Prof. Dr. Şükran KONCA



BİDGE Yayınları

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PREFACE

Recent Researchs in Operator Theory and Approximation Theory brings together studies that reflect current developments in operator theory and approximation theory, two closely related areas of functional anaysis in mathematics. The chapters included in this book focus on the investigation of operator classes on ordered structures and the approximation properties of Bernstein-type operators constructed via the k-Gamma function. In addition, the structural properties of new classes of operators defined on Riesz spaces and f-algebras are studied. By presenting recent theoretical results from these fields, this book aims to contribute to the ongoing development of mathematical research and to serve as a useful reference for researchers, graduate students, and readers interested in functional analysis, operator theory, and approximation theory. I would like to express my sincere gratitude to all authors for their valuable contributions.

Prof. Dr. Şükran KONCA

Editor

İÇİNDEKİLER

THE CLASS OF DEMI MULTIPLICATIVE ORDER-TO-ORDER CONTINUOUS OPERATORS 1

BAHRİ TURAN, BİROL ALTIN

THE CLASS OF DEMI-UNBOUNDED ORDER –ORDER CONTINUOUS OPERATORS 13

BİROL ALTIN, BAHRİ TURAN

ON A DURRMEYER-TYPE GENUINE BERNSTEIN OPERATOR SEQUENCE CONSTRUCTED USING THE k-GAMMA FUNCTION 25

ÜLKÜ DİNLEMEZ KANTAR, İSMET YÜKSEL

THE VORONOVSKAJA THEOREM FOR A DURRMEYER-TYPE GENUINE BERNSTEIN OPERATOR SEQUENCE CONSTRUCTED USING THE k-GAMA FUNCTION 38

ÜLKÜ DİNLEMEZ KANTAR

BÖLÜM 1

THE CLASS OF DEMI MULTIPLICATIVE ORDER-TO-ORDER CONTINUOUS OPERATORS

Bahri TURAN[1]
Birol ALTIN[2]

Introduction

The demi terminology goes back to Petryshyn's demicompact mappings (Petryshyn, 1966). It has since produced a number of operator classes obtained from a given class of operators by replacing the implication "$x_\alpha \to 0$ implies $\Psi x_\alpha \to 0$" with the weaker requirement that $x_\alpha \to 0$ together with $x_\alpha - \Psi x_\alpha \to 0$ already forces $x_\alpha \to 0$ in the stronger sense. The construction is available whenever the underlying space carries a pair of convergences, one of them weaker than the other, and it has been carried out for a variety of such pairs: for weak convergence and weak compactness (Krichen & O'Regan, 2019), for the Dunford–Pettis property (Benkhaled et al., 2023), and, in the order theoretic setting, for order weak demicompactness (Benkhaled et al., 2020), for Levi and quasi Levi operators on Banach lattices (Bal et al., 2026), for quasi KB-operators on locally solid vector lattices

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(Erkurşun-Özcan et al., 2026), for ab-continuity (Erkurşun-Özcan & Eryüksel, 2025), and for order-to-norm continuity on normed Riesz spaces (Keleş & Altın, 2023).

In an Archimedean f-algebra such a pair is available in purely order theoretic terms: multiplicative order convergence, introduced by Aydın (2020) and studied in detail by Gürkök and Turan (2025), is implied by order convergence but not conversely, the failure being governed by the absence of a multiplicative unit. This suggests taking mo as the input convergence and o as the output convergence. The resulting class, and above all its demi version, is the subject of this section.

Two features of the resulting theory deserve to be pointed out. First, c_0 is the standard example of an f-algebra on which the identity operator fails to be demi, while c and ℓ^∞ behave well. Second, the demi class does not collapse onto $\mathcal{L}_{mo,o}(A)$ merely because mo- and o-convergence happen to coincide in A: an order continuity assumption has to be added, and we give a counterexample showing that it cannot be dispensed with, see Theorem 2 and Example 2.

Preliminaries

All Riesz spaces in this section are Archimedean and all operators are linear. We refer to (Aliprantis & Burkinshaw, 2006) and (Zaanen, 1983) for unexplained terminology.

A net (x_α) in a Riesz space E is *order convergent* to $x \in E$, written $x_\alpha \xrightarrow{o} x$, if there is a net $(y_\beta)_{\beta\in B}$ in E with $y_\beta \downarrow 0$ such that for every β there exists α_0 with $|x - x_\alpha| \le y_\beta$ for all $\alpha \ge \alpha_0$. It is well known that order convergence is *linear*, that is, $x_\alpha \xrightarrow{o} 0$ and $y_\alpha \xrightarrow{o} 0$ imply $\lambda x_\alpha + \mu y_\alpha \xrightarrow{o} 0$ for all $\lambda, \mu \in \mathbb{R}$, and *solid*, that is, $|x_\alpha| \le |y_\alpha|$ for all α together with $y_\alpha \xrightarrow{o} 0$ implies $x_\alpha \xrightarrow{o} 0$; in

particular $x_\alpha \xrightarrow{o} 0$ if and only if $|x_\alpha| \xrightarrow{o} 0$. Both properties will be used throughout without further mention.

A Riesz space A equipped with an associative multiplication making it an algebra and in which $a \cdot b \in A^+$ whenever $a, b \in A^+$ is a *Riesz algebra*; it is an f*-algebra* if in addition $a \wedge b = 0$ implies $(a \cdot c) \wedge b = (c \cdot a) \wedge b = 0$ for all $c \in A^+$. Every Archimedean f-algebra is commutative (de Pagter, 1981). An algebra A is *semiprime* if $a^2 = 0$ implies $a = 0$.

Remark 1. (see de Pagter, 1981) Let A be an Archimedean f-algebra. Then $|a \cdot b| = |a| \cdot |b|$ for all $a, b \in A$, and multiplication by a fixed element is order continuous, that is, $b_\alpha \xrightarrow{o} b$ implies $a \cdot b_\alpha \xrightarrow{o} a \cdot b$ for every $a \in A$.

Definition 1. (Aydın, 2020) Let A be a commutative Riesz algebra. A net (x_α) in A is *multiplicative order convergent* to $x \in A$, written $x_\alpha \xrightarrow{mo} x$, if $|x_\alpha - x| \cdot a \xrightarrow{o} 0$ for every $a \in A^+$.

Remark 2. (see Aydın, 2020) Let A be an Archimedean f-algebra. Then:

(i) $x_\alpha \xrightarrow{o} 0$ implies $x_\alpha \xrightarrow{mo} 0$; the converse fails in general;

(ii) $x_\alpha \xrightarrow{mo} 0$ if and only if $|x_\alpha| \xrightarrow{mo} 0$;

(iii) if A has a multiplicative unit e, then mo-convergence and o-convergence coincide, since $|x_\alpha| \cdot e = |x_\alpha|$;

(iv) in c_0, with coordinatewise order and multiplication, the standard basis (e_n) satisfies $e_n \xrightarrow{mo} 0$ while (e_n) is not order convergent to zero (Gürkök & Turan, 2025, Example 2.2).

We write $\mathcal{L}_b(A,B)$, $\mathcal{L}_n(A,B)$ and $\mathcal{L}_{mo}(A,B)$ for the spaces of order bounded, order continuous and mo-continuous operators, respectively, an operator being *mo-continuous* when $x_\alpha \xrightarrow{mo} 0$ implies $\Psi x_\alpha \xrightarrow{mo} 0$, as in Aydın et al. (2025) and Gürkök and Turan (2025).

Definition 2. Let A be an f-algebra and F a Riesz space. An operator $\Psi: A \to F$ is called *multiplicative order-to-order continuous* (briefly, *mo-o continuous*) if

$$x_\alpha \overset{mo}{\to} 0 \quad \Rightarrow \quad \Psi x_\alpha \overset{o}{\to} 0$$

for every net (x_α) in A. The class of all such operators is denoted by $\mathcal{L}_{mo,o}(A,F)$, and by $\mathcal{L}_{mo,o}(A)$ when $F = A$.

Only the following elementary facts about this class will be needed.

Proposition 1. Let A be an f-algebra and F a Riesz space.

(i) $\mathcal{L}_{\mathrm{mo,o}}(\mathrm{A},\mathrm{F})$ is a linear subspace of $\mathcal{L}_{\mathrm{n}}(\mathrm{A},\mathrm{F})$; in particular every mo-o continuous operator is order bounded.

(ii) If A has a multiplicative unit, then $\mathcal{L}_{\mathrm{mo,o}}(\mathrm{A},\mathrm{F}) = \mathcal{L}_{\mathrm{n}}(\mathrm{A},\mathrm{F})$.

(iii) For $\mathrm{u} \in \mathrm{A}$ the multiplication operator $\pi_{\mathrm{u}}: \mathrm{A} \to \mathrm{A}$, $\pi_{\mathrm{u}}\mathrm{x} = \mathrm{u} \cdot \mathrm{x}$, belongs to $\mathcal{L}_{\mathrm{mo,o}}(\mathrm{A})$.

Proof. (i) That $\mathcal{L}_{mo,o}(A,F)$ is a subspace follows from the linearity of order convergence. If $x_\alpha \overset{o}{\to} 0$ then $x_\alpha \overset{mo}{\to} 0$ by Remark 2(i), hence $\Psi x_\alpha \overset{o}{\to} 0$; thus $\Psi \in \mathcal{L}_n(A,F)$, and order continuous operators are order bounded by Abramovich and Sirotkin (2005, Theorem 2.1).

(ii) Immediate from Remark 2(iii).

(iii) If $x_\alpha \overset{mo}{\to} 0$ then $|x_\alpha| \cdot |u| \overset{o}{\to} 0$, and $|\pi_u x_\alpha| = |x_\alpha| \cdot |u|$ by Remark 1. □

Demi multiplicative order-to-order continuous operators

Definition 3. Let A be an f-algebra. An operator $\Psi: A \to A$ is called *demi multiplicative order-to-order continuous* (briefly, a *d-moo operator*) if for every net (x_α) in A^+,

$$x_\alpha \overset{mo}{\to} 0 \quad \text{and} \quad x_\alpha - \Psi x_\alpha \overset{o}{\to} 0 \quad \Rightarrow \quad x_\alpha \overset{o}{\to} 0.$$

The class of all d-moo operators on A is denoted by $\mathfrak{D}\mathcal{L}_{mo,o}(A)$.

Theorem 1. Let A be an f-algebra. Then $\mathcal{L}_{\mathrm{mo,o}}(\mathrm{A}) \subseteq \mathfrak{D}\mathcal{L}_{\mathrm{mo,o}}(\mathrm{A})$.

Proof. Let $\Psi \in \mathcal{L}_{\mathrm{mo,o}}(\mathrm{A})$ and let (x_α) be a net in A^+ with $\mathrm{x}_\alpha \overset{\mathrm{mo}}{\to} 0$ and $\mathrm{x}_\alpha - \Psi\mathrm{x}_\alpha \overset{\mathrm{o}}{\to} 0$. Since Ψ is mo-o continuous we have $\Psi\mathrm{x}_\alpha \overset{\mathrm{o}}{\to} 0$, and therefore

$$x_\alpha = (x_\alpha - \Psi x_\alpha) + \Psi x_\alpha \overset{o}{\to} 0$$

by the linearity of order convergence. Hence $\Psi \in \mathfrak{D}\mathcal{L}_{mo,o}(A)$. □

The inclusion is proper in general; see Corollary 1(i) below.

Proposition 2. Let A and B be f-algebras.

(i) For every scalar $\beta \neq 1$, the operator $\beta\mathrm{I}$ belongs to $\mathfrak{D}\mathcal{L}_{\mathrm{mo,o}}(\mathrm{A})$.

(ii) If $\Psi\colon \mathrm{A} \to \mathrm{A}$ is such that $(\mathrm{I} - \Psi)^{-1}$ exists and is order continuous, then $\Psi \in \mathfrak{D}\mathcal{L}_{\mathrm{mo,o}}(\mathrm{A})$.

(iii) Let $\widehat{\mathrm{A}} = \mathrm{A} \oplus \mathrm{B}$ carry the coordinatewise order and multiplication, let $\mathrm{H} \in \mathcal{L}_{\mathrm{n}}(\mathrm{A}, \mathrm{B})$ and let $\beta \neq 1$. Then the operator

$$\widehat{\mathrm{H}} = \begin{bmatrix} 0 & 0 \\ \mathrm{H} & \beta\mathrm{I} \end{bmatrix}, \qquad \widehat{\mathrm{H}}(\mathrm{x}, \mathrm{y}) = (0,\ \mathrm{Hx} + \beta\mathrm{y}),$$

belongs to $\mathfrak{D}\mathcal{L}_{\mathrm{mo,o}}(\widehat{\mathrm{A}})$.

Proof. (i) If $\mathrm{x}_\alpha - \beta\mathrm{x}_\alpha = (1-\beta)\mathrm{x}_\alpha \overset{\mathrm{o}}{\to} 0$ then $\mathrm{x}_\alpha \overset{\mathrm{o}}{\to} 0$, since $1 - \beta \neq 0$.

(ii) If $(\mathrm{I} - \Psi)\mathrm{x}_\alpha \overset{\mathrm{o}}{\to} 0$ then $\mathrm{x}_\alpha = (\mathrm{I} - \Psi)^{-1}(\mathrm{I} - \Psi)\mathrm{x}_\alpha \overset{\mathrm{o}}{\to} 0$ by the order continuity of $(\mathrm{I} - \Psi)^{-1}$.

(iii) Let $\hat{\mathrm{x}}_\alpha = (\mathrm{x}_\alpha, \mathrm{y}_\alpha)$ be a net in $\widehat{\mathrm{A}}^+$ with $\hat{\mathrm{x}}_\alpha \overset{\mathrm{mo}}{\to} 0$ and $\hat{\mathrm{x}}_\alpha - \widehat{\mathrm{H}}\hat{\mathrm{x}}_\alpha \overset{\mathrm{o}}{\to} 0$. Since order convergence in $\widehat{\mathrm{A}}$ is coordinatewise and

$$\hat{x}_\alpha - \widehat{H}\hat{x}_\alpha = (x_\alpha,\ (1-\beta)y_\alpha - Hx_\alpha),$$

we obtain $x_\alpha \overset{o}{\to} 0$ in A and $(1-\beta)y_\alpha - Hx_\alpha \overset{o}{\to} 0$ in B. As H is order continuous, $Hx_\alpha \overset{o}{\to} 0$, whence $(1-\beta)y_\alpha \overset{o}{\to} 0$ and so $y_\alpha \overset{o}{\to} 0$. Therefore $\hat{x}_\alpha \overset{o}{\to} 0$. □

Example 1. Part (iii) of Proposition 2 fails for $\beta = 1$. Take $B = c_0$, let A be arbitrary, let $H = 0$ and $\beta = 1$, so that $\widehat{H}(x,y) = (0,y)$. Put $\hat{x}_n = (0, e_n) \in \hat{A}^+$. Then $\hat{x}_n \overset{mo}{\to} 0$ by Remark 2(iv) and $\hat{x}_n - \widehat{H}\hat{x}_n = 0 \overset{o}{\to} 0$, while $(\hat{x}_n)$ is not order convergent to zero. Hence $\widehat{H} \notin \mathfrak{D}\mathcal{L}_{mo,o}(\hat{A})$. The same example with $A = \{0\}$, for which $\hat{A} = c_0$ and $\widehat{H} = I$, shows that the identity operator need not be a d-moo operator.

Proposition 3. Let A be an f-algebra, $\Psi \in \mathcal{L}_{mo,o}(A)$ and $S \in \mathfrak{D}\mathcal{L}_{mo,o}(A)$. Then $S + \Psi$ and $S - \Psi$ belong to $\mathfrak{D}\mathcal{L}_{mo,o}(A)$.

Proof. Let (x_α) be a net in A^+ with $x_\alpha \overset{mo}{\to} 0$ and $x_\alpha - (S+\Psi)x_\alpha \overset{o}{\to} 0$. Since $\Psi \in \mathcal{L}_{mo,o}(A)$ we have $\Psi x_\alpha \overset{o}{\to} 0$, and

$$x_\alpha - Sx_\alpha = (x_\alpha - (S+\Psi)x_\alpha) + \Psi x_\alpha \overset{o}{\to} 0.$$

As $S \in \mathfrak{D}\mathcal{L}_{mo,o}(A)$, we conclude $x_\alpha \overset{o}{\to} 0$. The argument for $S - \Psi$ is identical. □

The sum of two d-moo operators need not be a d-moo operator, and $\mathfrak{D}\mathcal{L}_{mo,o}(A)$ need not be closed under scalar multiplication; see Corollary 1(ii).

Theorem 2. Let A be an f-algebra. The following statements are equivalent:

(i) mo-convergence and o-convergence coincide in A, that is, $x_\alpha \overset{mo}{\to} 0$ implies $x_\alpha \overset{o}{\to} 0$;

(ii) every operator $\Psi: A \to A$ is a d-moo operator;

(iii) $I \in \mathfrak{D}\mathcal{L}_{mo,o}(A)$;

(iv) $\mathcal{L}_{mo,o}(A) = \mathfrak{D}\mathcal{L}_{mo,o}(A) \cap \mathcal{L}_n(A)$.

In particular, all of these hold whenever A has a multiplicative unit.

Proof. (i)⇒(ii) and (ii)⇒(iii) are trivial, the first because the conclusion $x_\alpha \overset{o}{\to} 0$ is then available from the hypothesis $x_\alpha \overset{mo}{\to} 0$ alone.

(iii)⇒(i). Let $x_\alpha \overset{mo}{\to} 0$ in A. Then $|x_\alpha| \overset{mo}{\to} 0$ by Remark 2(ii), $(|x_\alpha|)$ is a net in A^+, and $|x_\alpha| - I|x_\alpha| = 0 \overset{o}{\to} 0$. Since $I \in \mathfrak{D}\mathcal{L}_{mo,o}(A)$ we get $|x_\alpha| \overset{o}{\to} 0$, hence $x_\alpha \overset{o}{\to} 0$.

(i)⇒(iv). By (ii) we have $\mathfrak{D}\mathcal{L}_{mo,o}(A) \cap \mathcal{L}_n(A) = \mathcal{L}_n(A)$, and by (i) together with Remark 2(i) an operator is mo-o continuous exactly when it is order continuous, that is, $\mathcal{L}_{mo,o}(A) = \mathcal{L}_n(A)$.

(iv)⇒(iii). The operator $\frac{1}{2}I$ is order continuous, and it belongs to $\mathfrak{D}\mathcal{L}_{mo,o}(A)$ by Proposition 2(i). Hence $\frac{1}{2}I \in \mathcal{L}_{mo,o}(A)$ by (iv), and since $\mathcal{L}_{mo,o}(A)$ is a linear subspace, $I = 2 \cdot \frac{1}{2}I \in \mathcal{L}_{mo,o}(A)$. Now Theorem 1 gives $I \in \mathfrak{D}\mathcal{L}_{mo,o}(A)$.

The last assertion follows from Remark 2(iii). □

The intersection with $\mathcal{L}_n(A)$ in Theorem 2(iv) cannot be omitted. Indeed, condition (i) is a statement about the convergences of A alone and says nothing about individual operators; an f-algebra in which mo- and o-convergence coincide may still carry order bounded operators that are not order continuous, as the following example shows.

Example 2. Let $A = c$ be the f-algebra of all convergent real sequences, with coordinatewise order and multiplication. Since c has the unit $e = (1,1,\dots)$, condition (i) of Theorem 2 holds, so $\mathfrak{D}\mathcal{L}_{mo,o}(c)$ consists of all operators on c. Define

$$\Psi: c \to c, \qquad \Psi x = \left(\lim_n x_n\right) e.$$

Then Ψ is positive and order bounded, and $\Psi \in \mathfrak{D}\mathcal{L}_{mo,o}(c)$. On the other hand, put $x_n = e - \sum_{i \le n} e_i$. Then $x_n \downarrow 0$ in c, while $\Psi x_n = e$ for every n. Hence Ψ is not order continuous and therefore,

by Proposition 1(i), $\Psi \notin \mathcal{L}_{mo,o}(c)$. Consequently $\mathcal{L}_{mo,o}(c) \subsetneq \mathfrak{D}\mathcal{L}_{mo,o}(c)$ although mo- and o-convergence coincide in c.

Proposition 4. Let A be an f-algebra and let $S, H, N: A \to A$ be operators.

(i) If $S \le H \le I$ and $H \in \mathfrak{D}\mathcal{L}_{mo,o}(A)$, then $S \in \mathfrak{D}\mathcal{L}_{mo,o}(A)$.

(ii) If $I \le S \le H$ and $S \in \mathfrak{D}\mathcal{L}_{mo,o}(A)$, then $H \in \mathfrak{D}\mathcal{L}_{mo,o}(A)$.

(iii) If $N \le S \le H \le I + N$ with $N \in \mathcal{L}_{mo,o}(A)$ and $H \in \mathfrak{D}\mathcal{L}_{mo,o}(A)$, then $S \in \mathfrak{D}\mathcal{L}_{mo,o}(A)$.

Proof. (i) Let (x_α) be a net in A^+ with $x_\alpha \overset{mo}{\to} 0$ and $x_\alpha - Sx_\alpha \overset{o}{\to} 0$. From $S \le H \le I$ we get $0 \le (I-H)x_\alpha \le (I-S)x_\alpha$ for every α, so $x_\alpha - Hx_\alpha \overset{o}{\to} 0$. Since $H \in \mathfrak{D}\mathcal{L}_{mo,o}(A)$, $x_\alpha \overset{o}{\to} 0$.

(ii) Let (x_α) be a net in A^+ with $x_\alpha \overset{mo}{\to} 0$ and $x_\alpha - Hx_\alpha \overset{o}{\to} 0$. From $I \le S \le H$ we get $0 \le (S-I)x_\alpha \le (H-I)x_\alpha$, hence $x_\alpha - Sx_\alpha \overset{o}{\to} 0$ and therefore $x_\alpha \overset{o}{\to} 0$, because $S \in \mathfrak{D}\mathcal{L}_{mo,o}(A)$.

(iii) The hypothesis gives $0 \le S - N \le H - N \le I$. By Proposition 3, $H - N \in \mathfrak{D}\mathcal{L}_{mo,o}(A)$, hence $S - N \in \mathfrak{D}\mathcal{L}_{mo,o}(A)$ by (i). Applying Proposition 3 once more to $S = (S-N) + N$ yields $S \in \mathfrak{D}\mathcal{L}_{mo,o}(A)$. □

Remark 3. Without the upper bound I the domination property fails: in $A = c_0$ we have $0 \le I \le 2I$ and $2I \in \mathfrak{D}\mathcal{L}_{mo,o}(c_0)$, while $I \notin \mathfrak{D}\mathcal{L}_{mo,o}(c_0)$ by Theorem 2 and Remark 2(iv).

Multiplication operators on c_0

Throughout this section $A = c_0$ carries the coordinatewise order and multiplication; it is a semiprime f-algebra without unit. For $u = (u_n) \in \ell^\infty$ the multiplication operator $\pi_u: c_0 \to c_0$ is defined by $\pi_u x = u \cdot x$; it is order bounded, and $\pi_e = I$, where $e = (1,1,\dots)$ denotes the constant sequence of ones. The following

theorem determines both classes completely for such operators and yields all the counterexamples announced above.

Theorem 3. Let $u = (u_n) \in \ell^\infty$. Then:

(i) $\pi_u \in \mathcal{L}_{mo,o}(c_0)$ if and only if $u \in c_0$;

(ii) $\pi_u \in \mathfrak{D}\mathcal{L}_{mo,o}(c_0)$ if and only if 1 is not the limit of any subsequence of the sequence (u_n).

Proof. Throughout we use that (e_n) is mo-null but not order null in c_0 (Remark 2(iv)).

(i) If $u \in c_0$, then $\pi_u \in \mathcal{L}_{mo,o}(c_0)$ by Proposition 1(iii). Conversely, suppose $u \notin c_0$ and choose $\varepsilon > 0$ and a subsequence (n_j) with $|u_{n_j}| \geq \varepsilon$ for all j. The sequence (e_{n_j}) is mo-null, and $|\pi_u e_{n_j}| = |u_{n_j}|\, e_{n_j} \geq \varepsilon e_{n_j} \geq 0$. If $\pi_u e_{n_j} \xrightarrow{o} 0$, then $\varepsilon e_{n_j} \xrightarrow{o} 0$ by the solidity of order convergence, a contradiction. Hence $\pi_u \notin \mathcal{L}_{mo,o}(c_0)$.

(ii) Suppose first that 1 is not the limit of any subsequence of (u_n), and choose $\delta > 0$ and $N \in \mathbb{N}$ with $|1 - u_n| \geq \delta$ for all $n > N$. Put $v = \sum_{i \leq N} e_i \in c_0$ and let $P = \pi_v$ be the corresponding band projection onto the first N coordinates. Let $(x_\alpha) \subseteq c_0^+$ satisfy $x_\alpha \xrightarrow{mo} 0$ and $y_\alpha := x_\alpha - \pi_u x_\alpha \xrightarrow{o} 0$. Since $v \in c_0$, Proposition 1(iii) gives $Px_\alpha \xrightarrow{o} 0$. For the remaining part, comparing coordinates $n > N$ gives

$$0 \leq (I - P)x_\alpha \leq \delta^{-1}\,|(I - P)y_\alpha| \leq \delta^{-1}|y_\alpha|,$$

because $(y_\alpha)_n = (1 - u_n)(x_\alpha)_n$. Hence $(I - P)x_\alpha \xrightarrow{o} 0$ by the solidity of order convergence, and therefore $x_\alpha = Px_\alpha + (I - P)x_\alpha \xrightarrow{o} 0$. Thus $\pi_u \in \mathfrak{D}\mathcal{L}_{mo,o}(c_0)$.

Conversely, suppose $u_{n_j} \to 1$ for some subsequence (n_j), and put $\varepsilon_j = |1 - u_{n_j}|$, so that $\varepsilon_j \to 0$. Consider $x_j = e_{n_j} \in c_0^+$; then $x_j \xrightarrow{mo} 0$. Moreover

$$|x_j - \pi_u x_j| = \varepsilon_j e_{n_j} \le w_j := \sum_{k \ge j} \varepsilon_k \, e_{n_k}.$$

Each w_j lies in c_0 because $\varepsilon_k \to 0$, and $w_j \downarrow$. If $0 \le z \le w_j$ for all j, then for each fixed coordinate i we have $(w_j)_i = 0$ for large j, whence $z = 0$; thus $w_j \downarrow 0$ and $x_j - \pi_u x_j \overset{o}{\to} 0$. Since $\left(e_{n_j}\right)$ is not order null, $\pi_u \notin \mathfrak{D}\mathcal{L}_{mo,o}(c_0)$. □

Corollary 1. In $A = c_0$ the following hold.

(i) $2I \in \mathfrak{D}\mathcal{L}_{mo,o}(c_0) \backslash \mathcal{L}_{mo,o}(c_0)$; hence the inclusion of Theorem 1 is proper.

(ii) $\frac{1}{2}I \in \mathfrak{D}\mathcal{L}_{mo,o}(c_0)$ while $\frac{1}{2}I + \frac{1}{2}I = I \notin \mathfrak{D}\mathcal{L}_{mo,o}(c_0)$ and $2 \cdot \frac{1}{2}I = I \notin \mathfrak{D}\mathcal{L}_{mo,o}(c_0)$; thus $\mathfrak{D}\mathcal{L}_{mo,o}(c_0)$ is neither closed under addition nor under multiplication by scalars.

(iii) $-I \in \mathfrak{D}\mathcal{L}_{mo,o}(c_0)$ while $|-I| = I \notin \mathfrak{D}\mathcal{L}_{mo,o}(c_0)$; thus $\mathfrak{D}\mathcal{L}_{mo,o}(c_0)$ is not a sublattice of $\mathcal{L}_b(c_0)$, even when restricted to order bounded operators.

(iv) Putting $v_n = \sum_{i \le n} e_i$ and $T_n = \pi_{v_n}$ we have $T_n \in \mathcal{L}_{mo,o}(c_0)$ for every n and $0 \le T_n \uparrow I$ in $\mathcal{L}_b(c_0)$, while $I \notin \mathfrak{D}\mathcal{L}_{mo,o}(c_0)$; thus $\mathfrak{D}\mathcal{L}_{mo,o}(c_0)$ is not order closed in $\mathcal{L}_b(c_0)$.

Proof. All four statements follow from Theorem 3, applied with the constant sequences $u = 2 \cdot e$, $u = \frac{1}{2}e$, $u = -e$, $u = e$, together with Theorem 1 for (i) and Proposition 1(iii) for (iv). □

References

Abramovich, Y., & Sirotkin, G. (2005). On order convergence of nets. *Positivity*, *9*(3), 287–292. https://doi.org/10.1007/s11117-004-7543-x

Aliprantis, C. D., & Burkinshaw, O. (2006). *Positive operators*. Springer. https://doi.org/10.1007/978-1-4020-5008-4

Aydın, A. (2020). Multiplicative order convergence in *f*-algebras. *Hacettepe Journal of Mathematics and Statistics*, *49*(3), 998–1005. https://doi.org/10.15672/hujms.512073

Aydın, A., Emelyanov, E., & Gorokhova, S. G. (2025). Multiplicative order continuous operators on Riesz algebras. *Matematički Vesnik*, *77*(2), 110–119. https://doi.org/10.57016/MV-R79EJZ17

Bal, Ö., Erkurşun-Özcan, N., & Yurtoğlu, A. S. (2026). Demi version of quasi Levi and Levi operators on Banach lattices. *Miskolc Mathematical Notes*, *27*(1), 77–87. https://doi.org/10.18514/MMN.2026.5123

Benkhaled, H., Elleuch, A., & Jeribi, A. (2020). The class of order weakly demicompact operators. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, *114*(2), 1–8. https://doi.org/10.1007/s13398-020-00808-4

Benkhaled, H., Hajji, M., & Jeribi, A. (2023). On the class of demi Dunford–Pettis operators. *Rendiconti del Circolo Matematico di Palermo Series 2*, *72*(2), 901–911. https://doi.org/10.1007/s12215-021-00702-x

de Pagter, B. (1981). *f-Algebras and orthomorphisms* [Doctoral dissertation, Leiden University].

Erkurşun-Özcan, N., & Eryüksel, E. H. (2025). Investigation of demi-*ab* continuous operators. *Journal of Mathematical Sciences*, *289*, 59–66. https://doi.org/10.1007/s10958-024-07091-3

Erkurşun-Özcan, N., Krichen, B., & Yurtoğlu, A. S. (2026). Quasi-*KB* operators and their demi versions on locally solid vector lattices. *Positivity, 30,* Article 21. https://doi.org/10.1007/s11117-026-01177-9

Gürkök, H., & Turan, B. (2025). On multiplicative order convergence and multiplicative order continuous operators. *Turkish Journal of Mathematics, 49*(5), 562–572. https://doi.org/10.55730/1300-0098.3608

Keleş, G. S., & Altın, B. (2023). The class of demi-order norm continuous operators. *Gazi University Journal of Science, 36*(4), 1693–1698. https://doi.org/10.35378/gujs.1098503

Krichen, B., & O'Regan, D. (2019). Weakly demicompact linear operators and axiomatic measures of weak noncompactness. *Mathematica Slovaca, 69*(6), 1403–1412. https://doi.org/10.1515/ms-2017-0317

Petryshyn, W. V. (1966). Construction of fixed points of demicompact mappings in Hilbert space. *Journal of Mathematical Analysis and Applications, 14*(2), 276–284. https://doi.org/10.1016/0022-247X(66)90027-8

Zaanen, A. C. (1983). *Riesz spaces II*. North-Holland.

BÖLÜM 2

THE CLASS DEMI UNBOUNDED ORDER-TO-ORDER CONTINUOUS OPERATORS

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Introduction

The prefix "demi" goes back to Petryshyn (1966), who introduced demicompact mappings in Hilbert space. The idea behind it is simple: rather than asking an operator T to improve the convergence of a net (x_α), one asks that, whenever x_α converges in a weak sense and $x_\alpha - Tx_\alpha$ converges in a strong sense, then x_α itself converges in the strong sense. This scheme has been applied to a number of operator classes on Banach spaces and Banach lattices, among them weakly demicompact operators (Krichen & O'Regan, 2019), demi Dunford–Pettis operators (Benkhaled et al., 2023), and order weakly demicompact operators (Benkhaled et al., 2020). Most recently, Keleş and Altın (2023) defined, on a normed Riesz space M, the demi-order norm continuous operators: a bounded operator T on M belongs to this class if, for every net (x_α) in M^+,

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$$x_\alpha \xrightarrow{o} 0 \text{ and } \| x_\alpha - Tx_\alpha \| \to 0 \quad \Rightarrow \quad \| x_\alpha \| \to 0.$$

Here the "weak" convergence is order convergence and the "strong" one is norm convergence, and the underlying operator class is that of order-to-norm continuous operators of Jalili et al. (2021). A central result of Keleş and Altın (2023) is that the identity is demi-order norm continuous if and only if the norm of M is order continuous.

The purpose of this chapter is to carry out the same construction entirely inside the order structure of a Riesz space, with no norm present. Two order convergences of different strength are available on any Riesz space: order convergence and its unbounded version, uo-convergence, introduced by Nakano (1948) and named by De Marr (1964); see Gao et al. (2017) for a systematic account. Order convergence always implies uo-convergence, and the two coincide on order bounded nets. This last fact dictates the shape of the definition: if one restricts attention to order bounded uo-null nets, the class of operators carrying them to order null nets is just the class $L_n(E)$ of order continuous operators, and the corresponding demi condition is void, because the hypothesis already contains the conclusion. The construction becomes meaningful only when arbitrary uo-null nets are admitted. We therefore let uo-convergence play the role of the weak convergence and order convergence the role of the strong one.

In Bahramnezhad and Haghnejad Azar (2017), an operator $T: E \to F$ between two Riesz spaces is called a strongly order continuous operator if $Tx_\alpha \xrightarrow{o} 0$ in F whenever $x_\alpha \xrightarrow{uo} 0$ in E. In this chapter, in order to make the types of convergence appearing in the definition more explicit, we shall refer to such operators as unbounded order-to-order continuous operators. We call an order bounded operator T *demi uo-to-order continuous* if, for every net (x_α) in E^+, the conditions $x_\alpha \xrightarrow{uo} 0$ and $x_\alpha - Tx_\alpha \xrightarrow{o} 0$ imply

$x_\alpha \xrightarrow{o} 0$. The results of Keleş and Altın (2023) then transfer almost verbatim, with one structural difference that we find instructive.

Preliminaries

Throughout, all Riesz spaces are Archimedean, and all operators are linear. For unexplained terminology we refer to (Aliprantis & Burkinshaw, 2006). For a Riesz space E we write $L_b(E)$ for the space of order bounded operators on E, $L_n(E)$ for the space of order continuous operators, $E^{\sim}$ for the order dual and $E_n^{\sim}$ for the order continuous dual.

A net (x_α) in E is *order convergent* to $x \in E$, written $x_\alpha \xrightarrow{o} x$, if there is a net (y_β) in E, possibly indexed by a different directed set, with $y_\beta \downarrow 0$ and such that for every β there is α_0 with $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_0$. This is the definition of Abramovich and Sirotkin (2005), which is also the one adopted by Gao et al. (2017), Keleş and Altın (2023) and Turan et al. (2022). We record the elementary properties we need.

(O1) If $x_\alpha \xrightarrow{o} 0$ and $z_\alpha \xrightarrow{o} 0$ then $x_\alpha + z_\alpha \xrightarrow{o} 0$ and $\lambda x_\alpha \xrightarrow{o} 0$ for every $\lambda \in \mathbb{R}$.

(O2) If $|z_\alpha| \leq |x_\alpha|$ for all α and $x_\alpha \xrightarrow{o} 0$, then $z_\alpha \xrightarrow{o} 0$.

(O3) Order convergence depends only on tails: an order convergent net is eventually order bounded, and (x_α) is order null if and only if some (equivalently, every) tail of it is order null.

(O4) If $\lambda_\alpha \to 0$ in $\mathbb{R}$ and $u \in E$, then $\lambda_\alpha u \xrightarrow{o} 0$; this uses only that E is Archimedean.

A net (x_α) is unbounded order convergent (uo-convergent) to x, written $x_\alpha \xrightarrow{uo} x$, if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for every $u \in E^+$ (Gao et al., 2017). We record the properties of uo-convergence used below.

(U1) $x_\alpha \xrightarrow{uo} 0$ if and only if $|x_\alpha| \xrightarrow{uo} 0$, and order convergence implies uo-convergence. If (x_α) is order bounded, say $|x_\alpha| \le u$, then $|x_\alpha| \wedge u = |x_\alpha|$, so for order bounded nets uo-convergence and order convergence coincide.

(U2) In c_0, the sequence (e_n) of unit vectors is uo-null but not order null. Indeed, for $u \in c_0^+$ put $y_m = (0, \dots, 0, u_m, u_{m+1}, \dots) \in c_0$; then $y_m \downarrow 0$ and $e_n \wedge u = u_n e_n \le y_m$ for all $n \ge m$, so $e_n \xrightarrow{uo} 0$. On the other hand, if (e_n) were order null, it would be eventually order bounded by (O3), say $e_n \le v \in c_0$ for $n \ge n_0$; this gives $v_n \ge 1$ for all $n \ge n_0$, which is impossible.

(U3) Uo-convergence and order convergence coincide on E (that is, every uo-null net in E is order null) if and only if E is finite dimensional (Emelyanov et al., 2019; Turan et al., 2022).

(U4) Let $E \oplus F$ carry the coordinate-wise order. Then $(x_\alpha, y_\alpha) \xrightarrow{o} 0$ in $E \oplus F$ if and only if $x_\alpha \xrightarrow{o} 0$ in E and $y_\alpha \xrightarrow{o} 0$ in F, and the same holds for uo-convergence. Indeed, $|(x, y)| = (|x|, |y|)$ and $|(x, y)| \wedge (u, v) = (|x| \wedge u, |y| \wedge v)$; if $a_\beta \downarrow 0$ in E and $b_\gamma \downarrow 0$ in F, then $(a_\beta, b_\gamma) \downarrow 0$ in $E \oplus F$ over the product directed set; conversely, each coordinate of a net decreasing to 0 in $E \oplus F$ decreases to 0.

Demi Uo-to-Order Continuous Operators

Definition 1. An operator $T: E \to E$ is called *strongly order continuous* if $x_\alpha \xrightarrow{uo} 0$ in E implies $Tx_\alpha \xrightarrow{o} 0$ (Bahramnezhad & Haghnejad Azar, 2017).

The class of all strongly order continuous operators on E is denoted by $L_{uo,o}(E)$.

To highlight the class of strongly order continuous operators, we will designate them as *unbounded order-to-order continuous* operators.

Proposition 1. Let E be a Riesz space.

(a) $L_{uo,o}(E)$ is a linear subspace of $L_n(E)$.

(b) $I \in L_{uo,o}(E)$ if and only if E is finite dimensional. Consequently, if E is infinite dimensional then $\lambda I \notin L_{uo,o}(E)$ for every $\lambda \neq 0$, and if E is finite dimensional then $L_{uo,o}(E) = L_n(E)$.

(c) For $n \in \mathbb{N}$ let $P_n: c_0 \to c_0$ be the projection $P_n x = (x_1, \dots, x_n, 0, 0, \dots)$. Then $P_n \in L_{uo,o}(c_0)$.

Proof. (a) $L_{uo,o}(E)$ is a linear subspace by (O1). Let $T \in L_{uo,o}(E)$. If $x_\alpha \overset{o}{\to} 0$ then $x_\alpha \overset{uo}{\to} 0$ by (U1), so $Tx_\alpha \overset{o}{\to} 0$. To see that T is order bounded, fix $u \in E^+$ and consider the net (x/n) indexed by the pairs $(n, x) \in \mathbb{N} \times [0, u]$, directed by the first coordinate. Since $0 \leq x/n \leq u/n$, this net is order null by (O2) and (O4), hence uo-null by (U1), so (Tx/n) is order null. By (O3) there are $n_0 \in \mathbb{N}$ and $v \in E^+$ with $|Tx|/n \leq v$ for all $n \geq n_0$ and $x \in [0, u]$; taking $n = n_0$ gives $T[0, u] \subseteq [-n_0 v, n_0 v]$. Hence $T \in L_n(E)$.

(b) By definition, $I \in L_{uo,o}(E)$ means that every uo-null net in E is order null, so the first assertion is (U3). If $\lambda I \in L_{uo,o}(E)$ for some $\lambda \neq 0$, then $I = \lambda^{-1}(\lambda I) \in L_{uo,o}(E)$ by (a). If E is finite dimensional, uo-null nets are order null by (U3), so every $T \in L_n(E)$ belongs to $L_{uo,o}(E)$; together with (a) this gives $L_{uo,o}(E) = L_n(E)$.

(c) Let $x_\alpha \overset{uo}{\to} 0$ in c_0 and write $x_\alpha = (x_\alpha(i))$. For each i, $|x_\alpha| \wedge e_i = \min(|x_\alpha(i)|, 1) e_i \overset{o}{\to} 0$. Since a net decreasing to 0 in c_0 decreases to 0 in every coordinate, this forces $\min(|x_\alpha(i)|, 1) \to 0$, that is, $x_\alpha(i) \to 0$ in $\mathbb{R}$. Hence $P_n x_\alpha = \Sigma_{i=1}^n x_\alpha(i) e_i \overset{o}{\to} 0$ by (O4) and (O1). □

Definition 2. An order bounded operator $T: E \to E$ is called *demi unbounded order-to-order continuous* (briefly, *demi uo-to-order continuous*) if for every net (x_α) in E^+,

$$x_\alpha \overset{uo}{\to} 0 \text{ and } x_\alpha - Tx_\alpha \overset{o}{\to} 0 \quad \Rightarrow \quad x_\alpha \overset{o}{\to} 0.$$

The class of all such operators is denoted by $\mathfrak{D}L_{uo,o}(E)$.

Since $x_\alpha \overset{uo}{\to} 0$ if and only if $|x_\alpha| \overset{uo}{\to} 0$, restricting to positive nets in the first hypothesis is harmless; it is the second hypothesis that is affected by the sign of x_α.

Example 1. For every $\beta \neq 1$, $\beta I \in \mathfrak{D}L_{uo,o}(E)$: if $x_\alpha - \beta x_\alpha = (1-\beta)x_\alpha \overset{o}{\to} 0$ then $x_\alpha \overset{o}{\to} 0$ by (O1).

The case $\beta = 1$ is governed by the dimension of E. The following theorem is the analogue of Keleş and Altın (2023, Theorem 2.5), with "order continuous norm" replaced by "finite dimensional".

Theorem 1. For a Riesz space E the following are equivalent.

(i) Every $T \in L_b(E)$ belongs to $\mathfrak{D}L_{uo,o}(E)$.

(ii) $I \in \mathfrak{D}L_{uo,o}(E)$.

(iii) E is finite dimensional.

Proof. (i)⇒(ii) is trivial. (ii)⇒(iii): Let $x_\alpha \overset{uo}{\to} 0$ in E. Then $|x_\alpha| \overset{uo}{\to} 0$, $|x_\alpha| - I|x_\alpha| = 0 \overset{o}{\to} 0$, and (ii) gives $|x_\alpha| \overset{o}{\to} 0$, that is $x_\alpha \overset{o}{\to} 0$. Thus uo-convergence and order convergence coincide in E, and E is finite dimensional by (U3). (iii)⇒(i): if E is finite dimensional and $x_\alpha \overset{uo}{\to} 0$ then $x_\alpha \overset{o}{\to} 0$ by (U3), so the conclusion of the demi condition holds regardless of T. □

Example 2. By Theorem 1, $I \notin \mathfrak{D}L_{uo,o}(E)$ for every infinite dimensional Riesz space E; so every infinite dimensional Riesz space shows that Example 1 fails for $\beta = 1$, and that $\mathfrak{D}L_{uo,o}(E)$ is in general a proper subset of $L_b(E)$. A concrete witness in $E = c_0$ is the net (e_n): it is uo-null by (U2), $e_n - Ie_n = 0$, but (e_n) is not order null by (U2).

Theorem 2. $L_{uo,o}(E) \subset \mathfrak{D}L_{uo,o}(E)$.

Proof. Let $T \in L_{uo,o}(E)$; it is order bounded by Proposition 1(a). Let (x_α) in E^+ satisfy $x_\alpha \overset{uo}{\to} 0$ and $x_\alpha - Tx_\alpha \overset{o}{\to} 0$. Then $Tx_\alpha \overset{o}{\to} 0$, and $x_\alpha = (x_\alpha - Tx_\alpha) + Tx_\alpha \overset{o}{\to} 0$ by (O1). □

The converse fails whenever E is infinite dimensional: $\frac{1}{2}I \in \mathfrak{D}L_{uo,o}(E)$ by Example 1, but $\frac{1}{2}I \notin L_{uo,o}(E)$ by Proposition 1(b).

Theorem 3. *If $T \in L_{uo,o}(E)$ and $S \in \mathfrak{D}L_{uo,o}(E)$, then $T + S$ and $S - T$ belong to $\mathfrak{D}L_{uo,o}(E)$.*

Proof. Let (x_α) in E^+ with $x_\alpha \overset{uo}{\to} 0$ and $x_\alpha - (T + S)x_\alpha \overset{o}{\to} 0$. Then $Tx_\alpha \overset{o}{\to} 0$ and

$$x_\alpha - Sx_\alpha = (x_\alpha - (T + S)x_\alpha) + Tx_\alpha \overset{o}{\to} 0,$$

so $x_\alpha \overset{o}{\to} 0$ because $S \in \mathfrak{D}L_{uo,o}(E)$. The case $S - T$ follows by replacing T with $-T$. □

Corollary 1. Let E be infinite dimensional. Then $\mathrm{I} + \mathrm{T} \notin \mathfrak{D}\mathrm{L}_{\mathrm{uo,o}}(\mathrm{E})$ for every $\mathrm{T} \in \mathrm{L}_{\mathrm{uo,o}}(\mathrm{E})$. In particular $\mathrm{I} + \mathrm{P}_\mathrm{k} \notin \mathfrak{D}\mathrm{L}_{\mathrm{uo,o}}(\mathrm{c}_0)$ for every $\mathrm{k} \in \mathbb{N}$, with P_k as in Proposition 1(c); this is the analogue of Keleş and Altın (2023, Example 2.3).

Proof. If $I + T \in \mathfrak{D}L_{uo,o}(E)$ with $T \in L_{uo,o}(E)$, then $I = (I + T) - T \in \mathfrak{D}L_{uo,o}(E)$ by Theorem 3, contradicting Theorem 1.

The particular case follows since $P_k \in L_{uo,o}(c_0)$ by Proposition 1(c). □

Example 3. $\mathfrak{D}L_{uo,o}(E)$ is not closed under addition or scalar multiplication in general: for infinite dimensional E, $\frac{1}{2}I \in \mathfrak{D}L_{uo,o}(E)$ by Example 1, but $\frac{1}{2}I + \frac{1}{2}I = 2 \cdot \frac{1}{2}I = I \notin \mathfrak{D}L_{uo,o}(E)$ by Theorem 1.

The next theorem is the analogue of Keleş and Altın (2023, Theorem 2.3).

Theorem 4. For a Riesz space E the following are equivalent.

(i) E is finite dimensional.

(ii) $\mathrm{L_{uo,o}(E)} = \mathfrak{D}\mathrm{L_{uo,o}(E)} \cap \mathrm{L_n(E)}$.

Proof. *(i)⇒(ii):* By Proposition 1(b) and Theorem 1, $L_{uo,o}(E) = L_n(E)$ and $\mathfrak{D}L_{uo,o}(E) = L_b(E)$, so both sides equal $L_n(E)$.

(ii)⇒(i): $\frac{1}{2}I$ is order continuous and belongs to $\mathfrak{D}L_{uo,o}(E)$ by Example 1, so (ii) gives $\frac{1}{2}I \in L_{uo,o}(E)$, hence $I \in L_{uo,o}(E)$ and E is finite dimensional by Proposition 1(b). □

Remark 1. In Keleş and Altın (2023, Theorem 2.3) the corresponding statement reads $L_{on}(M) = \mathfrak{D}L_{on}(M)$, without intersecting with a smaller class, because on a normed Riesz space with order continuous norm every bounded operator is order-to-norm continuous. Here the analogous step would require $L_n(E) = L_b(E)$, which fails in general (on $C[0,1]$ the operator $\delta_t \otimes \mathbf{1}$ is order bounded but not order continuous, as $C[0,1]_n^{\sim} = \{0\}$). This is why $L_n(E)$ appears in Theorem 4(ii).

Theorem 5. Let E and F be Riesz spaces.

(i) If $T \in L_b(E)$ is such that $I - T$ is bijective and $(I - T)^{-1}$ is order continuous, then $T \in \mathfrak{D}L_{uo,o}(E)$.

(ii) Let $\hat{E} = E \oplus F$ with the coordinate-wise order, let $T \in L_n(E, F)$ and $\beta \neq 1$, and define $\hat{T}_\beta : \hat{E} \to \hat{E}$ by $\hat{T}_\beta(x, y) = (0, Tx + \beta y)$, that is, $\hat{T}_\beta = \begin{pmatrix} 0 & 0 \\ T & \beta I \end{pmatrix}$. Then $\hat{T}_\beta \in \mathfrak{D}L_{uo,o}(\hat{E})$.

Proof. *(i)* If $x_\alpha \overset{uo}{\to} 0$ and $(I - T)x_\alpha \overset{o}{\to} 0$, then $x_\alpha = (I - T)^{-1}(I - T)x_\alpha \overset{o}{\to} 0$ by order continuity of $(I - T)^{-1}$.

(ii) $\hat{T}_\beta$ is order bounded. Let (x_α, y_α) be a net in $\hat{E}^+$ with $(x_\alpha, y_\alpha) \overset{uo}{\to} 0$ and

$$(x_\alpha, y_\alpha) - \hat{T}_\beta(x_\alpha, y_\alpha) = (x_\alpha,\ (1 - \beta)y_\alpha - Tx_\alpha) \overset{o}{\to} 0.$$

By (U4), $x_\alpha \overset{o}{\to} 0$ in E and $(1 - \beta)y_\alpha - Tx_\alpha \overset{o}{\to} 0$ in F. Since T is order continuous, $Tx_\alpha \overset{o}{\to} 0$, hence $(1 - \beta)y_\alpha \overset{o}{\to} 0$ and $y_\alpha \overset{o}{\to} 0$ by (O1). By (U4) again, $(x_\alpha, y_\alpha) \overset{o}{\to} 0$. □

Example 4. Theorem 5(ii) fails for $\beta = 1$. Let $F = c_0$, let E be any Riesz space, $T \in L_n(E, c_0)$, and $\hat{x}_n = (0, e_n) \in \hat{E}^+$. Then $\hat{x}_n \overset{uo}{\to} 0$ by (U2) and (U4), $\hat{x}_n - \hat{T}_1\hat{x}_n = (0, e_n - e_n) = 0$, but $\hat{x}_n$ is not order null by (U2) and (U4). Hence $\hat{T}_1 \notin \mathfrak{D}L_{uo,o}(\hat{E})$.

We now turn to domination. As in Keleş and Altın (2023, Example 2.7), the class is not solid downward in general: for infinite dimensional E, $0 \leq I \leq 2I$ and $2I \in \mathfrak{D}L_{uo,o}(E)$ by Example 1, but $I \notin \mathfrak{D}L_{uo,o}(E)$ by Theorem 1. Under an additional upper bound the domination property does hold, and, unlike the normed case, no lattice norm is needed, only (O2).

Theorem 6. *Let $S, T \in L_b(E)$ with $0 \leq S \leq T \leq I$. If $T \in \mathfrak{D}L_{uo,o}(E)$, then $S \in \mathfrak{D}L_{uo,o}(E)$.*

Proof. Let (x_α) in E^+ with $x_\alpha \overset{uo}{\to} 0$ and $(I-S)x_\alpha \overset{o}{\to} 0$. From $S \le T \le I$ we get $0 \le (I-T)x_\alpha \le (I-S)x_\alpha$, so $(I-T)x_\alpha \overset{o}{\to} 0$ by (O2). Since $T \in \mathfrak{D}L_{uo,o}(E)$, $x_\alpha \overset{o}{\to} 0$. □

Theorem 7. *Let $S,T \in L_b(E)$ with $I \le S \le T$. If $S \in \mathfrak{D}L_{uo,o}(E)$, then $T \in \mathfrak{D}L_{uo,o}(E)$.*

Proof. If $x_\alpha \overset{uo}{\to} 0$ in E^+ and $(T-I)x_\alpha \overset{o}{\to} 0$, then $0 \le (S-I)x_\alpha \le (T-I)x_\alpha$ gives $(S-I)x_\alpha \overset{o}{\to} 0$ by (O2), whence $x_\alpha \overset{o}{\to} 0$. □

Theorem 8. *Let $N,S,T \in L_b(E)$ with $N \le S \le T \le I+N$. If $N \in L_{uo,o}(E)$ and $T \in \mathfrak{D}L_{uo,o}(E)$, then $S \in \mathfrak{D}L_{uo,o}(E)$.*

Proof. The hypothesis gives $0 \le S-N \le T-N \le I$. By Theorem 3, $T-N \in \mathfrak{D}L_{uo,o}(E)$; by Theorem 6, $S-N \in \mathfrak{D}L_{uo,o}(E)$; and by Theorem 3 again, $S = (S-N)+N \in \mathfrak{D}L_{uo,o}(E)$. □

We close with the analogue of Keleş and Altın (2023, Example 2.9).

Example 5 ($\mathfrak{D}L_{uo,o}(E)$ is not order closed in $L_b(E)$). Let $E = c_0$ and P_n as in Proposition 1(c). Then $P_n \in L_{uo,o}(c_0) \subset \mathfrak{D}L_{uo,o}(c_0)$ by Proposition 1(c) and Theorem 2, and $0 \le P_n \uparrow I$ in $L_b(c_0)$, so $P_n \overset{o}{\to} I$ in $L_b(c_0)$, while $I \notin \mathfrak{D}L_{uo,o}(c_0)$ by Theorem 1.

References

Abramovich, Y. A., & Sirotkin, G. (2005). On order convergence of nets. *Positivity, 9*(3), 287–292. https://doi.org/10.1007/s11117-004-7543-x

Aliprantis, C. D., & Burkinshaw, O. (2006). *Positive operators*. Springer. https://doi.org/10.1007/978-1-4020-5008-4

Bahramnezhad, A., & Haghnejad Azar, K. (2017). *Strongly order continuous operators on Riesz spaces* [Preprint]. arXiv. https://doi.org/10.48550/arXiv.1712.04275

Benkhaled, H., Elleuch, A., & Jeribi, A. (2020). The class of order weakly demicompact operators. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, *114*(2), Article 76. https://doi.org/10.1007/s13398-020-00808-4

Benkhaled, H., Hajji, M., & Jeribi, A. (2023). On the class of demi Dunford–Pettis operators. *Rendiconti del Circolo Matematico di Palermo Series 2, 72*(2), 901–911. https://doi.org/10.1007/s12215-021-00702-x

De Marr, R. (1964). Partially ordered linear spaces and locally convex linear topological spaces. *Illinois Journal of Mathematics*, *8*(4), 601–606. https://doi.org/10.1215/ijm/1256059459

Emelyanov, E. Y., Gorokhova, S. G., & Kutateladze, S. S. (2019). Unbounded order convergence and the Gordon theorem. *Vladikavkazskii Matematicheskii Zhurnal*, *21*(4), 56–62. https://doi.org/10.23671/VNC.2019.21.44624

Gao, N., Troitsky, V. G., & Xanthos, F. (2017). Uo-convergence and its applications to Cesàro means in Banach lattices. *Israel Journal of Mathematics, 220*(2), 649–689. https://doi.org/10.1007/s11856-017-1530-y

Jalili, A., Haghnejad Azar, K., & Moghimi, M. B. (2021). Order-to-topology continuous operators. *Positivity*, *25*(4), 1313–1322. https://doi.org/10.1007/s11117-021-00817-6

Keleş, G. S., & Altın, B. (2023). The class of demi-order norm continuous operators. *Gazi University Journal of Science*, *36*(4), 1693–1698. https://doi.org/10.35378/gujs.1098503

Krichen, B., & O'Regan, D. (2019). Weakly demicompact linear operators and axiomatic measures of weak noncompactness. *Mathematica Slovaca*, *69*(6), 1403–1412. https://doi.org/10.1515/ms-2017-0317

Nakano, H. (1948). Ergodic theorems in semi-ordered linear spaces. *Annals of Mathematics*, *49*(2), 538–556. https://doi.org/10.2307/1969044

Petryshyn, W. V. (1966). Construction of fixed points of demicompact mappings in Hilbert space. *Journal of Mathematical Analysis and Applications*, *14*(2), 276–284. https://doi.org/10.1016/0022-247X(66)90027-8

Turan, B., Altın, B., & Gürkök, T. (2022). On unbounded order continuous operators. *Turkish Journal of Mathematics*, *46*(8), 3391–3399. https://doi.org/10.55730/1300-0098.3339

BÖLÜM 3

ON A DURRMEYER-TYPE GENUINE BERNSTEIN OPERATOR SEQUENCE CONSTRUCTED USING THE k-GAMMA FUNCTION

Ülkü DİNLEMEZ KANTAR[1]
İsmet YÜKSEL[2]

Introduction

In approximation theory, Bernstein (1912) first constructed a sequence of polynomials to prove the Weierstrass theorem. These sequences are known as Bernstein polynomials. Later, Durrmeyer (1967) defined an integral modification of the Bernstein operators. Today, these operators serve as the foundation for defining sequences of integral-type operators.

Chen (1987) defined a sequence of operators of the form

$$D_n(f(t),x) = f(0)b_{n,0}(x) + f(1)b_{n,n}(x) + [n-1]_q \sum_{j=1}^{n-1} b_{n,j}(x) \int_0^1 b_{n-2,j-1}(t) f(t)dt \qquad (1)$$

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where

$$b_{n,j}(x) = \binom{n}{j} x^n (1-x)^{n-j}, x \in [0,1] \quad (2)$$

is a basis function, and f is a continuous function.

The asymptotic approximation of operators (1) (Abel et al., 2026) and the λ-parametric extension of the q-analogue (Dinlemez Kantar & Yüksel, 2026; Yüksel & Dinlemez Kantar, 2026) were defined, and some theorems about the approximation properties were obtained.

In this section, a parametric generalization of the operators defined in (1)—constructed with the aid of the k-beta function—is introduced, and several approximation results are obtained.

First, the definitions and notations used in the literature arepresented.

Daz and Pariguan (2007) defined the k-gamma function as

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} dt \, , Re(z) > 0 \quad (3)$$

where $k \in \mathbb{R}^+$.

For $k = 1$, it reduces to the integral form of the Euler function $\Gamma(z)$, which is an extension of the factorial concept to complex numbers, as it is currently used by Chaudhry & Zubair (2001).

Some of the equalities between $\Gamma(z)$ and $\Gamma_k(z)$ are given below:

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma(z), \quad (4)$$

$$\Gamma_k(z+k) = z\Gamma_k(z), \quad (5)$$

$$\Gamma_k(k) = 1. \quad (6)$$

Using equations (3)–(6), İçöz & Demir (2022) defined a generalization of the Gamma operators using the k-gamma function and obtained some approximate results.

Wang (2016) defined the *k*-beta function as

$$\mathrm{B}_k(j,n) = \int_0^\infty \frac{t^{j-1}}{(1+t^k)^{\frac{s+n}{k}}}\,dt. \tag{7}$$

The following equations exist

$$\mathrm{B}_k(j,n) = \frac{\Gamma_k(j)\Gamma_k(n)}{\Gamma_k(j+n)}, \tag{8}$$

$$\mathrm{B}_k(j,n) = \int_0^1 t^{j-1}(1-t^k)^{\frac{n-k}{k}}\,dt. \tag{9}$$

Using formulas (3)–(9), we defined a sequence of Durrmeyer-type genuine Bernstein operators, generated via the k-gamma function, expressed in the form

$$U_n^k(f(t),x) = f(0)b_{n,0}(x) + f(1)b_{n,n}(x) + (n-1)\sum_{j=1}^{n-1} b_{n,j}(x)\int_0^1 b_{n-2,j-1}^k(t)f(t^k)dt \tag{10}$$

where

$$\langle {n \atop j} \rangle_k := \frac{\Gamma_k(n+1)}{\Gamma_k(j+1)\Gamma_k(n-j+k)} \tag{11}$$

is denoted as

$$b_{n,j}^k(x) = \langle {n \atop j} \rangle_k\, x^j(1-x^k)^{\frac{n-j}{k}}. \tag{12}$$

Since the sequence of operators defined by (10) reduces to the binomial coefficient $\langle {n \atop j} \rangle_1 = \binom{n}{j}$ for $k = 1$ in the expression (11) and to the basis function (2) in the formula (12), it simplifies to the sequence of operators given in (1).

The lemmas required for the theorem to be obtained in this study is presented in the following section.

Auxiliary results

Lemma 1. The following equalities are obtained.

i) $\int_0^1 b_{n-2,j-1}^k(x)dx = \frac{1}{n-1}$,

ii) $\int_0^1 b_{n-2,j-1}^k(x)x^k dx = \frac{j}{(n-1)(n-1+k)}$,

iii) $\int_0^1 b_{n-2,j-1}^k(x)x^{2k} dx = \frac{j(j+k)}{(n-1)(n-1+k)(n-1+2k)}$,

iv) $\int_0^1 b_{n-2,j-1}^k(x)x^{3k} dx = \frac{j(j+k)(j+2k)}{(n-1)(n-1+k)(n-1+2k)(n-1+3k)}$,

v) $\int_0^1 b_{n-2,j-1}^k(x)x^{4k} dx = \frac{j(j+k)(j+2k)(j+3k)}{(n-1)(n-1+k)(n-1+2k)(n-1+3k)(n-1+4k)}$.

Proof. i) Using formulas (5), (9), (11), and (12), the following equality is achieved.

$$\int_0^1 b_{n-2,j-1}^k(x)dx = \int_0^1 \langle {n-2 \atop j-1} \rangle_k \, x^{j-1}(1-x^k)^{\frac{n-2-(j-1)}{k}} dx,$$

$$= \frac{\Gamma_k(n-1)}{\Gamma_k(j)\Gamma_k(n-j-1+k)} B_k(j, n-j-1+k)$$

$$= \frac{1}{n-1}.$$

ii) Using the appropriate k-gamma and k-beta functions as in i), the following equality is obtained.

$$\int_0^1 b_{n-2,j-1}^k(x)x^k dx = \int_0^1 \langle {n-2 \atop j-1} \rangle_k \, x^{j+k-1}(1-x^k)^{\frac{n-j-1}{k}} dx,$$

$$= \frac{\Gamma_k(n-1)}{\Gamma_k(j)\Gamma_k(n-j-1+k)} B_k(j+k, n-j-1+k),$$

$$= \frac{\Gamma_k(n-1)}{\Gamma_k(j)\Gamma_k(n-j-1+k)} . \frac{\Gamma_k(j+k)\Gamma_k(n-j-1+k)}{\Gamma_k(n-1+2k)},$$

$$= \frac{j}{(n-1)(n-1+k)}.$$

The equalities iii), iv), and v) are derived using the procedures followed in i) and ii).

Lemma 2. For $e_j(t) = t^j$ test functions ,where j = 0, 1, 2, the following equations are obtained for the U_n^k operators defined in (10):

i) $U_n^k(e_0; x) = 1$,

ii) $U_n^k(e_1; x) = \frac{nx}{n-1+k} + \frac{(k-1)x^n}{n-1+k}$,

iii) $U_n^k(e_2; x) = \frac{n(n-1)}{(n-1+k)(n-1+2k)}x^2 + \frac{(1+k)n}{(n-1+k)(n-1+2k)}x$

$+ \frac{(k-1)(2n+2k-1)x^n}{(n-1+k)(n-1+2k)}$.

Proof. i) By setting $f(t) = e_0(t)$ in the operator sequence (10) and using part i) of Lemma 1, the following equality is derived,

$$U_n^k(e_0; x) = b_{n,0}(x) + b_{n,n}(x) + (n - 1)\sum_{j=1}^{n-1} b_{n,j}(x)\int_0^1 b_{n-2,j-1}^k(t)dt$$

$$= \sum_{j=0}^{n} b_{n,j}(x) = 1.$$

ii) Again, by setting $f(t) = e_1(t)$ in the operator sequence (10) and using part ii) of Lemma 1, the following equality is attained.

$$U_n^k(e_1, x) = b_{n,n}(x) + (n-1)\sum_{j=1}^{n-1} b_{n,j}(x)\int_0^1 b_{n-2,j-1}^k(t)t^k dt$$

$$= b_{n,n}(x) + \frac{1}{n-1+k}\sum_{j=1}^{n-1} b_{n,j}(x)j$$

$$= b_{n,n}(x) + \frac{1}{n-1+k}\sum_{j=1}^{n-1} \frac{n!}{(j-1)!\,(n-j)!}x^j(1-x)^{n-j}$$

$$= x^n + \frac{nx}{n-1+k}\left(\sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!}x^j(1-x)^{n-1-j} - x^{n-1}\right)$$

$$= x^n + \frac{nx}{n-1+k}(1 - x^{n-1})$$

$$= \frac{nx}{n-1+k} + \frac{(k-1)x^n}{n-1+k}.$$

iii) By setting $f(t) = e_2(t)$ in the sequence of operators (10) and applying part iii) of Lemma 1, the following equality is found.

$$U_n^k(e_2, x) = b_{n,n}(x) + (n-1)\sum_{j=1}^{n-1} b_{n,j}(x) \int_0^1 b_{n-2,j-1}^k(t) t^{2k} dt$$

$$= x^n + \frac{1}{(n-1+k)(n-1+2k)} \left(\sum_{j=1}^{n-1} b_{n,j}(x) j(j+k)\right)$$

$$= x^n + \frac{1}{(n-1+k)(n-1+2k)} \sum_{j=2}^{n-1} \frac{n!}{(j-2)!(n-j)!} x^j (1-x)^{n-j}$$

$$+ \frac{(1+k)}{(n-1+k)(n-1+2k)} \sum_{j=1}^{n-1} \frac{n!}{(j-1)!(n-j)!} x^j (1-x)^{n-j}$$

$$= x^n + \frac{n(n-1)x^2}{(n-1+k)(n-1+2k)} \sum_{j=0}^{n-3} \frac{(n-2)!}{j!(n-2-j)!} x^j (1-x)^{n-2-j}$$

$$+ \frac{(1+k)nx}{(n-1+k)(n-1+2k)} \sum_{j=0}^{n-2} \frac{(n-1)!}{j!(n-1-j)!} x^j (1-x)^{n-1-j}$$

$$= x^n + \frac{n(n-1)x^2}{(n-1+k)(n-1+2k)}$$

$$\times \left(\sum_{j=0}^{n-2} \frac{(n-2)!}{j!(n-2-j)!} x^j (1-x)^{n-2-j} - x^{n-2}\right)$$

$$+ \frac{(1+k)nx}{(n-1+k)(n-1+2k)} \left(\sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!} x^j (1-x)^{n-1-j} - x^{n-1}\right)$$

$$= x^n + \frac{n(n-1)x^2}{(n-1+k)(n-1+2k)}(1 - x^{n-2})$$

$$+ \frac{(1+k)nx}{(n-1+k)(n-1+2k)}(1 - x^{n-1})$$

$$= \frac{n(n-1)x^2}{(n-1+k)(n-1+2k)} + \frac{(1+k)nx}{(n-1+k)(n-1+2k)} + \frac{(k-1)(2n+2k-1)x^n}{(n-1+k)(n-1+2k)}.$$

Since the proof of Lemma 3 given below relies on the linearity of the operator sequence (10) and the results of Lemma 2, it is left to the researcher.

Lemma 3. The following equations hold:

i) $U_n^k((t-x),x) = \frac{(1-k)(x-x^n)}{n-1+k}$,

ii) $U_n^k((t-x)^2,x) = \frac{2k^2-3k+1-(1+k)n}{(n-1+k)(n-1+2k)}x^2 + \frac{(1+k)n}{(n-1+k)(n-1+2k)}x$

$$+\left(\frac{(2n+2k-1)(k-1)}{(n-1+k)(n-1+2k)} - \frac{2(k-1)x}{(n-1+k)}\right)x^n.$$

Lemma 4. Let $f(x) \in C[0,1]$. In this case, the following limit exists

$$\lim_{n\to\infty}\left\|U_n^k(f(t),x) - f(x)\right\| = 0$$

where $\|f\| = maks\{|f(x)|:\ x \in [0,1]\}$.

Proof. i) Using result i) of Lemma 2, the following limit exists,

$$\lim_{n\to\infty}\left\|U_n^k(1,x) - 1\right\| = 0.$$

ii) Similarly, by using result ii) of Lemma 2, the following limit is obtained

$$\lim_{n\to\infty}\left\|U_n^k(t,x) - x\right\| = \lim_{n\to\infty}\left\|\frac{nx}{n-1+k} + \frac{(k-1)x^n}{n-1+k} - x\right\|$$

$$= \lim_{n\to\infty}\left\|\frac{(1-k)x}{n-1+k} + \frac{(k-1)x^n}{n-1+k}\right\| = 0.$$

iii) Using part iii) of Lemma 2, the result can be obtained just like parts i) and ii). Thus, Lemma 4 is proved by the Korovkin theorem (Korovkin, 1960).

Lemma 5. The following limits are available:

i) $\lim_{n\to\infty} nU_n^k(t-x,x) = (1-k)x,$

ii) $\lim_{n\to\infty} nU_n^k((t-x)^2,x) = (1+k)(x-x^2).$

Proof. The results of Lemma 5 can be easily obtained by using the results of Lemma 3.

The approximation property

Here, we will present the theorem that gives the rate of convergence using the continuity modulus.

Let the first continuity modulus be given by

$$\omega(f;\varepsilon) := max\{|f(t)-f(x)|,\ \ t,x \in [0,1], |t-x| \le \varepsilon\} \qquad (13)$$

for the function $f(x) \in C[0,1]$.

The first continuity modulus given by (13) satisfies the following equation:

$$|f(t)-f(x)| \le \omega(f;\varepsilon)\left(1+\frac{|t-x|}{\varepsilon}\right). \qquad (14)$$

Theorem 1. Let $U_n^k(f;x)$ be a Durrmeyer-type sequence of genuine Bernstein operators generated by the k −gamma function, and let $f(x) \in C[0,1]$. Let $\vartheta_n^k = \left(\frac{10n+2k^2+13k+5}{(n-1+k)(n-1+2k)}\right)^{\frac{1}{2}}$,

then the inequality holds:

$\left|U_n^k(f;x)-f(x)\right| \le 2\omega\left(f;\ \vartheta_n^k\right).$

Proof. Using the linearity of the operator $U_n^k(f(t),x)$ defined in (10), then taking the absolute value of both sides of the equation, and finally applying inequality (14), we obtain

$$\left|U_n^k(f(t);x)-f(x)\right| = \left|U_n^k(f(t)-f(x);x)\right|$$

$$\le U_n^k(|f(t)-f(x)|;x)$$

$$\leq U_n^k\left(\omega(f;\varepsilon)\left(1+\frac{|t-x|}{\varepsilon}\right);x\right)$$

$$=\omega(f;\varepsilon)\left(1+\frac{1}{\varepsilon}U_n^k(|t-x|;x)\right).$$

If the Cauchy–Schwarz inequality is then applied to the expression $U_n^k(|t-x|;x)$,

$$\left|U_n^k(f(t);x)-f(x)\right|\leq\omega(f;\varepsilon)\left(1+\frac{1}{\varepsilon}\sqrt{U_n^k((t-x)^2;x)}\right)$$

is obtained. Using result ii) obtained in Lemma 3, the following inequality holds:

$$\begin{aligned}U_n^k((t-x)^2;x)&=\frac{2k^2-3k+1-(1+k)n}{(n-1+k)(n-1+2k)}x^2\\&\quad+\frac{(1+k)n}{(n-1+k)(n-1+2k)}x\\&\quad+\left(\frac{(2n+2k-1)(k-1)}{(n-1+k)(n-1+2k)}-\frac{2(k-1)x}{(n-1+k)}\right)x^n\\&\leq\frac{2k^2+3k+1+(1+k)n}{(n-1+k)(n-1+2k)}+\frac{(1+k)n}{(n-1+k)(n-1+2k)}\\&\quad+\frac{(2n+2k+1)(k+1)}{(n-1+k)(n-1+2k)}+\frac{2(k+1)(n-1+2k)}{(n-1+k)(n-1+2k)}\\&\leq\frac{2k^2+3k+1+5(2n+2k+1)}{(n-1+k)(n-1+2k)}\\&\leq\frac{10n+2k^2+13k+5}{(n-1+k)(n-1+2k)}.\end{aligned}$$

If $\vartheta_n^k=\left(\frac{10n+2k^2+13k+5}{(n-1+k)(n-1+2k)}\right)^{\frac{1}{2}}$ is chosen here,

then the inequality holds:

$$\left|U_n^k(f(t);x)-f(x)\right|\leq 2\omega\left(f;\vartheta_n^k\right).$$

The Graphical Results

In this study, the convergence properties of genuine k −Bernstein-Durrmeyer operators are examined. To support the theoretical results obtained, the convergence of these operators to a function f is illustrated with some graphical examples.

Example. Let the function $f(x) = x^{16}\tan(x^2)$ be defined for $x \in [0,1]$. First, the graph below shows the convergence of the $U_n^k(f;x)$ operators to the function $f(x)$ for values of $k = 0.1$, $k = 0.4$, and $k = 5$, where $n = 5$ (Figure 1). Second, it shows the convergence of the function $f(x)$ for the values $n = 5$, $n = 10$, and $n = 20$ corresponding to $k = 0.2$ (Figure 2).

Figure 1. Convergence of the $U_n^k(f;x)$ operators to the function $f(x)$ for different values of k

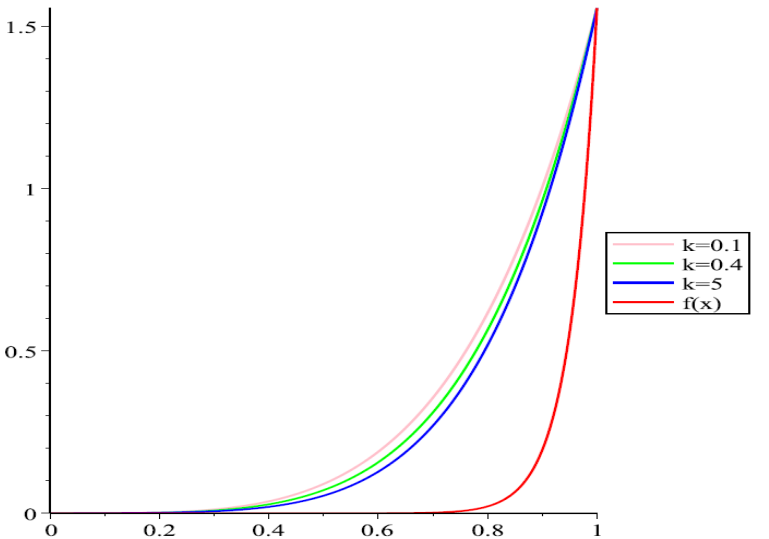
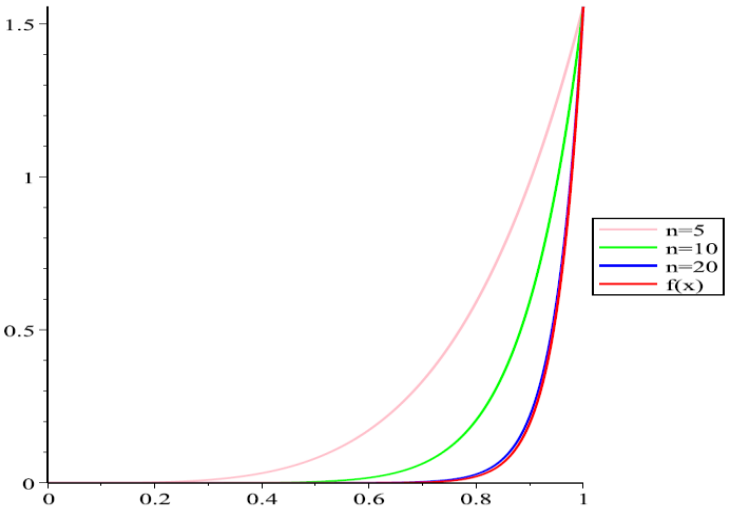


Figure 2. Convergence of the $U_n^k(f;x)$ operators to the function $f(x)$ for different values of n



Therefore, for any real number $k > 0$, the sequence of operators $U_n^k(f;x)$ on $[0,1]$ can be observed to converge to the function $f \in C[0,1]$ as k increases.

References

Abel, U., Acu, A. M., Heilmann, M., & Raşa, I. (2026). Asymptotic Expansions for Generalized Bernstein–Durrmeyer and Genuine Bernstein–Durrmeyer Operators. *Complex Analysis and Operator Theory*, 20(70), 2-15,.

Bernstein, S. N. (1912). Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités. *Comm. Soc. Math. Charkow Ser. 2 t., 13*, 1-2.

Chen, W. (1987). On the modified Durrmeyer-Bernstein operators. *Report of the fifth Chinese Conference on Approximation Theory.* Zhen Zhou, China.

Diaz, R., & Pariguan, E. (2007). On hypergeometric functions and Pochhammer k-symbol. *Divulgaciones Matem´aticas*, 15 (2), 179–192.

Dinlemez Kantar, Ü., & Yüksel, İ. (2026). Genuıne q- Bernsteın – Durrmeyer operatör dizisinin λ- parametrik genelleştirmesi üzerine. In *Current Topics in Applied Mathematics and Analysis* (p. 110). Ankara: Bidge.

Durrmeyer, J. L. (1967). Une formule d'inversion de la transformée de Laplace: applications à la théorie des moments. *Comptes Rendus de l'Académie des Sciences de Paris, 265*, 1447-1450.

İçöz, G., & Demir, S. (2022). Approximation Properties of a New Type of Gamma Operator Defined with the Help of k-Gamma Function. *Journal of Function Spaces*, 2022(1), 5493056.

Wang, W. S. (2016). Some properties of k-gamma and k-beta functions. *In ITM Web of Conferences.* EDP Sciences.

Yüksel, İ., & Dinlemez Kantar, Ü. (2026). λ-parametreli genuine q-Bernsteın –Durrmeyer operator dizisi icin Voronovskaja tip yaklaşım teoremi. In *Current Topics in Applied Mathematics and Analysis* (p. 92). Ankara: Bidge.

BÖLÜM 4

THE VORONOVSKAJA THEOREM FOR A DURRMEYER-TYPE GENUINE BERNSTEIN OPERATOR SEQUENCE CONSTRUCTED USING THE *k*-GAMA FUNCTION

Ülkü DİNLEMEZ KANTAR [1]

Introduction

The following base function

$$b_{n,j}(x) = \binom{n}{j} x^n (1-x)^{n-j} \tag{1}$$

for $n \in \mathbb{N}$, $x \in [0,1]$ and the

$$B_n(f,x) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right) b_{n,j}(x)$$

sequence operators on $f \in C[0,1]$ were defined by Bernstein (1912). Durrmeyer (1967) defined the integral-type variant of the Bernstein operator sequence as

$$D_n(f,x) = (\mathrm{n}+1) \sum_{j=0}^{n} b_{n,j}(x) \int_0^1 b_{n,j}(t)\, f(\mathrm{t}) dt.$$

Such studies can be found in (Lin et al., 2024; Cai et al., 2021).

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Chen (1987) defined the following Genuine Bernstein operator sequence.

$$D_n(f(t),x) = f(0)b_{n,0}(x) + f(1)b_{n,n}(x)$$
$$+[n-1]_q \sum_{j=1}^{n-1} b_{n,j}(x) \int_0^1 b_{n-2,j-1}(t)f(t)dt. \qquad (2)$$

Recently, the asymptotic approximation of the (2)-operator series has also been studied in (Abel et al., 2026; Torun & Dinlemez Kantar, 2025). The λ-parametric extension of the q-analog of the (2) operator sequence was defined, and some theorems regarding its approximation properties were established (Dinlemez Kantar & Yüksel, 2026; Yüksel & Dinlemez Kantar, 2026).

Firstly, the definitions and notation used in the literature are provided.

Daz and Pariguan (2007) defined the k-gamma function as

$$\Gamma_k(z) = \int_0^\infty t^{z-1} e^{-\frac{t^k}{k}} dt \, , Re(z) > 0 \qquad (3)$$

where $k \in \mathbb{R}^+$.

For $k = 1$, it reduces to the integral form of the Euler function $\Gamma(z)$, which is an extension of the factorial concept to complex numbers, as it is currently used by Chaudhry & Zubair (2001).

Some of the equalities between $\Gamma(z)$ and $\Gamma_k(z)$ are given below.

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma(z), \qquad (4)$$

$$\Gamma_k(z+k) = z\Gamma_k(z), \qquad (5)$$

$$\Gamma_k(k) = 1. \qquad (6)$$

Using equations (3)–(6), İçöz & Demir (2022) defined a generalization of the Gamma operators using the *k*-Gamma function and obtained some approximate results.

Wang (2016) defined the k-beta function as

$$\mathrm{B}_k(j,n) = \int_0^{\infty} \frac{t^{j-1}}{(1+t^k)^{\frac{s+n}{k}}}\, dt. \tag{7}$$

The following equations exist

$$\mathrm{B}_k(j,n) = \frac{\Gamma_k(j)\Gamma_k(n)}{\Gamma_k(j+n)}, \tag{8}$$

$$\mathrm{B}_k(j,n) = \int_0^1 t^{j-1}(1-t^k)^{\frac{n-k}{k}}\, dt. \tag{9}$$

Using the sequence of operators (1) derived from formulas (3)–(9), Dinlemez Kantar & Yüksel (2026) defined the sequence of Durrmeyer-type genuine Bernstein operators generated by the *k*-gamma function as

$$U_n^k(f(t),x) = f(0)b_{n,0}(x) + f(1)b_{n,n}(x) + (n-1)\sum_{j=1}^{n-1} b_{n,j}(x)\int_0^1 b_{n-2,j-1}^k(t) f(t^k)dt \tag{10}$$

where

$$\langle {n \atop j} \rangle_k := \frac{\Gamma_k(n+1)}{\Gamma_k(j+1)\Gamma_k(n-j+k)} \tag{11}$$

and

$$b_{n,j}^k(x) = \langle {n \atop j} \rangle_k\, x^j(1-x^k)^{\frac{n-j}{k}}. \tag{12}$$

In this section, we examine the Voronovskaya-type asymptotic convergence behavior of the sequence of operators defined by (10).

Now, we present the lemmas necessary to prove the Voronovskaya-type theorem.

Auxiliary results

Lemma 1. The following equations are obtained

i) $\int_0^1 b_{n-2,j-1}^k(x)dx = \frac{1}{n-1}$,

ii) $\int_0^1 b_{n-2,j-1}^k(x)x^k dx = \frac{j}{(n-1)(n-1+k)}$,

iii) $\int_0^1 b_{n-2,j-1}^k(x)x^{2k} dx = \frac{j(j+k)}{(n-1)(n-1+k)(n-1+2k)}$,

iv) $\int_0^1 b_{n-2,j-1}^k(x)x^{3k} dx = \frac{j(j+k)(j+2k)}{(n-1)(n-1+k)(n-1+2k)(n-1+3k)}$,

v) $\int_0^1 b_{n-2,j-1}^k(x)x^{4k} dx = \frac{j(j+k)(j+2k)(j+3k)}{(n-1)(n-1+k)(n-1+2k)(n-1+3k)(n-1+4k)}$.

Proof.

i) Using equations (8), (11), and (12), the following is obtained

$$\int_0^1 b_{n-2,j-1}^k(x)dx = \int_0^1 \langle {n-2 \atop j-1} \rangle_k \, x^{j-1}(1-x^k)^{\frac{n-2-(j-1)}{k}} dx,$$

$$= \frac{\Gamma_k(n-1)}{\Gamma_k(j)\Gamma_k(n-j-1+k)} B_k(j, n-j-1+k)$$

$$= \frac{1}{n-1}.$$

ii) Using the appropriate *k*-gamma and *k*-beta functions employed in i), the following equation is obtained

$$\int_0^1 b_{n-2,j-1}^k(x)x^k dx = \int_0^1 \langle {n-2 \atop j-1} \rangle_k \, x^{j+k-1}(1-x^k)^{\frac{n-j-1}{k}} dx,$$

$$= \frac{\Gamma_k(n-1)}{\Gamma_k(j)\Gamma_k(n-j-1+k)} B_k(j+k, n-j-1+k),$$

$$= \frac{\Gamma_k(n-1)}{\Gamma_k(j)\Gamma_k(n-j-1+k)} . \frac{\Gamma_k(j+k)\Gamma_k(n-j-1+k)}{\Gamma_k(n-1+2k)},$$

$$= \frac{j}{(n-1)(n-1+k)}.$$

Equations iii), iv), and v) are derived using the procedures followed in i) and ii).

Lemma 2. For $j = 0,1,2,3,4,$ the following equalities are obtained for the operators. U_n^k defined in (10), corresponding to the test functions $e_j(t) = t^j$:

i) $U_n^k(e_0; x) = 1,$

ii) $U_n^k(e_1; x) = \frac{n}{n-1+k}x \quad + \frac{k-1}{n-1+k}x^n,$

iii) $U_n^k(e_2; x) = \frac{n(n-1)}{(n-1+k)(n-1+2k)}x^2 + \frac{(1+k)n}{(n-1+k)(n-1+2k)}x$

$$+ \frac{(k-1)(2n+2k-1)}{(n-1+k)(n-1+2k)}x^n,$$

iv) $U_n^k(e_3; x) = \frac{n(n-1)(n-2)}{(n-1+k)(n-1+2k)(n-1+3k)}x^3 + \frac{3(1+k)n(n-1)}{(n-1+k)(n-1+2k)(n-1+3k)}x^2$

$$+ \frac{(1+3k+2k^2)n}{(n-1+k)(n-1+2k)(n-1+3k)}x$$

$$+ \frac{(k-1)(-5k-3n+9kn+6k^2+3n^2+1)}{(n-1+k)(n-1+2k)(n-1+3k)}x^n,$$

v) $U_n^k(e_4; x) = \frac{n(n-1)(n-2)(n-3)}{(n-1+k)(n-1+2k)(n-1+3k)(n-1+4k)}x^4$

$$+ \frac{6(1+k)n(n-1)(n-2)}{(n-1+k)(n-1+2k)(n-1+3k)(n-1+4k)}x^3$$

$$+ \frac{(11k^2+18k+7)n(n-1)}{(n-1+k)(n-1+2k)(n-1+3k)(n-1+4k)}x^2$$

$$+ \frac{(6k^3+11k^2+6k+1)n}{(n-1+k)(n-1+2k)(n-1+3k)(n-1+4k)}x$$

$$+ \frac{(k-1)(4k+2n-1)(-5k-2n+8kn+6k^2+2n^2+1)}{(n-1+k)(n-1+2k)(n-1+3k)(n-1+4k)}x^n.$$

Proof.

i) Statements i) - iii) have been proven by Dinlemez Kantar and Yüksel (2026) in the book chapter titled “On the Durrmeyer-type

genuine Bernstein operator sequence constructed using the k-gamma function."

iv) By taking $f(t) = e_3(t)$ in the operator sequence (10) and using part (iv) of Lemma 1,

$$U_n^k(e_3;x) = b_{n,n}(x) + (n-1)\sum_{j=1}^{n-1} b_{n,j}(x)\int_0^1 b_{n-2,j-1}^k(t)t^{3k}dt,$$

$$= x^n + \frac{1}{(n-1+k)(n-1+2k)(n-1+3k)}\sum_{j=1}^{n-1} b_{n,j}(x)j(j+k)(j+2k)$$

is obtained. In the final equation, using the equation for

$$j(j+k)(j+2k) = j(j-1)(j-2) + (3+3k)j(j-1) + (1+3k+2k^2)j$$

yields

$$U_n^k(e_3;x) = x^n + \frac{1}{(n-1+k)(n-1+2k)(n-1+3k)}\left\{\sum_{j=3}^{n-1}\frac{n!}{(j-3)!(n-j)!}x^j(1-x)^{n-j}\right.$$

$$+(3+3k)\sum_{j=2}^{n-1}\frac{n!}{(j-2)!(n-j)!}x^j(1-x)^{n-j}$$

$$\left.+(1+3k+2k^2)\sum_{j=1}^{n-1}\frac{n!}{(j-1)!(n-j)!}x^j(1-x)^{n-j}\right\}$$

$$U_n^k(e_3;x) = x^n + \frac{1}{(n-1+k)(n-1+2k)(n-1+3k)}\left\{\sum_{j=0}^{n-4}\frac{n!}{j!(n-j-3)!}x^{j+3}(1-x)^{n-j-3}\right.$$

$$+(3+3k)\sum_{j=0}^{n-3}\frac{n!}{j!(n-j-2)!}x^{j+2}(1-x)^{n-j-2}$$

$$\left.+(1+3k+2k^2)\sum_{j=0}^{n-2}\frac{n!}{j!(n-j-1)!}x^{j+1}(1-x)^{n-j-1}\right\}$$

$$= x^n + \frac{n(n-1)(n-2)x^3}{(n-1+k)(n-1+2k)(n-1+3k)}(1-x^{n-3})$$

$$+\frac{3(1+k)n(n-1)x^2}{(n-1+k)(n-1+2k)(n-1+3k)}(1-x^{n-2})$$

$$+\frac{(1+3k+2k^2)nx}{(n-1+k)(n-1+2k)(n-1+3k)}(1-x^{n-1})$$

$$=\frac{n(n-1)(n-2)}{(n-1+k)(n-1+2k)(n-1+3k)}x^3+\frac{3(1+k)n(n-1)}{(n-1+k)(n-1+2k)(n-1+3k)}x^2$$

$$+\frac{(1+3k+2k^2)n}{(n-1+k)(n-1+2k)(n-1+3k)}x$$

$$+\frac{(k-1)(-5k-3n+9kn+6k^2+3n^2+1)}{(n-1+k)(n-1+2k)(n-1+3k)}x^n.$$

v) Again, the desired result is obtained by taking $f(t)=e_4(t)$ in the operator sequence (10) and using part iv) of Lemma 1 along with equation

$$j(j+k)(j+2k)(j+3k)=j(j-1)(j-2)(j-3)$$
$$+(6+6k)j(j-1)(j-2)$$
$$+(7+18k+11k^2)j(j-1)$$
$$+(1+6k+11k^2+6k^3)j,$$

The proof of Lemma 3, provided below, is left to the researcher, as it relies on the linearity of the sequence of operators (10) and the results of Lemma 2.

Lemma 3. The following equations hold.

(i) $U_n^k((t-x),x)=\frac{(1-k)(x-x^n)}{n-1+k}$,

(ii) $U_n^k((t-x)^2,x)=\frac{2k^2-3k+1-(1+k)n}{(n-1+k)(n-1+2k)}x^2+\frac{(1+k)n}{(n-1+k)(n-1+2k)}x$
$$+\left(\frac{(2n+2k-1)(k-1)}{(n-1+k)(n-1+2k)}-\frac{2(k-1)}{(n-1+k)}x\right)x^n,$$

(iii) $U_n^k((t-x)^4,x)$
$$=\frac{1-10k+35k^2-50k^3+24k^4-(4-4k+38k^2+46k^3)n+(3+6k+3k^2)n^2}{(n-1+k)(n-1+2k)(n-1+3k)(n-1+4k)}x^4$$

$$+\frac{6n(k+1)(k-n-kn+12k^2+1)}{(n-1+k)(n-1+2k)(n-1+3k)(n-1+4k)}x^3$$

$$+\frac{-n(k+1)(19k-3n-3kn+32k^2+3)}{(n-1+k)(n-1+2k)(n-1+3k)(n-1+4k)}x^2$$

$$+\frac{(1+6k+11k^2+6k^3)n}{(n-1+k)(n-1+2k)(n-1+3k)(n-1+4k)}x$$

$$+\Big[\frac{(k-1)(4k+2n-1)(-5k-2n+8kn+6k^2+2n^2+1)}{(n-1+k)(n-1+2k)(n-1+3k)(n-1+4k)}$$

$$-4\frac{(k-1)(-5k-3n+9kn+6k^2+3n^2+1)}{(n-1+k)(n-1+2k)(n-1+3k)}x$$

$$+6\frac{(k-1)(2n+2k-1)}{(n-1+k)(n-1+2k)}x^2-4\frac{(k-1)}{n-1+k}x^3\Big]x^n.$$

Lemma 4. The following limits exist.

i) $\lim_{n\to\infty} nU_n^k(t-x,x)=(1-k)x,$

ii) $\lim_{n\to\infty} nU_n^k((t-x)^2,x)=(1+k)(x-x^2),$

iii) $\lim_{n\to\infty} n^2U_n^k((t-x)^4,x)=3(1+k)^2x^2(x-1)^2.$

Proof. The results of Lemma 4 are obtained by using the results of Lemma 3.

Main Result

In this section, we present the asymptotic approximation of the operator sequence $U_n^k(f(t),x)$ using the following theorem.

Theorem. For $f(x)\in C[0,1]$, the sequence of operators $U_n^k(f(t),x)$ has the following limit

$$\lim_{n\to\infty} n\big(U_n^k(f(t),x)-f(x)\big)=(1-k)xf'(x)+\frac{(1+k)(x-x^2)}{2}f''(x).$$

Proof. The Taylor series expansion of the function f at the point x is.

$$f(t) = f(x) + f'(x)(t-x) + \frac{f''(x)}{2}(t-x)^2 + \sigma(t,x)(t-x)^2,$$

where $\sigma(t,x) \to 0$ as $t \to x$. If the operator sequence U_n^k is applied to this equation, the following equation follows from the linearity of the operator sequence

$$U_n^k(f(t),x) - f(x) = f'(x)U_n^k(t-x,x) + \frac{f''(x)}{2}U_n^k((t-x)^2,x) + U_n^k(\sigma(t,x)(t-x)^2,x).$$

In the last equation, if both sides are multiplied by n and the limit is taken as $n \to \infty$, we get

$$\lim_{n\to\infty} n\Big(U_n^k(f(t),x) - f(x)\Big) = f'(x)\lim_{n\to\infty} nU_n^k(t-x,x) + \frac{f''(x)}{2}\lim_{n\to\infty} nU_n^k((t-x)^2,x) + \lim_{n\to\infty} nU_n^k(\sigma(t,x)(t-x)^2,x). \quad (13)$$

If the Cauchy–Schwarz inequality is applied to the last term on the right-hand side of equation (13), we have

$$\lim_{n\to\infty} nU_n^k(\sigma(t,x)(t-x)^2,x) \le \sqrt{\lim_{n\to\infty} U_n^k\left(\left(\sigma(t,x)\right)^2,x\right)} \times \sqrt{\lim_{n\to\infty} n^2U_n^k\left((t-x)^4,x\right)} \quad (14)$$

By Lemma 4, the limits in equation (13) exist, respectively

$$\lim_{n\to\infty} nU_n^k(t-x,x) = (1-k)x,$$

$$\lim_{n\to\infty} nU_n^k((t-x)^2,x) = (1+k)(x-x^2),$$

$$\lim_{n\to\infty} U_n^k(\sigma(t,x),x) = 0.$$

For $x \in [0,1]$, the following limit is finite

$$\lim_{n\to\infty} n^2 U_n^k\left((t-x)^4, x\right) = 3(1+k)^2x^2(x-1)^2.$$

Therefore,

$$\lim_{n\to\infty} nU_n^k(\sigma(t,x)(t-x)^2, x) = 0$$

is obtained from inequality (3). Thus,

$$\lim_{n\to\infty} n\left(U_n^k(f(t),x) - f(x)\right) = (1-k)xf'(x) + \frac{(1+k)(x-x^2)}{2}f''(x)$$

is obtained. Thereby, the proof is complete.

Graphical Results

In this study, a Voronovskaja-type asymptotic approximation theorem is obtained for a sequence of genuine-type k-Bernstein-Durrmeyer operators. To support the theoretical results obtained, graphs illustrating the convergence behavior of the sequence of operators

$$n(U_n^k(f(t),x) - f(x))$$

toward the asymptotic value

$$(1-k)xf'(x)\frac{(1+k)(x-x^2)}{2}f''(x)$$

as $n \to \infty$ is presented.

Example. Let the function $f(x) = (x^8 - \mathrm{x} - 1)\sin(\pi x^2)$ be defined for $x \in [0,1]$. Firstly, the graph below shows the convergence of the function $(0.8)xf'(x)\frac{(1.2)(x-x^2)}{2}f''(x)$ to $k = 0.2$—that is, for the values of the $n(U_n^{0.2}(f(t),x) - f(x))$ operator sequence corresponding to $n = 5$, $n = 10$, and $n = 50$ (Figure 1). Secondly, the graph shows the convergence of the function

$-xf'(x)\frac{3(x-x^2)}{2}f''(x)$ for $k = 2$—that is, for the values of the sequence of $n(U_n^2(f(t),x) - f(x))$ operator corresponding to $n = 5$, $n = 10$, and $n = 50$ (Figure 2).

Figure 1. $n(U_n^{0.2}(f(t),x) - f(x))$ *convergence of the sequence of operators to the asymptotic value of* $(0.8)xf'(x)\frac{(1.2)(x-x^2)}{2}f''(x)$

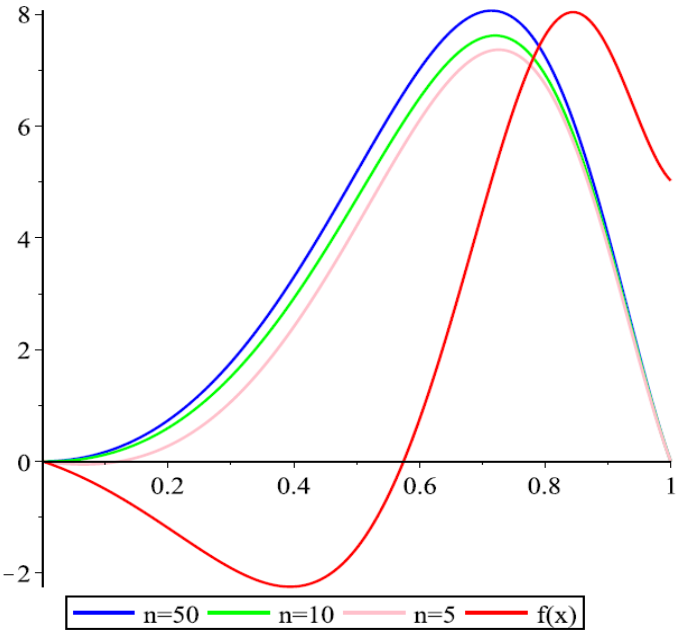
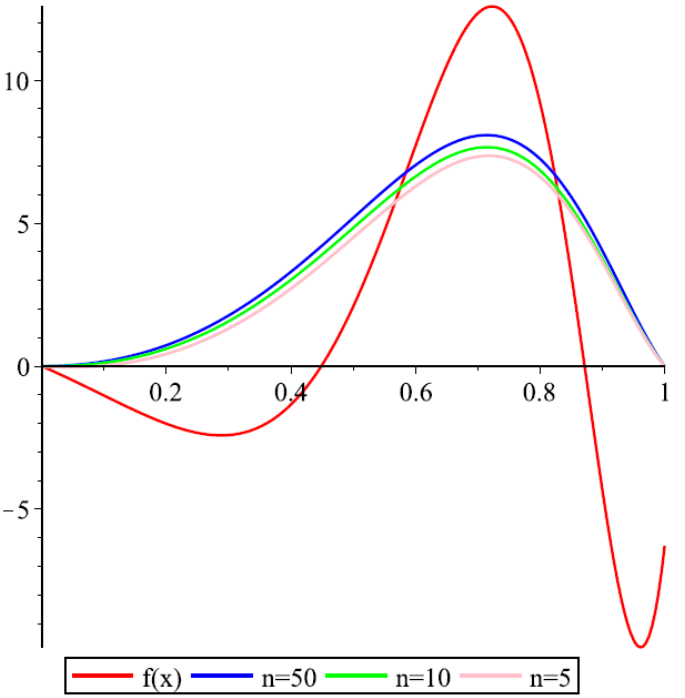


Figure 2. Convergence of the sequence of operators $n(U_n^2(f(t),x)-f(x))$ *to the asymptotic value of* $-xf'(x)\frac{3(x-x^2)}{2}f''(x)$



Hence, for any real number $k>0$, the sequence of operators $n(U_n^k(f(t),x)-f(x))$ is observed to tend toward the asymptotic value $(1-k)xf'(x)\frac{(1+k)(x-x^2)}{2}f''(x)$ on the interval [0,1].

References

Lin, Z., Torun, G., Kangal, E., Dinlemez Kantar, Ü., & Q. B., C. (2024). On the Properties of the Modified λ-Bernstein-Stancu Operators. *Symmetry*, 1276.

Abel, U., Acu, A. M., Heilmann, M., & Raşa, I. (2026). Asymptotic Expansions for Generalized Bernstein–Durrmeyer and Genuine Bernstein–Durrmeyer Operators. *Complex Analysis and Operator Theory*, 20(70), 1-15.

Bernstein, S. N. (1912). Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités. *Comm. Soc. Math. Charkow Ser. 2 t., 13*, 1-2.

Cai, Q.-B., Torun, G., & Kantar Dinlemez, Ü. (2021). Approximation Properties of Generalized λ-Bernstein–Stancu-Type Operators. *2021*(1), 1-17.

Chaudhry, M., & Zubair, S. (2001). *On a Class of Incomplete Gamma Functions with Applications.* Chapman & Hall/ CRC.

Chen, W. (1987). On the modified Durrmeyer-Bernstein operators. *Report of the fifth Chinese Conference on Approximation Theory.* Zhen Zhou, China.

Diaz, R., & Pariguan, E. (2007). On hypergeometric functions and Pochhammer k-symbol. *Divulgaciones Matem´aticas*, 15 (2), 179–192.

Dinlemez Kantar, Ü., & Torun, G. (2025). (Λ,M)-Bernstein-Durrmeyer-Stancu Operatör Dizisinin Yakınsaklığı Üzerine. In *Matematiksel Yapılar: Analiz Ve Geometri Perspektiviyle* (p. 161). Ankara: Bidge.

Dinlemez Kantar, Ü., & Yüksel, İ. (2026). Genuıne q- Bernsteın – Durrmeyer operatör dizisinin λ- parametrik genelleştirmesi üzerine. In *Current Topics in Applied Mathematics and Analysis* (p. 110). Ankara: Bidge.

Dinlemez Kantar, Ü., & Yüksel, İ. (2026). k- Gamma Fonksiyonu İle Oluşturulmuş Durrmeyer Tipli Genuine Bernstein Operatör Dizisi Üzerine. Ankara: Bidge.

Durrmeyer, J. L. (197). *Une formule d inversion de la transformee laplace, applications a la theorie des moments.* Paris: These de 3e cycle, Faculte des Sciences de I Universite de Paris.

İçöz, G., & Demir, S. (2022). Approximation Properties of a New Type of Gamma Operator Defined with the Help of k-Gamma Function. *Journal of Function Spaces*, 2022(1), 5493056.

Wang, W. S. (2016). Some properties of k-gamma and k-beta functions. *In ITM Web of Conferences* (Vol. 7, p. 07003). EDP Sciences.

Yüksel, İ., & Dinlemez Kantar, Ü. (2026). λ-parametreli genuıne q-Bernsteın –Durrmeyer operator dizisi icin Voronovskaja tip yaklaşım teoremi. In *Current Topics in Applied Mathematics and Analysis* (p. 92). Ankara: Bidge.


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