CURRENT TOPICS IN APPLIED MATHEMATICS AND ANALYSIS

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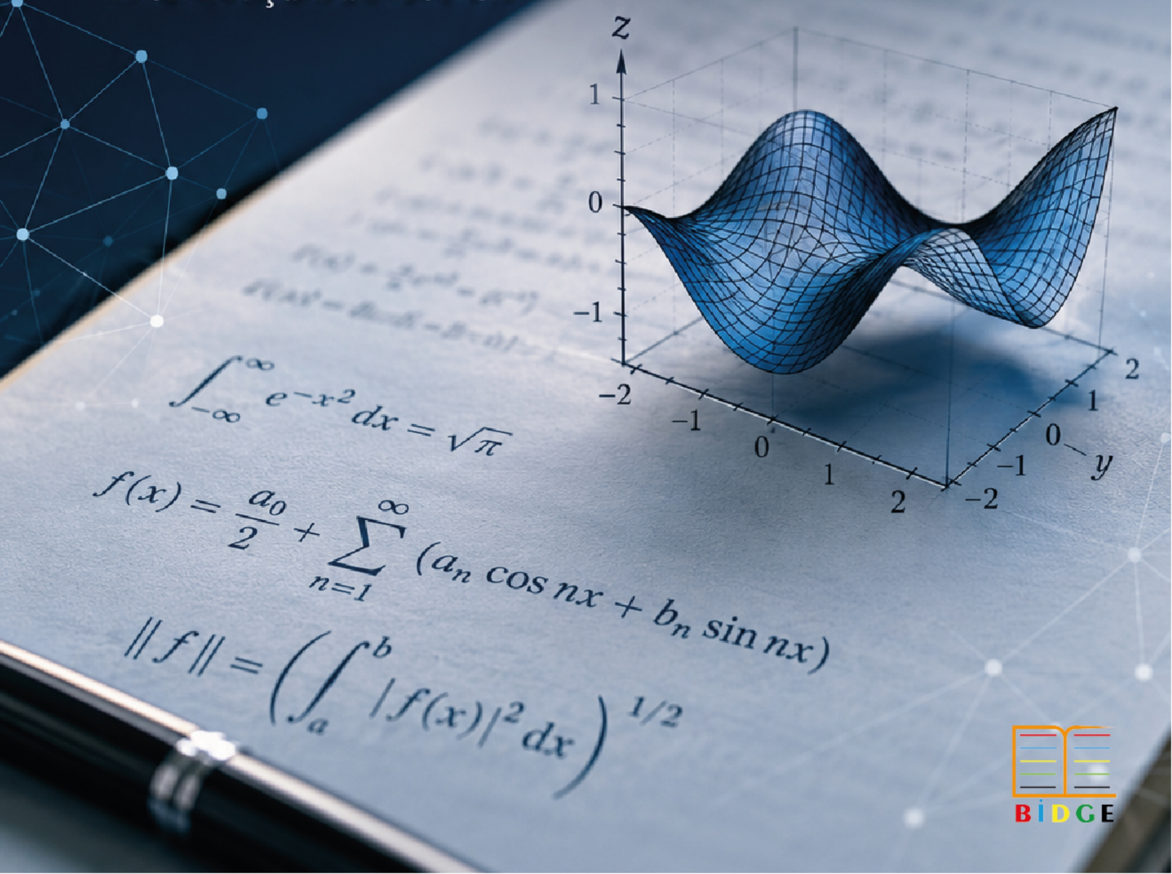

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FOREWORD

This book, "*Current Topics in Applied Mathematics and Analysis*" brings together current research topics in the fields of applied mathematics and mathematical analysis. The studies presented in this book combine both theoretical developments and applied mathematical methods, reflecting interdisciplinary interaction.

We thank everyone who contributed to the creation of this book and hope that it will be a useful resource for graduate students and researchers and contribute to future studies.

Prof. Dr. Şükran KONCA

CONTENTS

CHAPTER 1

A Multiple Scales Method for Solving the Nonlinear Sawada-Kotera Equation

MURAT KOPARAN[1]

Introduction

The multiple scales expansion method is a perturbation method. First applied (Zakharov, Kuznetsov,1986), the multiscale expansion method was developed by reducing the KdV equation to the NLS equation and applying it to a class of nonlinear formation equations. Using this method, they showed that integrable systems can be reduced to other integrable systems. Even if the system we initially consider is not integrable, the reduced system obtained by applying the method is either integrable or non-integrable. However, if the method is applied to an integrable system, the system obtained at the end of the analysis is always an integrable system. This is the main purpose of applying the multiscale expansion method to integrable systems.

The Korteweg-de Vries (KdV) equation was derived in various physical contexts. The Korteweg-de Vries (KdV) equation can also be used to model the one-sided breeding of small-amplitude,

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long-wavelength gravity waves in a shallow channel. In all cases, the Korteweg-de Vries (KdV) equation is attained at a definite approximation degree, so it can't be accepted that it represents physical reality with perfect accuracy. Thus, one important question that emerges is what happens to the solutions of Korteweg-de Vries (KdV) when perturbations, resulting from the terms neglected in its derivation, are applied. For example, whether the perturbed Korteweg-de Vries (KdV) equation has a solitary-wave-type solution can be asked. The answer would be related to the physical origin and nature of the perturbation. The theory of nonlinear dispersive wave motion has recently been the subject of extensive study. We do not attempt to characterize the general form of nonlinear dispersive wave equations (Wazwaz, 2002; Wazwaz, 2008; Groves, 2004). Nonlinear phenomena play a crucial role in applied mathematics and physics (Ablowitz, 2011; Bruckner, Düll, Schneider, 2014; Hong, Kwak, Yang, 2022; Debnath, 1997).

The multiple scales analysis of the Korteweg-de Vries (KdV) equation causes the Nonlinear Schrödinger (NLS) equation for the regulated amplitude to be widely known (Calegora, Degasperis, Xiaosdo, 2001; Degasperis, 1997; Osborne, Boffetta, 1989; Osborne, Boffetta, 1991). In (Zakharov, Kuznetsov, 1986), a much deeper correspondence between these integrable equations not only at the level of the equation, but also at the level of the linear spectral problem, by showing that a multiple scales analysis of the Schrödinger spectral problem leads to the Zakharov-Shabat problem for the Nonlinear Schrödinger (NLS) equation. The nonlinear Schrödinger (NLS) equation is an example of a universal nonlinear model because it describes a wide variety of physical systems. As a result, the equation may be used to describe a wide range of nonlinear physical events (Seadawy, 2012; Liu, Kengne, 2019).

1 Multiple Scales Method

In this section, we consider the application of the multiple scales method to NEEs. Using the technique of (Zakharov, Kuznetsov,1986), getting the NLS-type equations from KdV-type equations was shown step by step.

Let us consider the general evolution equation in the following form.

$$u_t = K(u, u_x, u_y \ldots) \tag{1}$$

where $K[u]$ is a function of u and its derivatives with respect to the x-spatial variables. The

well-known example of this type is the KdV equation.

Let $L[\partial_x, \partial_y]u$ be the linear part of $K[u]$. So, using, $K[u]$ we can reach the dispersion relation for Eq. (1). Substituting the wave solution space

$$\begin{aligned} u_k &= Ae^{i(kx+ry-\omega(k,r)t)} \\ &\equiv Ae^{i\theta} \end{aligned} \tag{2}$$

into the linear part of Eq. (1)

$$u_t = L[\partial_x, \partial_y]u \tag{3}$$

we get the dispersion relation

$$\omega(k,r) = iL[ik, ir] \tag{4}$$

Then, the dispersion relation (4) is substituted in Eq. (1). We assume the following series expansions for the solution of Eq. (1):

$$u(x,y,t)=\sum_{n=1}^{\infty}\varepsilon^{n}U_{n}(x,t,\xi,\tau) \tag{5}$$

Based on this solution, we also define slow space ξ and multiple time variables τ with respect to the scaling parameter $\varepsilon > 0$, respectively, as follows.

$$\begin{aligned}\xi &= \varepsilon\left(x-\frac{d\omega(k,r)}{dk}t\right)\\ \tau &= -\tfrac{1}{2}\varepsilon^{2}\left(\frac{d^{2}\omega(k,r)}{dk^{2}}\right)t\end{aligned} \tag{6}$$

A nonlinear equation modulates the amplitude of this plane wave solution in such a way that it may be considered dependent upon the slow variables. If we choose the slow variables' different forms, we can derive higher-order NLS equations. The multiple scales analysis starts with the assumption:

$$u(x,y,t)=U(x,y,t,\xi,\tau) \tag{7}$$

and solution is in the form

$$U(x, y, t, \xi, \tau) = (\varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \ldots) \tag{8}$$

Then, considering transformation (7) and solution (8), using dispersion relation (4) and slow variables (6), we get u and its derivatives with respect to ε in Eq. (1). And we substitute these terms with (7) and (8) in the Eq. (1). Collecting all terms with the same order of ε together, the left-hand side of Eq. (1) is converted into a polynomial in ε. Then setting each coefficient of this polynomial to zero, we get a set of algebraic equations. Using wave solution space (2) and dispersion relation (4), these equations can be solved by iteration and by use of Maple. So, we can get NLS-type equations from Eq. (1). Also, from this procedure, we can reach numerical solutions of fKdV-type equations.

2 The Fifth Order KdV Equation (fKdV)

The multiple scales expansion method is used to study the standard fifth-order KdV equation (fKdV) of the form (Wazwaz, 2006; Ito, 1980)

$$u_t + u_{xxxxx} + \gamma u u_{xxx} + \beta\, u_x u_{xx} + \alpha u^2 u_x = 0 \tag{9}$$

where α, β, and γ are arbitrary nonzero and real parameters, and u = u(x, t) is a sufficiently often differentiable function. The parameters α, β, and γ are arbitrary, which will drastically change the characteristics of the fKdV equation (9). Lots of forms of the fKdV equation can be constructed by changing these parameters, but four forms of the fKdV that are of particular interest, namely:

The Lax equation (Lax,1968) is given by

$$u_t + u_{xxxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x = 0 \tag{10}$$

and characterized by β = 2 γ, $\alpha = \frac{3}{10}\gamma^2$.

The Sawada–Kotera (SK) (Sawada, Kotera, 1974) equations is given by

$$u_t + u_{xxxxx} + 5uu_{xxx} + 5u_xu_{xx} + 5u^2u_x = 0 \tag{11}$$

and characterized by β = γ, $\alpha = \frac{1}{5}\gamma^2$.

The Sawada–Kotera–Parker–Dye (SKPD) equation

$$u_t + u_{xxxxx} - 15uu_{xxx} - 15u_xu_{xx} + 45u^2u_x = 0 \tag{12}$$

obtained by (Parker, Dye, 2002) where γ = -15 was selected to make β = -15 and α = 45 to obtain a second form of the SK equation.

The Kaup–Kupershmidt (Kaup, 1980; Kupershmidt, 1984) equation

$$u_t + u_{xxxxx} + 10uu_{xxx} + 25\, u_xu_{xx} + 20u^2u_x = 0 \tag{13}$$

and characterized by $\beta = \frac{2}{5}\gamma$, $\alpha = \frac{1}{5}\gamma^2$.

The Kaup–Kupershmidt–Parker–Dye equation (KKPD) (Parker, Dye, 2002; Kaup, 1980; Kupershmidt, 1984) equation

$$u_t + u_{xxxxx} - 15uu_{xxx} - \frac{75}{2}u_xu_{xx} + 45u^2u_x = 0 \tag{14}$$

established by (Parker, Dye, 2002) where γ = -15 was selected that makes $\beta = -\frac{75}{2}$ and α = 45 to obtain another form for the KK equation.

The Ito equation (Ito,1980)

$$u_t + u_{xxxxx} + 3uu_{xxx} + 6\, u_x u_{xx} + 2u^2 u_x = 0 \tag{15}$$

and characterized by β = 2 γ, $\alpha = \frac{2}{9}\gamma^2$.

It was found that the Lax, SK, and KK equations belong to the completely integrable hierarchy of higher-order KdV equations. These three equations have infinite sets of conservation laws (Göktaş, Hereman, 1997; Hereman, Nuseir, 1997). However, the Ito equation is not completely integrable but has a limited number of conservation laws (Wazwaz, 2007).

3 Applications

In this section, we will apply the multiscale expansion method to the (1+1)-dimensional fifth-order nonlinear Sawada-Kotera (SK) (11) where u_{kx} denotes the partial derivative $\partial^k u / \partial x^k$ equation.

To find the dispersion relation for (11), we consider the linear part of (11) in the form

$$u_t = u_{xxxxx} \tag{16}$$

and linear differential equation (16) satisfies the solution

$$u(x,t) = e^{i\theta}, \theta = kx - w(k)t \tag{17}$$

Substituting the solution (17) into the linear differential equation (16), we get

$$we^{i(kx-wt)} = k^5 e^{i(kx-wt)} \tag{18}$$

and from this, we reach the

$$w(k) = k^5 \tag{19}$$

dispersion relation. Thus, the solution of linear differential equation (16) is as follows:

$$u(x,t) = e^{i(kx-k^5 t)} \tag{20}$$

Let the solution of Eq. (11) be in the form

$$u(x,t) = U(x,t,\xi,\tau) \quad , \quad U(x,t,\xi,\tau) = \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \ldots \tag{21}$$

and slow variables are in the form

$$\begin{aligned} \xi &= \varepsilon(x - \tfrac{dw(k)}{dk} t) \\ &= \varepsilon(x - 5k^4 t) \\ \tau &= -\tfrac{1}{2}\varepsilon^2 (\tfrac{d^2 w(k)}{dk^2} t) \\ &= -10\varepsilon^2 k^3 t \end{aligned} \tag{22}$$

In this case, the derivative terms in equation (11) are obtained as follows using (20-22).

$$
\begin{aligned}
D_x &= \partial_x + \varepsilon\partial_\xi \\
D_t &= \partial_t - 5\varepsilon k^4\partial_\xi - 10\varepsilon^2 k^3\partial_\tau \\
D_{xx} &= \partial_{xx} + 2\varepsilon\partial_{x\xi} + \varepsilon^2\partial_{\xi\xi} \\
D_{xxx} &= \partial_{xxx} + 3\varepsilon\partial_{xx\xi} + 3\varepsilon^2\partial_{x\xi\xi} + \varepsilon^3\partial_{\xi\xi\xi} \\
D_{xxxx} &= \partial_{xxxx} + 4\varepsilon\partial_{xxx\xi} + 6\varepsilon^2\partial_{xx\xi\xi} + 4\varepsilon^3\partial_{x\xi\xi\xi} + \varepsilon^4\partial_{\xi\xi\xi\xi} \\
D_{xxxxx} &= \partial_{xxxxx} + 5\varepsilon\partial_{xxxx\xi} + 10\varepsilon^2\partial_{xxx\xi\xi} + 10\varepsilon^3\partial_{xx\xi\xi\xi} \\
&\quad + 5\varepsilon^4\partial_{x\xi\xi\xi\xi} + \varepsilon^5\partial_{\xi\xi\xi\xi\xi}
\end{aligned} \tag{23}
$$

Then, taking into account the transformation (11) and the solution (21), using the dispersion relation (19) and the slow variables (22), we obtain u and its derivatives with respect to ε in Eq. (11). We replace these terms with (20) and (21) in the Eq. (11). By adding together all terms with the same ε-order, the left-hand side of Eq. (11) is turned into a polynomial in ε. After setting each coefficient of this polynomial equal to zero, we obtain a set of algebraic equations. If we let $\varepsilon \to 0$ and vanish the terms at minimal powers of epsilon, by considering the case $n \geq 1$, we obtain the following:

$$\varepsilon \quad = \quad u_{1t} + u_{1xxxxx} = 0 \tag{24}$$

$$\varepsilon^2 \quad = \quad u_{2t} - 5k^4 u_{1\xi} + 5u_1 u_{1xxx} + 5u_{1x}u_{1xx} + u_{2xxxxx} + 5u_{1xxxx\xi} = 0 \tag{25}$$

$$\varepsilon^3 \quad = \quad u_{3t} - 5k^4 u_{2\xi} - 10k^3 u_{1\tau} + 5u_1^2 u_{1x} + 10u_{1xxx\xi\xi}$$
$$+5u_1\left(u_{2xxx} + 3u_{1xx\xi}\right) + 5u_2 u_{1xxx} - u_{3xxxxxx} \qquad (26)$$
$$+5u_{1x}\left(u_{2xx} + 2u_{1x\xi}\right) + 5u_{1xx}\left(u_{2x} + u_{1\xi}\right) + 5u_{2xxxx\xi} = 0$$

Then, we can find the solution of (24) as follows

$$u_1(x,t,\xi,\tau) = v_1(\xi,\tau)e^{i(kx-k^5t)} + v_{-1}(\xi,\tau)e^{-i(kx-k^5t)} \qquad (27)$$

where v_{-1} is complex conjugate of v_1. Substituting the solution (27) into (25), the solution of (25) is in the form,

$$u_2(x,t,\xi,\tau) = v_2(\xi,\tau)e^{2i(kx-k^5t)} + v_{-2}(\xi,\tau)e^{-2i(kx-k^5t)} + f_1(\xi,\tau) \qquad (28)$$

where $f_1(\xi,\tau)$ is integration constant. Thus, we get

$$v_2(\xi,\tau) \quad = \quad \frac{v_1^2(\xi,\tau)}{3k^2} \quad , \quad v_{-2}(\xi,\tau) \quad = \quad \frac{v_{-1}^2(\xi,\tau)}{3k^2} \qquad (29)$$

where v_{-1} is the complex conjugate of v_1 and v_{-2} is the complex conjugate of v_2. Substituting solutions (26), (28), and (29) into the (26), we find the solution of (26) in the form

$$u_3(x,t,\xi,\tau) \quad = \quad v_3(\xi,\tau)e^{3i(kx-k^5t)} + v_{-3}(\xi,\tau)e^{-3i(kx-k^5t)}$$
$$+ f_2(\xi,\tau)e^{2i(kx-k^5t)} + f_3(\xi,\tau)e^{-2i(kx-k^5t)} \qquad (30)$$

where $f_2(\xi,\tau)$ and $f_3(\xi,\tau)$ are integration constants. Then we get

$$v_3(\xi,\tau) = \frac{-10v_1^3(\xi,\tau)}{123k^4} \quad , \quad v_{-3}(\xi,\tau) = \frac{-10v_{-1}^3(\xi,\tau)}{123k^4} \tag{31}$$

and

$$\begin{aligned} f_1(\xi,\tau) &= \frac{3v_1 v_{-1}}{k^2} \\ f_2(\xi,\tau) &= \frac{10iv_{-1}v_{-1\xi}}{17k^3} \\ f_3(\xi,\tau) &= -\frac{10iv_1 v_{1\xi}}{17k^3} \end{aligned} \tag{32}$$

where v_{-3} and f_3 are the complex conjugates of v_3 and, f_2 respectively. Thus, the solutions of (31) - (32) are obtained as

$$\begin{aligned} u_1 &= v_1(\xi,\tau)e^{i\theta} + v_{-1}(\xi,\tau)e^{-i\theta} \\ u_2 &= k^{-2}(3v_1(\xi,\tau)v_{-1}(\xi,\tau) + \tfrac{1}{3}(v_1^2(\xi,\tau)e^{2i\theta} + v_{-1}^2(\xi,\tau)e^{-2i\theta})) \\ u_3 &= k^{-3}(\tfrac{10}{17}iv_{-1}(\xi,\tau)v_{-1\xi}(\xi,\tau)e^{-2i\theta} - \tfrac{10}{17}iv_1(\xi,\tau)v_{1\xi}(\xi,\tau)e^{2i\theta}) \\ &\quad - \tfrac{10}{123}k^{-4}(v_1^3(\xi,\tau)e^{3i\theta} + v_{-1}^3(\xi,\tau)e^{-3i\theta}) \end{aligned} \tag{33}$$

where

$$\theta = \left(kx - k^5 t\right) \tag{34}$$

Finally, substituting the solutions (33) into (26), we get

$$iv_{1\tau} = v_{1\xi\xi} - \tfrac{2}{k^2} v_1 v_{-1}$$

$$iv_{1\tau} = v_{1\xi\xi} - \tfrac{2}{k^2} v_1 \left|v_1\right|^2 \tag{35}$$

Describing as $q = \frac{v_1}{k}$ and $q_{-1} = \frac{v_{-1}}{k}$ (35) equation we get the $(1+1)$-dimensional NLS type equations in the form

$$iq_\tau = q_{\xi\xi} - 2q\left|q^2\right| \tag{36}$$

Also, the numerical solution of the (1+1)-dimensional Sawada-Kotera (SK) equation (11) is found as

$$\begin{aligned} u(x,t) = {} & \varepsilon k(q(\xi,\tau)e^{i(kx-k^5t)} + q_{-1}(\xi,\tau)e^{-i(kx-k^5t)}) \\ & +\varepsilon^2(3q(\xi,\tau)q_{-1}(\xi,\tau) + \tfrac{1}{3}q^2(\xi,\tau)e^{2i(kx-k^5t)} \\ & +\tfrac{1}{3}q_{-1}^2(\xi,\tau)e^{-2i(kx-k^5t)}) + k\varepsilon^3(\tfrac{10}{17}iq_{-1}(\xi,\tau)q_{-1\xi}(\xi,\tau)e^{-2i(kx-k^5t)} \\ & -\tfrac{10}{17}iq(\xi,\tau)q_\xi(\xi,\tau)e^{2i(kx-k^5t)}) \\ & -\tfrac{10}{123}k\varepsilon^3(q_1^3(\xi,\tau)e^{3i(kx-k^5t)} + q_{-1}^3(\xi,\tau)e^{-3i(kx-k^5t)}) \end{aligned} \tag{37}$$

where q is the solution of the NLS equation.

4 Results and Discussions

The multiple scales method, which is a perturbation method, is used to find approximate solutions of nonlinear evolution equations. This method allows us to find the solution of the given nonlinear equation depending on the solution of the linear part by using multiple scales, which are defined as the slow variables and depend on a parameter, in time and space variables. First, Zakharov and Kuznetsov showed that integrable systems can be reduced to other integrable systems using this method. If the initially taken system is not integrable, it is seen that the reduced system is either integrable or non-integrable as a result of the application of the method. However, if the method is applied to an integrable system, it is seen that the system obtained as a result of the analysis is always integrable. This is the main purpose of applying multiple-scale methods to integrable systems. In this study, we examined how only NLS-type equations are derived from fKdV-type equations using the multiple scales method. We hope this conclusion serves as a valuable resource for researchers, scientists, and engineers interested in nonlinear wave theory, inspiring a deeper desire to explore further and reveal the mysteries still held within the intricate language of the NLS equation. At the same time, we think that the results obtained will form the basis of numerical calculations. Also, the method can be applied to many different NLEE equations.

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CHAPTER 2

Multiple Scales Method for the Lax Equation

MURAT KOPARAN[1]

Introduction

The multiple scales expansion method is a perturbation method. First applied by (Zakharov, Kuznetsov,1986), the multiscale expansion method was developed by reducing the KdV equation to the NLS equation and applying it to a class of nonlinear formation equations. Using this method, they showed that integrable systems can be reduced to other integrable systems. Even if the system we initially take is not an integrable system, the system reduced as a result of applying the method is either integrable or non-integrable. However, if the method is applied to an integrable system, the system obtained at the end of the analysis is always an integrable system. This is the main purpose of applying the multiscale expansion method to integrable systems.

The Korteweg-de Vries (KdV) equation was derived in various physical contexts. The Korteweg-de Vries (KdV) equation can also be used to model the one-sided breeding of small-amplitude, long-wavelength gravity waves in a shallow channel. In all cases, the

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Korteweg-de Vries (KdV) equation is attained at a definite approximation degree, so it can't be accepted that it represents physical reality with perfect accuracy. Thus, one important question that emerges is what happens to the solutions of Korteweg-de Vries (KdV) when perturbations, resulting from the terms neglected in its derivation, are applied. For example, whether the perturbed Korteweg-de Vries (KdV) equation has a solitary-wave-type solution can be asked. The answer would be related to the physical origin and nature of the perturbation. The theory of nonlinear dispersive wave motion has recently undergone much study. We do not attempt to characterize the general form of nonlinear dispersive wave equations (Wazwaz, 2002; Wazwaz, 2008; Groves, 2004). Nonlinear phenomena play a crucial role in applied mathematics and physics (Ablowitz, 2011; Bruckner, Düll, Schneider, 2014; Hong, Kwak, Yang, 2022; Debnath, 1997).

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1 Multiple Scales Method

In this section, we consider the application of the multiple scales method to NEEs. Using the technique of Zakharov and Kuznetsov [3], getting the NLS-type equations from KdV-type equations was shown step by step.

Let us consider the general evolution equation in the following form.

$$u_t = K(u, u_x, u_y \ldots) \tag{1}$$

where $K[u]$ is a function of u and its derivatives with respect to the x-spatial variables. The

well-known example of this type is the KdV equation.

Let $L[\partial_x, \partial_y]u$ be the linear part of $K[u]$. So, using, $K[u]$ we can reach the dispersion relation for Eq. (1). Substituting the wave solution space

$$\begin{aligned} u_k &= Ae^{i(kx+ry-\omega(k,r)t)} \\ &\equiv Ae^{i\theta} \end{aligned} \tag{2}$$

into the linear part of Eq. (1)

$$u_t = L[\partial_x, \partial_y]u \tag{3}$$

we get the dispersion relation

$$\omega(k,r) = iL[ik, ir] \tag{4}$$

Then, the dispersion relation (4) is substituted in Eq. (1). We assume the following series expansions for the solution of Eq. (1):

$$u(x,y,t)=\sum_{n=1}^{\infty}\varepsilon^n U_n(x,t,\xi,\tau) \tag{5}$$

Based on this solution, we also define slow space ξ and multiple time variables τ with respect to the scaling parameter ε > 0, respectively, as follows.

$$\begin{aligned}\xi &= \varepsilon\left(x-\frac{d\omega(k,r)}{dk}t\right)\\ \tau &= -\tfrac{1}{2}\varepsilon^2\left(\frac{d^2\omega(k,r)}{dk^2}\right)t\end{aligned} \tag{6}$$

A nonlinear equation modulates the amplitude of this plane wave solution in such a way that it may be considered dependent upon the slow variables. If we choose the slow variables' different forms, we can derive higher-order NLS equations. The multiple scales analysis starts with the assumption:

$$u(x,y,t)=U(x,y,t,\xi,\tau) \tag{7}$$

and solution is in the form

$$U(x, y, t, \xi, \tau) = (\varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \ldots) \tag{8}$$

Then, considering transformation (7) and solution (8), using dispersion relation (4) and slow variables (6), we get u and its derivatives with respect to ε in Eq. (1). And we substitute these terms with (7) and (8) in the Eq. (1). Collecting all terms with the same order of ε together, the left-hand side of Eq. (1) is converted into a polynomial in ε. Then setting each coefficient of this polynomial to zero, we get a set of algebraic equations. Using wave solution space (2) and dispersion relation (4), these equations can be solved by iteration and by use of Maple. So, we can get NLS-type equations from Eq. (1). Also, from this procedure, we can reach numerical solutions of fKdV-type equations.

2 The Fifth Order KdV Equation (fKdV)

The multiple scales expansion method is used to study the standard fifth-order KdV equation (fKdV) of the form (Wazwaz, 2006; Ito, 1980)

$$u_t + u_{xxxxx} + \gamma u u_{xxx} + \beta\, u_x u_{xx} + \alpha u^2 u_x = 0 \tag{9}$$

where α, β, and γ are arbitrary nonzero and real parameters, and u = u(x, t) is a sufficiently often differentiable function. The parameters α, β, and γ are arbitrary, which will drastically change the characteristics of the fKdV equation (9). Lots of forms of the fKdV equation can be constructed by changing these parameters, but four forms of the fKdV that are of particular interest, namely:

The Lax equation (Lax,1968) is given by

$$u_t + u_{xxxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x = 0 \tag{10}$$

and characterized by β = 2 γ, $\alpha = \frac{3}{10}\gamma^2$.

The Sawada–Kotera (SK) (Sawada, Kotera, 1974) equations is given by

$$u_t + u_{xxxxx} + 5uu_{xxx} + 5u_xu_{xx} + 5u^2u_x = 0 \tag{11}$$

and characterized by β = γ, $\alpha = \frac{1}{5}\gamma^2$.

The Sawada–Kotera–Parker–Dye (SKPD) equation

$$u_t + u_{xxxxx} - 15uu_{xxx} - 15u_xu_{xx} + 45u^2u_x = 0 \tag{12}$$

obtained by (Parker, Dye, 2002) where γ = -15 was selected to make β = -15 and α = 45 to obtain a second form of the SK equation.

The Kaup–Kupershmidt (Kaup, 1980; Kupershmidt, 1984) equation

$$u_t + u_{xxxxx} + 10uu_{xxx} + 25\,u_xu_{xx} + 20u^2u_x = 0 \tag{13}$$

and characterized by $\beta = \frac{2}{5}\gamma$, $\alpha = \frac{1}{5}\gamma^2$.

The Kaup–Kupershmidt–Parker–Dye equation (KKPD) (Parker, Dye, 2002; Kaup, 1980; Kupershmidt, 1984) equation

$$u_t + u_{xxxxx} - 15uu_{xxx} - \frac{75}{2}u_xu_{xx} + 45u^2u_x = 0 \tag{14}$$

established by (Parker, Dye, 2002) where γ = -15 was selected that makes $\beta = -\frac{75}{2}$ and α = 45 to obtain another form for the KK equation.

The Ito equation (Ito,1980)

$$u_t + u_{xxxxx} + 3uu_{xxx} + 6\,u_x u_{xx} + 2u^2 u_x = 0 \tag{15}$$

and characterized by $\beta = 2\,\gamma$, $\alpha = \frac{2}{9}\gamma^2$.

It was found that the Lax, SK, and KK equations belong to the completely integrable hierarchy of higher-order KdV equations. These three equations have infinite sets of conservation laws (Göktaş, Hereman, 1997; Hereman, Nuseir, 1997). However, the Ito equation is not completely integrable but has a limited number of conservation laws (Wazwaz, 2007).

3 Applications

In this section, we will apply the multiscale expansion method to the (1+1)-dimensional fifth-order nonlinear Lax equation (10), where u_{kx} denotes the partial derivative $\partial^k u / \partial x^k$ equation.

To find the dispersion relation for (10), we consider the linear part of (10) in the form

$$u_t = u_{xxxxx} \tag{16}$$

and linear differential equation (16) satisfies the solution

$$u(x,t) = e^{i\theta}, \theta = kx - w(k)t \tag{17}$$

Substituting the solution (17) into the linear differential equation (16), we get

$$we^{i(kx-wt)} = k^5 e^{i(kx-wt)} \tag{18}$$

and from this, we reach the

$$w(k) = k^5 \tag{19}$$

dispersion relation. Thus, the solution of linear differential equation (16) is as follows:

$$u(x,t) = e^{i(kx-k^5t)} \tag{20}$$

Let the solution of Eq. (11) be in the form

$$u(x,t)=U(x,t,\xi,\tau) \quad , \quad U(x,t,\xi,\tau)=\varepsilon U_1+\varepsilon^2 U_2+\varepsilon^3 U_3+... \tag{21}$$

and slow variables are in the form

$$\begin{aligned}
\xi &= \varepsilon(x-\tfrac{dw(k)}{dk}t) \\
&= \varepsilon(x-5k^4t) \\
\tau &= -\tfrac{1}{2}\varepsilon^2(\tfrac{d^2w(k)}{dk^2}t) \\
&= -10\varepsilon^2k^3t
\end{aligned} \tag{22}$$

In this case, the derivative terms in equation (11) are obtained as follows using (20-22).

$$D_x = \partial_x + \varepsilon\partial_\xi$$

$$D_t = \partial_t - 5\varepsilon k^4\partial_\xi - 10\varepsilon^2 k^3\partial_\tau$$

$$D_{xx} = \partial_{xx} + 2\varepsilon\partial_{x\xi} + \varepsilon^2\partial_{\xi\xi}$$

$$D_{xxx} = \partial_{xxx} + 3\varepsilon\partial_{xx\xi} + 3\varepsilon^2\partial_{x\xi\xi} + \varepsilon^3\partial_{\xi\xi\xi}$$

$$D_{xxxx} = \partial_{xxxx} + 4\varepsilon\partial_{xxx\xi} + 6\varepsilon^2\partial_{xx\xi\xi} + 4\varepsilon^3\partial_{x\xi\xi\xi} + \varepsilon^4\partial_{\xi\xi\xi\xi}$$

$$D_{xxxxx} = \partial_{xxxxx} + 5\varepsilon\partial_{xxxx\xi} + 10\varepsilon^2\partial_{xxx\xi\xi} + 10\varepsilon^3\partial_{xx\xi\xi\xi} + 5\varepsilon^4\partial_{x\xi\xi\xi\xi} + \varepsilon^5\partial_{\xi\xi\xi\xi\xi} \tag{23}$$

Then, taking into account the transformation (10) and the solution (21), using the dispersion relation (19) and the slow variables (22), we obtain u and its derivatives with respect to ε in Eq. (10). We replace these terms with (20) and (21) in the Eq. (10). By adding together all terms with the same ε-order, the left-hand side of Eq. (10) is turned into a polynomial in ε. After setting each coefficient of this polynomial equal to zero, we obtain a set of algebraic equations. If we let $\varepsilon \to 0$ and vanish the terms at minimal powers of epsilon, by considering the case $n \geq 1$, we obtain the following:

$$\varepsilon = u_{1t} + u_{1xxxxx} = 0 \tag{24}$$

$$\varepsilon^2 = u_{2t} - 5k^4 u_{1\xi} + 10u_1 u_{1xxx} + 20u_{1x}u_{1xx} + u_{2xxxxx} + 5u_{1xxxx\xi} = 0 \tag{25}$$

$$\varepsilon^3 \quad = \quad u_{3t} - 5k^4 u_{2\xi} - 10k^3 u_{1\tau} + 30u_1^2 u_{1x} + 10u_{1xxx\xi\xi}$$
$$+10u_1\left(u_{2xxx} + 3u_{1xx\xi}\right) + 10u_2 u_{1xxx} - u_{3xxxxx} \qquad (26)$$
$$+20u_{1x}\left(u_{2xx} + 2u_{1x\xi}\right) + 20u_{1xx}\left(u_{2x} + u_{1\xi}\right) + 5u_{2xxxx\xi} = 0$$

Then, we can find the solution of (24) as follows

$$u_1(x,t,\xi,\tau) = v_1(\xi,\tau)e^{i(kx-k^5t)} + v_{-1}(\xi,\tau)e^{-i(kx-k^5t)} \qquad (27)$$

where v_{-1} is the complex conjugate of v_1. Substituting the solution (27) into (25), the solution of (25) is in the form,

$$u_2(x,t,\xi,\tau) = v_2(\xi,\tau)e^{2i(kx-k^5t)} + v_{-2}(\xi,\tau)e^{-2i(kx-k^5t)} + f_1(\xi,\tau) \qquad (28)$$

where $f_1(\xi,\tau)$ is an integration constant. Thus, we get

$$v_2(\xi,\tau) \quad = \quad \frac{v_1^2(\xi,\tau)}{k^2} \quad , \quad v_{-2}(\xi,\tau) \quad = \quad \frac{v_{-1}^2(\xi,\tau)}{k^2} \qquad (29)$$

where v_{-1} is the complex conjugate of v_1 and v_{-2} is the complex conjugate of v_2. Substituting solutions (26), (28), and (29) into (26), we find the solution of (26) in the form

$$u_3(x,t,\xi,\tau) \quad = \quad v_3(\xi,\tau)e^{3i(kx-k^5t)} + v_{-3}(\xi,\tau)e^{-3i(kx-k^5t)}$$
$$+ f_2(\xi,\tau)e^{2i(kx-k^5t)} + f_3(\xi,\tau)e^{-2i(kx-k^5t)} \qquad (30)$$

where $f_2(\xi,\tau)$ and $f_3(\xi,\tau)$ are integration constants. Then we get

$$v_3(\xi,\tau) = \frac{-3v_1^3(\xi,\tau)}{4k^4}, \quad v_{-3}(\xi,\tau) = \frac{-3v_{-1}^3(\xi,\tau)}{4k^4} \tag{31}$$

and

$$\begin{aligned} f_1(\xi,\tau) &= \frac{2v_1 v_{-1}}{k^2} \\ f_2(\xi,\tau) &= \frac{2iv_{-1}v_{-1\xi}}{k^3} \\ f_3(\xi,\tau) &= -\frac{2iv_1 v_{1\xi}}{k^3} \end{aligned} \tag{32}$$

where v_{-3} and f_3 are the complex conjugates of v_3 and, f_2 respectively. Thus, the solutions of (31) - (32) are obtained as

$$\begin{aligned} u_1 &= v_1(\xi,\tau)e^{i\theta} + v_{-1}(\xi,\tau)e^{-i\theta} \\ u_2 &= k^{-2}(2v_1(\xi,\tau)v_{-1}(\xi,\tau) + 2(v_1^2(\xi,\tau)e^{2i\theta} + v_{-1}^2(\xi,\tau)e^{-2i\theta})) \\ u_3 &= k^{-3}(2iv_{-1}(\xi,\tau)v_{-1\xi}(\xi,\tau)e^{-2i\theta} - 2iv_1(\xi,\tau)v_{1\xi}(\xi,\tau)e^{2i\theta}) \\ &\quad -\frac{3}{4}k^{-4}(v_1^3(\xi,\tau)e^{3i\theta} + v_{-1}^3(\xi,\tau)e^{-3i\theta}) \end{aligned} \tag{33}$$

where

$$\theta = \left(kx - k^5 t\right) \tag{34}$$

Finally, substituting the solutions (33) into (26), we get

$$iv_{1\tau} = v_{1\xi\xi} - \tfrac{2}{k^2} v_1 v_{-1}$$

$$iv_{1\tau} = v_{1\xi\xi} - \tfrac{2}{k^2} v_1 \left|v_1\right|^2 \tag{35}$$

Describing as $q = \frac{v_1}{k}$ and $q_{-1} = \frac{v_{-1}}{k}$ (35) equation, we get the $(1+1)$-dimensional NLS-type equations in the form

$$iq_\tau = q_{\xi\xi} - 2q\left|q^2\right| \tag{36}$$

Also, the numerical solution of the (1+1)-dimensional Lax equation (10) is found as

$$\begin{aligned} u(x,t) \;=\; & \varepsilon k(q(\xi,\tau)e^{i(kx-k^5t)} + q_{-1}(\xi,\tau)e^{-i(kx-k^5t)}) \\ & +\varepsilon^2(2q(\xi,\tau)q_{-1}(\xi,\tau) + q^2(\xi,\tau)e^{2i(kx-k^5t)} \\ & +q_{-1}^2(\xi,\tau)e^{-2i(kx-k^5t)}) + k\varepsilon^3(2iq_{-1}(\xi,\tau)q_{-1\xi}(\xi,\tau)e^{-2i(kx-k^5t)} \\ & -2iq(\xi,\tau)q_\xi(\xi,\tau)e^{2i(kx-k^5t)}) \\ & -\tfrac{3}{4}k\varepsilon^3(q_1^3(\xi,\tau)e^{3i(kx-k^5t)} + q_{-1}^3(\xi,\tau)e^{-3i(kx-k^5t)}) \end{aligned} \tag{37}$$

where q is the solution of the NLS equation.

4 Results and Discussions

The multiple scales method, which is a perturbation method, is used to find approximate solutions of nonlinear evolution equations. This method allows us to find the solution of the given nonlinear equation depending on the solution of the linear part by using the multiple scales, which are defined as the slow variables and depend on a parameter, in time and space variables. First, Zakharov and Kuznetsov showed that integrable systems can be reduced to other integrable systems using this method. If the initially taken system is not integrable, it is seen that the reduced system is either integrable or non-integrable as a result of the application of the method. However, if the method is applied to an integrable system, it is seen that the system obtained as a result of the analysis is always integrable. This is the main purpose of applying the multiple-scale methods to integrable systems. In this study, we examined how only NLS-type equations are derived from fKdV-type equations using the multiple scales method. We hope this conclusion serves as a valuable resource for researchers, scientists, and engineers interested in nonlinear wave theory, inspiring a deeper desire to explore further and reveal the mysteries still held within the intricate language of the NLS equation. At the same time, we think that the obtained results will form the basis of numerical calculations. Also, the method can be applied to many different NLEE equations.

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CHAPTER 3

Approximate Solution of the Kaup-Kupershmidt (KK) equation

MURAT KOPARAN[1]

Introduction

The multiple-scale expansion method is a perturbation method. First applied (Zakharov, Kuznetsov,1986), the multiple scales expansion method was developed by reducing the KdV equation to the NLS equation and applying it to a class of nonlinear formation equations. Using this method, they showed that integrable systems can be reduced to other integrable systems. Even if the system we initially took is not an integrable system, the system reduced as a result of applying the method is either integrable or non-integrable. However, if the method is applied to an integrable system, the system obtained at the end of the analysis is always an integrable system. This is the main purpose of applying the multiscale expansion method to integrable systems.

Known for its distinguished role in the development of nonlinear physics, the Korteweg-de Vries (KdV) equation has been

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derived in a variety of physical contexts. For example, it can be thought of as modeling the unilateral propagation of long-wavelength gravitational waves of small amplitude in a shallow channel. In any case, the Korteweg-de Vries equation is obtained with a certain degree of approximation and, therefore, cannot be considered to represent physical reality with perfect accuracy. Therefore, an important question arises about what would happen to the solutions of the Korteweg-de Vries equation when perturbations from neglected terms in the derivation come into play. For instance, would the perturbed Korteweg-de Vries equation still have a well-localized single-wave solution? The answer, sure, would depend on the nature and physical origin of the perturbation. Currently, studies are underway on the nonlinear fifth-order KdV-type equations, as they can describe real properties in various scientific applications and engineering fields, and have practical and physical significance. The theory of nonlinear dispersive wave motion has recently been the subject of extensive study. We do not attempt to characterize the general form of nonlinear dispersive wave equations (Wazwaz, 2002; Wazwaz, 2008; Groves, 2004). Nonlinear phenomena play a crucial role in applied mathematics and physics (Ablowitz, 2011; Bruckner, Düll, Schneider, 2014; Hong, Kwak, Yang, 2022; Debnath, 1997).

The nonlinear Schrödinger (NLS) equation is an example of a universal nonlinear model because it describes a wide variety of physical systems. As a result, the equation may be used to describe a wide range of nonlinear physical events. The multiple scales analysis of the Korteweg-de Vries (KdV) equation causes the Nonlinear Schrödinger (NLS) equation for the regulated amplitude to be widely known (Calegora, Degasperis, Xiaosdo, 2001; Degasperis, 1997; Osborne, Boffetta, 1989; Osborne, Boffetta, 1991). In (Zakharov, Kuznetsov, 1986), a much deeper correspondence between these integrable equations is indicated, not only at the level of the equation, but also at the level of the linear spectral problem by showing that a multiple scales analysis of the Schrödinger spectral problem leads to the Zakharov-Shabat problem for the Nonlinear Schrödinger (NLS) equation. The nonlinear Schrödinger (NLS) equation is an example of a universal nonlinear model because it describes a wide variety of physical systems. As a result, the equation may be used to describe a wide range of nonlinear physical events (Seadawy, 2012; Liu, Kengne, 2019).

1 Multiple Scales Method

In this section, we consider the application of the multiple scales method to NEEs. Using the technique of (Zakharov, Kuznetsov, 1986), getting the NLS-type equations from KdV-type equations was shown step by step.

Let us consider the general evolution equation in the following form.

$$u_t = K(u, u_x, u_y \ldots) \tag{1}$$

where $K[u]$ is a function of u and its derivatives with respect to the x-spatial variables. The

well-known example of this type is the KdV equation.

Let $L[\partial_x, \partial_y]u$ be the linear part of $K[u]$. So, using, $K[u]$ we can reach the dispersion relation for Eq. (1). Substituting the wave solution space

$$\begin{aligned} u_k &= Ae^{i(kx+ry-\omega(k,r)t)} \\ &\equiv Ae^{i\theta} \end{aligned} \tag{2}$$

into the linear part of Eq. (1)

$$u_t = L\left\lfloor \partial_x, \partial_y \right\rfloor u \tag{3}$$

we get the dispersion relation

$$\omega(k,r) = iL[ik, ir] \tag{4}$$

Then, the dispersion relation (4) is substituted in Eq. (1). We assume the following series expansions for the solution of Eq. (1):

$$u(x,y,t) = \sum_{n=1}^{\infty} \varepsilon^n U_n(x,t,\xi,\tau) \tag{5}$$

Based on this solution, we also define slow space ξ and multiple time variables τ with respect to the scaling parameter $\varepsilon > 0$, respectively, as follows.

$$\begin{aligned} \xi &= \varepsilon\left(x - \frac{d\omega(k,r)}{dk} t \right) \\ \tau &= -\tfrac{1}{2}\varepsilon^2 \left(\frac{d^2\omega(k,r)}{dk^2} \right) t \end{aligned} \tag{6}$$

A nonlinear equation modulates the amplitude of this plane wave solution in such a way that it may be considered dependent upon the slow variables. If we choose the slow variables' different forms, we can derive higher-order NLS equations. The multiple scales analysis starts with the assumption:

$$u(x,y,t)=U(x,y,t,\xi,\tau) \tag{7}$$

and solution is in the form

$$U(x,y,t,\xi,\tau)=(\varepsilon U_1+\varepsilon^2 U_2+\varepsilon^3 U_3+...) \tag{8}$$

Then, considering transformation (7) and solution (8), using dispersion relation (4) and slow variables (6), we get u and its derivatives with respect to ε in Eq. (1). And we substitute these terms with (7) and (8) in the Eq. (1). Collecting all terms with the same order of ε together, the left-hand side of Eq. (1) is converted into a polynomial in ε. Then setting each coefficient of this polynomial to zero, we get a set of algebraic equations. Using wave solution space (2) and dispersion relation (4), these equations can be solved by iteration and by use of Maple. So, we can get NLS-type equations from Eq. (1). Also, from this procedure, we can reach numerical solutions of fKdV-type equations.

2 The Fifth Order KdV Equation (fKdV)

The multiple scales expansion method is used to study the standard fifth-order KdV equation (fKdV) of the form (Wazwaz, 2006; Ito, 1980)

$$u_t + u_{xxxxx} + \gamma u u_{xxx} + \beta\, u_x u_{xx} + \alpha u^2 u_x = 0 \qquad (9)$$

where α, β, and γ are arbitrary nonzero and real parameters, and u = u(x, t) is a sufficiently often differentiable function. The parameters α, β, and γ are arbitrary, which will drastically change the characteristics of the fKdV equation (9). Lots of forms of the fKdV equation can be constructed by changing these parameters, but four forms of the fKdV that are of particular interest, namely:

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and characterized by β = 2 γ, $\alpha = \frac{3}{10}\gamma^2$.

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and characterized by $\beta = \frac{2}{5}\gamma$, $\alpha = \frac{1}{5}\gamma^2$.

The Kaup–Kupershmidt–Parker–Dye equation (KKPD) (Parker, Dye, 2002; Kaup, 1980; Kupershmidt, 1984) equation

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$$u_t + u_{xxxxx} + 3uu_{xxx} + 6\, u_x u_{xx} + 2u^2 u_x = 0 \tag{15}$$

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It was found that the Lax, SK, and KK equations belong to the completely integrable hierarchy of higher-order KdV equations. These three equations have infinite sets of conservation laws (Göktaş, Hereman, 1997; Hereman, Nuseir, 1997). However, the Ito equation is not completely integrable but has a limited number of conservation laws (Wazwaz, 2007).

3 Applications

In this section, we will apply the multiscale expansion method to the (1+1)-dimensional fifth-order nonlinear Kaup–Kupershmidt (KK) (13), where u_{kx} denotes the partial derivative $\partial^k u / \partial x^k$ equation.

To find the dispersion relation for (11), we consider the linear part of (11) in the form

$$u_t = u_{xxxxx} \tag{16}$$

and linear differential equation (16) satisfies the solution

$$u(x,t) = e^{i\theta}, \theta = kx - w(k)t \tag{17}$$

Substituting the solution (17) into the linear differential equation (16), we get

$$we^{i(kx-wt)} = k^5 e^{i(kx-wt)} \tag{18}$$

and from this, we reach the

$$w(k) = k^5 \tag{19}$$

dispersion relation. Thus, the solution of linear differential equation (16) is as follows:

$$u(x,t) = e^{i(kx-k^5 t)} \tag{20}$$

Let the solution of Eq. (11) be in the form

$$u(x,t) = U(x,t,\xi,\tau) \quad , \quad U(x,t,\xi,\tau) = \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \ldots \tag{21}$$

and slow variables are in the form

$$
\begin{aligned}
\xi &= \varepsilon(x - \tfrac{dw(k)}{dk}t) \\
&= \varepsilon(x - 5k^4 t) \\
\tau &= -\tfrac{1}{2}\varepsilon^2(\tfrac{d^2w(k)}{dk^2}t) \\
&= -10\varepsilon^2 k^3 t
\end{aligned} \tag{22}
$$

In this case, the derivative terms in equation (11) are obtained as follows using (20-22).

$$
\begin{aligned}
D_x &= \partial_x + \varepsilon\partial_\xi \\
D_t &= \partial_t - 5\varepsilon k^4 \partial_\xi - 10\varepsilon^2 k^3 \partial_\tau \\
D_{xx} &= \partial_{xx} + 2\varepsilon\partial_{x\xi} + \varepsilon^2\partial_{\xi\xi} \\
D_{xxx} &= \partial_{xxx} + 3\varepsilon\partial_{xx\xi} + 3\varepsilon^2\partial_{x\xi\xi} + \varepsilon^3\partial_{\xi\xi\xi} \\
D_{xxxx} &= \partial_{xxxx} + 4\varepsilon\partial_{xxx\xi} + 6\varepsilon^2\partial_{xx\xi\xi} + 4\varepsilon^3\partial_{x\xi\xi\xi} + \varepsilon^4\partial_{\xi\xi\xi\xi} \\
D_{xxxxx} &= \partial_{xxxxx} + 5\varepsilon\partial_{xxxx\xi} + 10\varepsilon^2\partial_{xxx\xi\xi} + 10\varepsilon^3\partial_{xx\xi\xi\xi} \\
&\quad + 5\varepsilon^4\partial_{x\xi\xi\xi\xi} + \varepsilon^5\partial_{\xi\xi\xi\xi\xi}
\end{aligned} \tag{23}
$$

Then, taking into account the transformation (13) and the solution (21), using the dispersion relation (19) and the slow variables (22), we obtain u and its derivatives with respect to ε in Eq. (13). We replace these terms with (20) and (21) in the Eq. (11). By adding together all terms with the same ε-order, the left-hand side of Eq.

(13) is turned into a polynomial in ε. After setting each coefficient of this polynomial equal to zero, we obtain a set of algebraic equations. If we let $\varepsilon \to 0$ and vanish the terms at minimal powers of epsilon, by considering the case $n \geq 1$, we obtain the following:

$$\varepsilon \quad = \quad u_{1t} + u_{1xxxxx} = 0 \tag{24}$$

$$\varepsilon^2 \quad = \quad u_{2t} - 5k^4 u_{1\xi} + 10u_1 u_{1xxx} + 25u_{1x}u_{1xx} + u_{2xxxxx} + 5u_{1xxxx\xi} = 0 \tag{25}$$

$$\begin{aligned}\varepsilon^3 \quad = \quad & u_{3t} - 5k^4 u_{2\xi} - 10k^3 u_{1\tau} + 20u_1^2 u_{1x} + 10u_{1xxx\xi\xi} \\ & +5u_1\left(u_{2xxx} + 3u_{1xx\xi}\right) + 10u_2 u_{1xxx} - u_{3xxxxx} \\ & +25u_{1x}\left(u_{2xx} + 2u_{1x\xi}\right) + 25u_{1xx}\left(u_{2x} + u_{1\xi}\right) + 5u_{2xxxx\xi} = 0\end{aligned} \tag{26}$$

Then, we can find the solution of (24) as follows

$$u_1(x,t,\xi,\tau) = v_1(\xi,\tau)e^{i(kx-k^5t)} + v_{-1}(\xi,\tau)e^{-i(kx-k^5t)} \tag{27}$$

where v_{-1} is the complex conjugate of v_1. Substituting the solution (27) into (25), the solution of (25) is in the form,

$$u_2(x,t,\xi,\tau) = v_2(\xi,\tau)e^{2i(kx-k^5t)} + v_{-2}(\xi,\tau)e^{-2i(kx-k^5t)} + f_1(\xi,\tau) \tag{28}$$

where $f_1(\xi,\tau)$ is an integration constant. Thus, we get

$$v_2(\xi,\tau) \quad = \quad \frac{7v_1^2(\xi,\tau)}{6k^2} \quad , \quad v_{-2}(\xi,\tau) \quad = \quad \frac{7v_{-1}^2(\xi,\tau)}{6k^2} \tag{29}$$

where v_{-1} is the complex conjugate of v_1 and v_{-2} is the complex conjugate of v_2. Substituting solutions (26), (28), and (29) into (26), we find the solution of (26) in the form

$$\begin{aligned} u_3(x,t,\xi,\tau) &= v_3(\xi,\tau)e^{3i(kx-k^5t)} + v_{-3}(\xi,\tau)e^{-3i(kx-k^5t)} \\ &\quad + f_2(\xi,\tau)e^{2i(kx-k^5t)} + f_3(\xi,\tau)e^{-2i(kx-k^5t)} \end{aligned} \tag{30}$$

where $f_2(\xi,\tau)$ and $f_3(\xi,\tau)$ are integration constants. Then we get

$$v_3(\xi,\tau) = \frac{13v_1^3(\xi,\tau)}{12k^4}, \quad v_{-3}(\xi,\tau) = \frac{13v_{-1}^3(\xi,\tau)}{12k^4} \tag{31}$$

and

$$\begin{aligned} f_1(\xi,\tau) &= \frac{6v_1v_{-1}}{k^2} \\ f_2(\xi,\tau) &= \frac{-7iv_{-1}v_{-1\xi}}{3k^3} \\ f_3(\xi,\tau) &= \frac{7iv_1v_{1\xi}}{3k^3} \end{aligned} \tag{32}$$

where v_{-3} and f_3 are the complex conjugates of v_3 and, f_2 respectively. Thus, the solutions of (31) - (32) are obtained as

$$
\begin{aligned}
u_1 &= v_1(\xi,\tau)e^{i\theta} + v_{-1}(\xi,\tau)e^{-i\theta} \\
u_2 &= k^{-2}(6v_1(\xi,\tau)v_{-1}(\xi,\tau) + \tfrac{7}{6}(v_1^2(\xi,\tau)e^{2i\theta} + v_{-1}^2(\xi,\tau)e^{-2i\theta})) \\
u_3 &= k^{-3}(-\tfrac{7}{3}iv_{-1}(\xi,\tau)v_{-1\xi}(\xi,\tau)e^{-2i\theta} + \tfrac{7}{3}iv_1(\xi,\tau)v_{1\xi}(\xi,\tau)e^{2i\theta}) \\
&\quad + \tfrac{13}{12}k^{-4}(v_1^3(\xi,\tau)e^{3i\theta} + v_{-1}^3(\xi,\tau)e^{-3i\theta})
\end{aligned} \tag{33}
$$

where

$$\theta = \left(kx - k^5 t\right) \tag{34}$$

Finally, substituting the solutions (33) into (26), we get

$$
\begin{aligned}
iv_{1\tau} &= v_{1\xi\xi} - \tfrac{2}{k^2}v_1 v_{-1} \\
iv_{1\tau} &= v_{1\xi\xi} - \tfrac{2}{k^2}v_1 \left|v_1\right|^2
\end{aligned} \tag{35}
$$

Describing as $q = \frac{v_1}{k}$ and $q_{-1} = \frac{v_{-1}}{k}$ (35) equation, we get the dimensional NLS-type equations in the form

$$iq_\tau = q_{\xi\xi} - 2q\left|q^2\right| \tag{36}$$

Also, the numerical solution of the (1+1)-dimensional Kaup-Kupershmidt (KK) equation (13) is found as

$$\begin{aligned} u(x,t) \quad = \quad & \varepsilon k(q(\xi,\tau)e^{i(kx-k^5t)} + q_{-1}(\xi,\tau)e^{-i(kx-k^5t)}) \\ & +\varepsilon^2(6q(\xi,\tau)q_{-1}(\xi,\tau) + \frac{7}{6}q^2(\xi,\tau)e^{2i(kx-k^5t)} \\ & +\frac{7}{6}q_{-1}^2(\xi,\tau)e^{-2i(kx-k^5t)}) + k\varepsilon^3(-\frac{7}{3}iq_{-1}(\xi,\tau)q_{-1\xi}(\xi,\tau)e^{-2i(kx-k^5t)} \\ & +\frac{7}{3}iq(\xi,\tau)q_{\xi}(\xi,\tau)e^{2i(kx-k^5t)}) \\ & +\frac{13}{12}k\varepsilon^3(q_1^3(\xi,\tau)e^{3i(kx-k^5t)} + q_{-1}^3(\xi,\tau)e^{-3i(kx-k^5t)}) \end{aligned} \tag{37}$$

where q is the solution of the NLS equation.

4 Results and Discussions

The multiple scales method, which is a perturbation method, is used to find approximate solutions of nonlinear evolution equations. This method allows us to find the solution of the given nonlinear equation depending on the solution of the linear part by using multiple scales, which are defined as the slow variables and depend on a parameter, in time and space variables. First, Zakharov and Kuznetsov showed that integrable systems can be reduced to other integrable systems using this method. If the initially taken system is not integrable, it is seen that the reduced system is either integrable or non-integrable because of the application of the method. However, if the method is applied to an integrable system, it is seen that the system obtained from the analysis is always integrable. This is the main purpose of applying multiple-scale methods to integrable systems. In this study, we examined how only NLS-type equations are derived from fKdV-type equations using the multiple scales method. We hope this conclusion serves as a valuable resource for researchers, scientists, and engineers interested in nonlinear wave theory, inspiring a deeper desire to explore further and reveal the mysteries still held within the intricate language of the NLS equation. At the same time, we think that the results obtained will form the basis of numerical calculations. Also, the method can be applied to many different NLEE equations.

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CHAPTER 4

CROSS-CORRELATION OF FUNCTIONS

AYŞE SANDIKÇI[1]

Introduction

Cross-correlation is a powerful mathematical tool used to measure how much two signals resemble each other. It works like a sliding dot product; we keep one function still, slide the second function over it, and calculate their overlap at every position. Because of this sliding mechanism, cross-correlation is widely used to search for a known, short pattern inside a much longer signal. Although it is structurally similar to convolution, it does not involve flipping the signal. Today, this operation plays a critical role in diverse areas such as image processing, pattern recognition, and neurophysiology. Furthermore, when a signal is correlated with itself, a process known as autocorrelation, the resulting function is bounded in magnitude by its value at zero shift. In particular, the autocorrelation magnitude attains its maximum at zero shift, which corresponds to the total energy of the signal.

Beyond its mathematical definition, cross-correlation occupies a central place in several areas of applied science. In signal

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processing, it provides the foundation for matched filtering and signal detection, while in statistics it is closely related to the study of dependence between random variables and stochastic processes. Cross-correlation is also widely used in image analysis and pattern recognition, where it serves as a measure of similarity between observed data and reference patterns. In time-series analysis, it is an important tool for investigating lag relationships between sequences of observations. Owing to these diverse applications, cross-correlation has become one of the most fundamental operations for similarity assessment across a broad range of scientific disciplines, (Akay & Boudreaux-Bartels, 2002), (Oppenheim & Schafer, 2010).

This chapter provides a unified exposition of the classical theory of cross-correlation. While the main definitions and results presented here are well established in the literature, they are adapted and organized with an emphasis on mathematical analysis. Particular attention is given to the existence of the cross-correlation operator for compactly supported and L^1 functions, together with examples illustrating the underlying theory. The contribution of the chapter lies in this systematic presentation rather than in the development of new theoretical results.

Before introducing the formal definition of cross-correlation and the criteria required for its mathematical existence and looking at concrete computational examples, it is necessary to define the core mathematical tools that make this analysis possible.

The Lebesgue space, denoted by $L^1(\mathbb{R})$, is defined as the set of complex-valued measurable functions on $\mathbb{R}$ that satisfy

$$\int_{-\infty}^{\infty} |h(y)| \, dy < \infty.$$

If $h \in L^1(\mathbb{R})$, the L^1 norm of h is defined by

$$\|h\|_{L^1} = \|h\|_1 = \int_{-\infty}^{\infty} |h(y)|\, dy < \infty.$$

Under the norm $\|\cdot\|_1$, the set of functions $L^1(\mathbb{R})$ forms a complete normed vector space.

Let $h \in L^1(\mathbb{R})$, let us define $\hat{h}$ (or $\mathcal{F}h$) by

$$\mathcal{F}h(z) = \hat{h}(z) = \int_{-\infty}^{\infty} h(x)e^{-2\pi ixz}dx\,, \quad z \in \mathbb{R}.$$

The expression $\hat{h}$ denotes the Fourier transform of h, (Gröchenig, 2001), (Debnath & Shah, 2015).

Let f and g be functions on $\mathbb{R}$. Then the convolution of f and g is the function $f * g$ defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(m)g(x-m)dm,$$

for any $x \in \mathbb{R}$ such that the integral exists, (Bracewell, 200), (Stade, 2005).

Definition and Properties of Cross-Correlation

Definition 1.1. Given two real or complex-valued functions f and g on $\mathbb{R}$, their cross-correlation, denoted by $f \star g$, is specified by the following integral for any $x \in \mathbb{R}$ where the integration converges

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(m)\overline{g(m-x)}dm.$$

In this context, the overline represents the complex conjugate of the function g, (Akay & Boudreaux-Bartels, 2002).

The main difference between convolution and cross-correlation is that convolution reverses the function g before shifting it, while cross-correlation shifts it directly and uses its complex conjugate. Consequently

$$(f \star g)(x) = (f * \tilde{g})(x),$$

where $\tilde{g}(x) = \overline{g(-x)}$. Thus, cross-correlation is equivalent to convolution with a time-reversed and conjugated version of g.

The cross-correlation operation satisfies several important algebraic properties, which are summarized below. In each case, it is assumed that f, g, and h are functions, and α and β are complex constants.

1. Cross-correlation does not satisfy the commutative law $(f \star g \neq g \star f)$. Commuting the operands results in a reflection of the correlation sequence across the ordinate axis, accompanied by a complex conjugation in the case of complex-valued signals, namely

$$(f \star g)(z) = \overline{(g \star f)(-z)}.$$

Indeed, by changing the variable to $m - z = u$, we get

$$(f \star g)(z) = \int_{-\infty}^{\infty} f(m)\overline{g(m-z)}dm = \int_{\mathbb{R}} f(u+z)\overline{g(u)}du$$

$$= \overline{\int_{\mathbb{R}} \overline{f(u+z)}g(u)du} = \overline{(g \star f)(-z)}.$$

2. If g is a Hermitian function, it satisfies the property $\overline{g(x)} = g(-x)$. When g is Hermitian, it introduces a beautiful simplification to the cross-correlation operation, reducing it to a standard convolution

$$(f \star g)(z) = (f * g)(z).$$

Indeed, using the Hermitian property of g, namely $\overline{g(m-z)} = g(z-m)$, we obtain

$$(f \star g)(z) = \int_{-\infty}^{\infty} f(m)\overline{g(m-z)}dm$$

$$= \int_{-\infty}^{\infty} f(m)g(z-m)dm$$

$$= (f * g)(z).$$

3. For Hermitian functions f and g, the cross-correlation is commutative:

$$(f \star g)(z) = \int_{-\infty}^{\infty} f(m)\overline{g(m-z)}dm = \int_{-\infty}^{\infty} f(m)g(z-m)dm$$

$$= \int_{-\infty}^{\infty} f(z-u)g(u)du = \int_{-\infty}^{\infty} \overline{f(u-z)}g(u)du$$

$$= (g \star f)(z).$$

4. Cross-correlation distributes over addition and respects scalar multiplication. Specifically, if the right side of the equation exists, then the left side is also well-defined and equals

$$\big((\alpha f + \beta g) \star h\big)(z) = \alpha(f \star h)(z) + \beta(g \star h)(z).$$

Indeed,

$$((\alpha f + \beta g) \star h)(z) = \int_{-\infty}^{\infty} (\alpha f + \beta g)(m)\,\overline{h(m-z)}\,dm$$

$$= \alpha \int_{-\infty}^{\infty} f(m)\,\overline{h(m-z)}\,dm + \beta \int_{-\infty}^{\infty} g(m)\,\overline{h(m-z)}\,dm$$

$= \alpha(f \star h)(z) + \beta(g \star h)(z).$

Example 1.2. Find $f \star g$, if $f(x) = e^{-|x|}$ and $g(x) = cosx.$

Solution. Since $cos\, x$ is a real-valued function, its complex conjugate remains unchanged, meaning $\overline{g(m-x)} = g(m-x)$. Substituting these into the formula yields

$$(f \star g)(x) = \int_{-\infty}^{\infty} e^{-|m|}\, \overline{\cos(m-x)}\, dm$$
$$= \int_{-\infty}^{\infty} e^{-|m|} cos(m-x) dm.$$

To simplify the integrand, we apply the trigonometric identity for the cosine of a difference, which states that

$$co\, s(m-x) = co\, s(m)\, co\, s(x) + si\, n(m)\, si\, n(x).$$

This allows us to expand the integral into two separate terms

$$(f \star g)(x) = \int_{-\infty}^{\infty} e^{-|m|} (co\, s(m)\, co\, s(x) + si\, n(m)\, si\, n(x)) dm.$$

Because the integration is performed with respect to m, the terms containing x act as constants and can be moved outside the integrals. The expression then becomes

$$(f \star g)(x) = cos(x) \int_{-\infty}^{\infty} e^{-|m|} co\, s(m)\, dm$$
$$+ sin(x) \int_{-\infty}^{\infty} e^{-|m|} si\, n(m)\, dm$$

We can analyze the behavior of these two integrals by looking at their symmetry. The integrand in the second term, $e^{-|m|} si\, n(m)$, is an odd function because replacing m with $-m$ negates the entire expression. Integrating an odd function over symmetric limits from $-\infty$ to ∞ results in zero, which eliminates the sine term

$$\int_{-\infty}^{\infty} e^{-|m|} si\, n(m)\, dm = 0.$$

For the remaining integral, $e^{-|m|} co\, s(m)$ is an even function, so we can write

$$\int_{-\infty}^{\infty} e^{-|m|} co\, s(m)\, dm = 2\int_{0}^{\infty} e^{-m} co\, s(m)\, dm.$$

Using the standard integral

$$\int_{0}^{\infty} e^{-ay} \cos by\, dy = \frac{a}{a^2+b^2}, \quad a > 0,$$

with $a = 1$ and $b = 1$, we obtain

$$\int_{0}^{\infty} e^{-m} co\, s(m)\, dm = \frac{1}{2}.$$

Therefore,

$$\int_{-\infty}^{\infty} e^{-|m|} co\, s(m)\, dm = 2 \cdot \frac{1}{2} = 1.$$

Substituting this result back gives

$$(f \star g)(x) = \cos(x).$$

Cross-Correlation of Two Functions with Compact Support

Another major difference between cross-correlation and basic arithmetic like addition or multiplication is the question of existence. If two functions $f(x)$ and $g(x)$ exist, their sum $(f + g)(x)$ and their product $(fg)(x)$ always exist as well. However, we cannot say the same for the cross-correlation $(f \star g)(x)$. To see this clearly, let us look at a very simple example. Suppose both functions are constant and equal to 1 for all x, so $f(x) = g(x) = 1$. If we put these into the cross-correlation formula, we get

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(m)\overline{g(m-x)}dm = \int_{-\infty}^{\infty} 1\, dm.$$

This integral is infinite. Even though our starting functions f and g are perfectly well-defined everywhere, their cross-correlation does not exist for even a single value of x.

This brings us to two important questions: When does $(f \star g)(x)$ actually exist? And what can we know about the nature of the resulting function $f \star g$ based on the properties of f and g? We will answer these questions with general rules and specific examples. First, we will look at functions that are zero outside a limited area, known as compactly supported functions, which will lead straight to our first useful result. To achieve this, we adapt a well-known proposition for convolution introduced by Eric Stade (Stade, 2005) and extend it to the context of cross-correlation.

Proposition 1.3. Assume that $a, b, c, d \in \mathbb{R}$, $a < b, c < d$, $d - c \leq b - a$, and

$$f(x) = \begin{cases} 0, & x < a, \\ F(x), & a < x < b, \\ 0, & x > b, \end{cases} \qquad g(x) = \begin{cases} 0, & x < c, \\ G(x), & c < x < d, \\ 0, & x > d. \end{cases}$$

(It does not matter how, or whether, $f(a)$, $f(b)$, $g(c)$, or $g(d)$ is defined). Then

$$
(f \star g)(x) = \begin{cases} 0, & x \le a-d, \\ \int_a^{x+d} F(y)\overline{G(y-x)}dy, & a-d < x < a-c, \\ \int_{x+c}^{x+d} F(y)\overline{G(y-x)}dy, & a-c \le x \le b-d, \\ \int_{x+c}^{b} F(y)\overline{G(y-x)}dy. & b-d < x < b-c, \\ 0, & x \ge b-c. \end{cases}
$$

In all cases, $(f \star g)(x)$ is well-defined whenever its corresponding integral converges.

Proof. Assume that $a < b, c < d,\ d-c \le b-a$. Since $f(y)$ is nonzero only for $a < y < b$, and $g(y-x)$ is nonzero only when

$$c < y - x < d,$$

the integrand is nonzero only on the intersection of the intervals

$$(a, b) \text{ and } (x+c, x+d).$$

Therefore,

$$(f \star g)(x) = \int_{max(a,x+c)}^{min(b,x+d)} F(y)\overline{G(y-x)}dy,$$

whenever the interval of integration is nonempty; otherwise,

$$(f \star g)(x) = 0.$$

The support of the cross-correlation is determined by the condition that the two intervals overlap. This gives

$$a - d < x < b - c.$$

Hence,

$$(f \star g)(x) = 0 \quad \text{for} \quad x \le a-d \text{ or } x \ge b-c.$$

Using the assumption $d - c \leq b - a$, the overlap can be described by three regions.

For $a - d < x < a - c$,

$$(f \star g)(x) = \int_{a}^{x+d} F(y)\overline{G(y-x)}dy,$$

for $a - c \leq x \leq b - d$,

$$(f \star g)(x) = \int_{x+c}^{x+d} F(y)\overline{G(y-x)}dy.$$

With the substitution $t = y - x$, this may also be written as

$$(f \star g)(x) = \int_{c}^{d} F(x+t)\overline{G(t)}dt.$$

Finally, for $b - d < x < b - c$,

$$(f \star g)(x) = \int_{x+c}^{b} F(y)\overline{G(y-x)}dy.$$

Combining these cases, the cross-correlation is

$$(f \star g)(x) = \begin{cases} 0, & x \leq a - d, \\ \int_{a}^{x+d} F(y)\overline{G(y-x)}dy, & a - d < x < a - c, \\ \int_{x+c}^{x+d} F(y)\overline{G(y-x)}dy, & a - c \leq x \leq b - d, \\ \int_{x+c}^{b} F(y)\overline{G(y-x)}dy. & b - d < x < b - c, \\ 0, & x \geq b - c. \end{cases}$$

This expression gives the cross-correlation of two functions whose supports are the finite intervals (a, b) and (c, d).

Looking at the regions above, we can see that the cross-correlation produces a non-zero result starting exactly at $x = a - d$ and ending exactly at $x = b - c$. Therefore, the support of the cross-correlation function $f \star g$ is the closed interval

$$supp(f \star g) = [a - d, b - c].$$

Remark. Notice how this differs from convolution. The support of the convolution $f * g$ is the sum of the intervals, $[a + c, b + d]$. However, because cross-correlation slides the template function without reversing it, its support is determined by the differences between the boundaries, $[a - d, b - c]$.

The above proposition has this useful consequence.

Corollary 1.4. If f and g are compactly supported, then so is $f \star g$.

Example 1.5. Consider the functions

$$f(x) = x\,\chi_{[-2,1]}(x), \quad g(x) = \chi_{[0,2]}(x),$$

where χ_A denotes the characteristic function of the set A. Since both functions are real-valued, the cross-correlation of f and g is defined by

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(y) g(y - x)\, dy.$$

Substituting the definitions of f and g, we obtain

$$(f \star g)(x) = \int_{-\infty}^{\infty} y\,\chi_{[-2,1]}(y)\chi_{[0,2]}(y - x)\, dy.$$

The condition $\chi_{[0,2]}(y - x) = 1$ is equivalent to

$$x \leq y \leq x + 2.$$

Therefore,

$$(f \star g)(x) = \int_{[-2,1]\cap[0,2]} y\, dy.$$

The intervals $[-2,1]$ and $[x, x+2]$ overlap only when

$$-4 < x < 1.$$

Hence, $(f \star\ g)(x) = 0$ for $x \leq -4$ and $x \geq 1$.

To evaluate the integral, we consider the possible overlap regions.

Case 1. $-4 < x \leq -2$

In this range,

$$[-2,1] \cap\ [x, x+2] = [-2, x+2].$$

Thus,

$$(f \star g)(x) = \int_{-2}^{x+2} y\, dy = \frac{x^2 + 4x}{2}.$$

Case 2. $-2 \leq x \leq -1$

Here,

$$[-2,1] \cap\ [x, x+2] = [x, x+2].$$

Therefore,

$$(f \star g)(x) = \int_{x}^{x+2} y\, dy = 2x + 2.$$

Case 3. $-1 \leq x < 1$

In this case,

$$[-2,1] \cap\ [x, x+2] = [x, 1].$$

Hence,

$$(f \star g)(x) = \int_{x}^{1} y\, dy = \frac{1 - x^2}{2}.$$

Combining the three cases yields

$$(f \star g)(x) = f(x) = \begin{cases} 0, & x \le -4, \\ \dfrac{x^2 + 4x}{2}, & -4 < x \le -2, \\ 2x + 2, & -2 \le x \le -1, \\ \dfrac{1 - x^2}{2}, & -1 \le x < 1, \\ 0, & x \ge 1. \end{cases}$$

Thus, the cross-correlation of $f(x) = x\,\chi_{[-2,1]}(x)$ and $g(x) = \chi_{[0,2]}(x)$ is the piecewise function given above and

$$supp(f \star g) = [-4,1].$$

When is the Cross-Correlation in $L^1(\mathbb{R})$?

When we study cross-correlation in the context of function spaces, a fundamental question arises. If two functions f and g belong to $L^1(\mathbb{R})$, does their cross-correlation $f \star g$ also belong to $L^1(\mathbb{R})$? The following theorem provides a positive answer to this question.

Theorem 1.6. Let $f, g \in L^1(\mathbb{R})$. Then $f \star g$ exists for almost every x, belongs to $L^1(\mathbb{R})$, and satisfies

$$\|f \star g\|_1 \le \|f\|_1 \|g\|_1.$$

Moreover,

$$\int_{-\infty}^{\infty} (f \star g)(x)\,dx = \left(\int_{-\infty}^{\infty} f(y)dy\right) \overline{\left(\int_{-\infty}^{\infty} g(u)\,du\right)}.$$

In particular, if g is real-valued, then

$$\int_{-\infty}^{\infty} (f \star g)(x)\,dx = \left(\int_{-\infty}^{\infty} f(y)dy\right) \left(\int_{-\infty}^{\infty} g(u)\,du\right).$$

Proof. Since $f, g \in L^1(\mathbb{R})$, Tonelli's theorem yields

$$\iint_{-\infty}^{\infty} |f(y)||g(y-x)|dydx = \int_{-\infty}^{\infty} |f(y)| \left(\int_{-\infty}^{\infty} |g(y-x)|dx \right) dy.$$

Using the change of variables $u = y - x$, we obtain

$$\int_{-\infty}^{\infty} |g(y-x)|dx = \int_{-\infty}^{\infty} |g(u)|du = \|g\|_1.$$

Hence

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y)||g(y-x)|dydx = \|f\|_1 \, \|g\|_1 < \infty.$$

Therefore the function $(y, x) \mapsto f(y) \, \overline{g(y-x)}$ is integrable on $\mathbb{R}^2$, and Fubini's theorem implies that $f \star g$ is defined for almost every x. Furthermore,

$$\begin{aligned} \|f \star g\|_1 &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f(y) \, \overline{g(y-x)} \, dy \right| dx \\ &\le \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y)||g(y-x)|dydx \\ &= \|f\|_1 \, \|g\|_1 < \infty. \end{aligned}$$

Consequently, $f \star g \in L^1(\mathbb{R})$.

Finally, by Fubini's theorem,

$$\begin{aligned} \int_{-\infty}^{\infty} (f \star g)(x)dx &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y) \, \overline{g(y-x)} \, dy \, dx \\ &= \int_{-\infty}^{\infty} f(y) \left(\int_{-\infty}^{\infty} \overline{g(y-x)} \, dx \right) dy. \end{aligned}$$

Again using the substitution $u = y - x$,

$$\int_{-\infty}^{\infty} \overline{g(y-x)}\, dx = \int_{-\infty}^{\infty} \overline{g(u)}\, du = \overline{\int_{-\infty}^{\infty} g(u)\, du}.$$

Therefore,

$$\int_{-\infty}^{\infty} (f \star g)(x)\, dx = \left(\int_{-\infty}^{\infty} f(y) dy\right) \overline{\left(\int_{-\infty}^{\infty} g(u)\, du\right)}.$$

This completes the proof.

The Fourier Transform of the Cross-Correlation

Theorem 1.7. Let $f, g \in L^1(\mathbb{R})$. Then

$$\mathcal{F}(f \star g)(\xi) = \hat{f}(\xi)\, \overline{\hat{g}(\xi)}.$$

Proof. By definition of Fourier transform, we write

$$\mathcal{F}(f \star g)(\xi) = \int_{-\infty}^{\infty} (f \star g)(x)\, e^{-2\pi i x\xi} dx.$$

Substituting the definition of the cross-correlation gives

$$\mathcal{F}(f \star g)(\xi) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y)\, \overline{g(y-x)}\, dy \right) e^{-2\pi i x\xi} dx.$$

Since $f, g \in L^1(\mathbb{R})$, Fubini's theorem allows us to interchange the order of integration

$$\mathcal{F}(f \star g)(\xi) = \int_{-\infty}^{\infty} f(y) \left(\int_{-\infty}^{\infty} \overline{g(y-x)}\, e^{-2\pi i x\xi} dx \right) dy.$$

Let $u = y - x$. Then

$$\int_{-\infty}^{\infty} \overline{g(y-x)}\, e^{-2\pi i x\xi} dx = \int_{-\infty}^{\infty} \overline{g(u)}\, e^{-2\pi i (y-u)\xi} du$$

$$= e^{-2\pi iy\xi} \int_{-\infty}^{\infty} \overline{g(u)}\, e^{2\pi iu\xi} du.$$

Substituting this into the previous expression yields

$$\mathcal{F}(f \star g)(\xi) = \left(\int_{-\infty}^{\infty} f(y)\, e^{-2\pi iy\xi}\, dy\right)\left(\int_{-\infty}^{\infty} \overline{g(u)}\, e^{2\pi iu\xi} du\right).$$

The first factor is $\hat{f}(\xi)$, while

$$\int_{-\infty}^{\infty} \overline{g(u)}\, e^{2\pi iu\xi} du = \overline{\int_{-\infty}^{\infty} g(u)\, e^{-2\pi iu\xi}\, du} = \overline{\hat{g}(\xi)}.$$

Hence,

$$\mathcal{F}(f \star g)(\xi) = \hat{f}(\xi)\, \overline{\hat{g}(\xi)}.$$

Results and Discussion

This chapter has reviewed and illustrated the main theoretical properties of the cross-correlation operator. Particular attention was given to existence conditions, the relationship with convolution, Fourier transform techniques, and representative examples that demonstrate the behavior of the operator in different settings.

Several examples were studied to illustrate both the analytical and geometric behavior of cross-correlation. In particular, compactly supported functions were shown to produce piecewise-defined correlation functions whose structure is determined by the overlap of the supports. These examples demonstrate how cross-correlation measures the similarity between shifted functions and how this similarity changes as the amount of overlap varies.

The results also highlight the close relationship between cross-correlation and convolution. Under suitable symmetry assumptions, the two operations coincide, while in the general case cross-correlation can be interpreted as a modified form of convolution involving reflection and complex conjugation. This connection allows many techniques developed for convolution to be applied to cross-correlation as well.

From an application perspective, cross-correlation plays an important role in signal processing, image analysis, pattern recognition, and time-series analysis. The theoretical results obtained in this chapter provide a mathematical foundation for such applications and facilitate both analytical and computational investigations.

Future work may focus on multidimensional extensions, generalized function settings, and numerical implementations based on Fourier transform techniques. Such developments would further broaden the applicability of the cross-correlation operator in both pure and applied mathematics.

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CHAPTER 5

DISCRETE ORLICZ SEQUENCE SPACES GENERATED BY YOUNG FUNCTIONS

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1. Introduction

Sequence spaces play a fundamental role in functional analysis, Banach space theory, approximation theory, and the study of linear operators. Also they, viewed as vector spaces, can be equipped with additional structures such as norms or inner products, leading to rich geometric and topological properties (Brezis, 2011).

A sequence is an ordered list of real numbers, such as

$$(x_k)_{k=1}^{\infty} = (x_1, x_2, x_3, \dots),$$

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where x_k represents the k-th term of the sequence (Rosen & Krithivasan, 2012). Let V be the collection of all real sequences. Together with the field $\mathbb{R}$, the set V forms a vector space under the usual operations of sequence addition and scalar multiplication (Lang, 2005). Indeed, these operations satisfy all vector space axioms (Kalita, Hazarika, 2024; Kreyszig, 1978).

A vector subspace can be obtained by imposing suitable membership conditions on the elements of V. One of the most important classes of sequence spaces is the family of p-summable sequence spaces ℓ^p, defined for $1 \leq p < \infty$ by

$$\ell^p = \left\{ x \in V : \sum_{k=1}^{\infty} | x_k |^p < \infty \right\}$$

(Idris et al., 2013). For the special case $p = \infty$, we define

$$\ell_\infty = \left\{ x \in V : \sup_{k \in \mathbb{N}} | x_k | < \infty \right\}.$$

It is straightforward to verify that each ℓ^p satisfies the axioms of a vector space. These spaces are commonly referred to as p-summable sequence spaces.

Let X be a real vector space. As stated in (Kreyszig, 1978), a norm on X is a mapping

$$\|\cdot\|_X : X \to \mathbb{R}$$

satisfying the following properties for all $x, y \in X$ and $\alpha \in \mathbb{R}$:

1. (Positivity) $\| x \|_X \geq 0$; $\| x \|_X = 0$ if and only if $x = 0$.
2. (Homogeneity) $\| \alpha x \|_X = | \alpha | \| x \|_X$.
3. (Triangle Inequality) $\| x + y \|_X \leq \| x \|_X + \| y \|_X$.

A pair $(X, \|\cdot\|_X)$ satisfying these properties is called a normed space.

A normed space X is called complete, or equivalently a Banach space, if every Cauchy sequence in X converges to an element of X (Berberian, 1961; Kreyszig, 1978).

A sequence $\left(x^{(m)}\right)$ converges to $x \in X$ if and only if

$$\| x^{(m)} - x \|_X \longrightarrow 0 \ \ (m \to \infty).$$

Similarly, $\left(x^{(m)}\right)$ is a Cauchy sequence whenever

$$\| x^{(m_1)} - x^{(m_2)} \|_X \longrightarrow 0 \ (m_1, m_2 \to \infty).$$

For a fixed $1 \leq p < \infty$, the space ℓ^p is equipped with the norm

$$\| x \|_p = \left(\sum_{k=1}^{\infty} | x_k |^p \right)^{1/p}, \qquad x \in \ell^p.$$

For $p = \infty$, the usual norm is

$$\| x \|_\infty = \sup_{k \in \mathbb{N}} | x_k |.$$

It is well known that every space ℓ^p, endowed with its usual norm, is complete and therefore forms a Banach space (Kreyszig, 1978).

We now examine the relationship between ℓ^s and ℓ^p when

$$1 \leq s < p \leq \infty.$$

Let $x \in \ell^s$. Then

$$\| x \|_s = \left(\sum_{k=1}^{\infty} | x_k |^s \right)^{1/s} < \infty.$$

Consequently,

$$1=\frac{\sum_{k=1}^{\infty}|x_k|^s}{\|x\|_s^s}=\sum_{k=1}^{\infty}\left(\frac{|x_k|}{\|x\|_s}\right)^s.$$

Since $0\leq\frac{|x_k|}{\|x\|_s}\leq 1$, for all $k\in\mathbb{N}$, it follows that

$$0\leq\left(\frac{|x_k|}{\|x\|_s}\right)^p\leq\left(\frac{|x_k|}{\|x\|_s}\right)^s\leq 1,$$

for all $k\in\mathbb{N}$ and $1\leq s<p\leq\infty$. Therefore,

$$\sum_{k=1}^{\infty}\left(\frac{|x_k|}{\|x\|_s}\right)^p\leq\sum_{k=1}^{\infty}\left(\frac{|x_k|}{\|x\|_s}\right)^s=1.$$

Multiplying both sides by $\|x\|_s^p$, we obtain

$$\|x\|_p^p=\sum_{k=1}^{\infty}|x_k|^p\leq\|x\|_s^p.$$

Hence,

$$\|x\|_p\leq\|x\|_s<\infty,$$

which proves that $\ell^s\subseteq\ell^p$. To see that this inclusion is proper, consider the sequence

$$y=(y_k)_{k=1}^{\infty},\qquad y_k=\frac{1}{k^{1/s}}.$$

Then observe that,

$$\sum_{k=1}^{\infty}|y_k|^s=\sum_{k=1}^{\infty}\frac{1}{k}=\infty,$$

the series is the harmonic series and diverges, showing that $y \notin \ell^s$. On the other hand,

$$\sum_{k=1}^{\infty} | y_k |^p = \sum_{k=1}^{\infty} \frac{1}{k^{p/s}} < \infty,$$

converges, since $\frac{p}{s} > 1$. Hence $y \in \ell^p$ but $y \notin \ell^s$. This shows that $\ell^s \subsetneq \ell^p$. Consequently, we have the chain of proper inclusions

$$\ell^1 \subsetneq \ell^s \subsetneq \ell^p \subsetneq \ell_\infty \text{ , for } 1 < s < p < \infty,$$

and the corresponding norms satisfy

$$\text{for every } x \in \ell^1, \ \| x \|_\infty \leq \| x \|_p \leq \| x \|_s \leq \| x \|_1.$$

The classical sequence spaces ℓ^p play a fundamental role in functional analysis and exhibit a rich hierarchy of inclusion relations (Konca et al., 2015). These relations reveal that stronger summability requirements produce smaller sequence spaces and naturally motivate the search for more flexible frameworks capable of describing a broader spectrum of growth behaviors. A natural extension of the spaces ℓ^p is obtained by replacing the power function t^p with a more general convex function, known as a Young function (Orlicz, 1932). This leads to the theory of Orlicz spaces, whose discrete counterparts, the Orlicz sequence spaces, generalize the classical ℓ^p-spaces and provide a versatile setting for studying summability beyond the traditional power-law framework (Orlicz, 1932; Stein & Shakarchi, 2005).

A central tool in the analysis of Orlicz spaces is the Luxemburg norm, which furnishes a natural norm structure even when the underlying growth function is not homogeneous. In this chapter, we visit discrete Orlicz sequence spaces generated by Young functions and examine their connections with the classical spaces ℓ^p. After reviewing the fundamental inclusion properties of ℓ^p-spaces, we introduce the Luxemburg norm and establish its basic properties. We then show how the spaces ℓ^p, $1 \leq p < \infty$, as well as the space ℓ_∞, arise within the Orlicz framework through appropriate choices of Young functions. Furthermore, when a Young function behaves near the origin like the power function t^β with $\beta \geq 1$, we prove the equivalence of the corresponding Luxemburg norm and the classical ℓ^β-norm. Consequently, the associated Orlicz sequence space coincides with ℓ^β, yielding completeness and inclusion results that demonstrate the decisive influence of the local growth of a Young function on the structural and topological properties of the corresponding sequence space.

2. Preliminary

2.1. Young Functions

Definition 1 Let Φ be a function defined on the interval $[0, \infty)$. We say that Φ is a Young function if it satisfies the following properties (see in Rao & Ren, 1991):

1. (Convexity) Φ is convex on $[0, \infty)$; that is, for every $x, y \geq 0$ and every $\lambda \in [0,1]$,

$$\Phi(\lambda x + (1 - \lambda)y) \leq \lambda\Phi(x) + (1 - \lambda)\Phi(y).$$

2. (Monotonicity) $\Phi$is strictly increasing on $[0, \infty)$; namely,

$$\Phi(t_1) < \Phi(t_2) \text{whenever} 0 \leq t_1 < t_2.$$

3. (Normalization at the Origin) Φ vanishes at the origin,

$$\Phi(0) = 0.$$

4. (Growth at Infinity) Φ tends to infinity as its argument tends to infinity,

$$\lim_{t\to\infty} \Phi(t) = \infty.$$

These conditions ensure that a Young function provides a suitable measure of growth and serves as the foundation for the construction of Orlicz spaces, which generalize the classical ℓ^p-spaces obtained from the power functions $\Phi(t) = t^p$, $1 \leq p < \infty$.

2.2. Discrete Orlicz Sequence Spaces and the Luxemburg Norm

Let Φ be a Young function. The discrete Orlicz sequence space associated with Φ is defined by

$$\ell^\Phi = \left\{ x = (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} \Phi\left(\frac{|x_n|}{\lambda}\right) < \infty \text{ for some } \lambda > 0 \right\}.$$

The space ℓ^Φ generalizes the classical sequence spaces ℓ^p. Since a Young function is not necessarily homogeneous, the usual norm construction used in ℓ^p-spaces is not applicable. Instead, one introduces the Luxemburg norm

$$\| x \|_\Phi = \inf \left\{ A > 0 : \sum_{n=1}^{\infty} \Phi \left(\frac{|x_n|}{A}\right) \leq 1 \right\}, x \in \ell^\Phi.$$

One may verify that $\|\cdot\|_\Phi$ is a norm on ℓ^Φ. Consequently, $(\ell^\Phi, \|\cdot\|_\Phi)$ is a normed vector space.

The Classical Spaces ℓ^p as Discrete Orlicz Spaces

For $1 \leq p < \infty$, consider the function $\Phi_p(t) = t^p,\ t \geq 0$. Since Φ_p is convex, strictly increasing, satisfies $\Phi_p(0) = 0$, and $\lim_{t\to\infty} \Phi_p(t) = \infty$, it is a Young function.

The corresponding discrete Orlicz space is

$$\ell^{\Phi_p} = \left\{ x = (x_n) : \sum_{n=1}^{\infty} |\, x_n \,|^p < \infty \right\} = \ell^p.$$

Moreover,

$$\|\, x \,\|_{\Phi_p} = \inf \left\{ A > 0 : \sum_{n=1}^{\infty} \left(\frac{|\, x_n \,|}{A} \right)^p \leq 1 \right\}.$$

Since

$$\sum_{n=1}^{\infty} \left(\frac{|\, x_n \,|}{A} \right)^p = \frac{1}{A^p} \sum_{n=1}^{\infty} |\, x_n \,|^p,$$

the above inequality is equivalent to

$$A^p \geq \sum_{n=1}^{\infty} |\, x_n \,|^p.$$

Hence,

$$\|\, x \,\|_{\Phi_p} = \left(\sum_{n=1}^{\infty} |\, x_n \,|^p \right)^{1/p} = \|\, x \,\|_p.$$

Therefore,

$$\ell^{\Phi_p} = \ell^p, \qquad \|\, x \,\|_{\Phi_p} = \|\, x \,\|_p,$$

showing that every classical space ℓ^p is a special case of a discrete Orlicz space.

An Extended-Valued Young Function for ℓ_∞

The classical space ℓ_∞ cannot be generated by an ordinary finite-valued Young function. Indeed, if $\Phi(t) < \infty$ for every $t \geq 0$, then the corresponding Orlicz condition imposes a summability requirement on the sequence. In contrast, the defining condition of ℓ_∞ requires only boundedness. Therefore, ℓ_∞ cannot be realized as a discrete Orlicz space generated by a finite-valued Young function.

Nevertheless, ℓ_∞ can be obtained by allowing extended-valued Young functions. Consider

$$\Phi_\infty(t) = \begin{cases} 0, & 0 \leq t < 1, \\ 1, & t = 1, \\ +\infty, & t > 1. \end{cases}$$

The effective domain of Φ_∞ is

$$\text{dom}\,(\Phi_\infty) = \{t \geq 0 : \Phi_\infty(t) < \infty\} = [0,1],$$

which is a convex set. Moreover, Φ_∞ is constant on $[0,1)$. Therefore, for every $t_1, t_2 \in [0,1]$ and every $\lambda \in [0,1]$,

$$\Phi_\infty(\lambda t_1 + (1-\lambda)t_2) \leq \lambda\Phi_\infty(t_1) + (1-\lambda)\Phi_\infty(t_2).$$

Hence Φ_∞ is convex in the extended-valued sense. Clearly,

$$\Phi_\infty(0) = 0,$$

and Φ_∞ may be regarded as an extended-valued Young function. For this choice of Φ_∞, the Luxemburg condition

$$\sum_{n=1}^{\infty} \Phi_\infty\left(\frac{|\, x_n \,|}{\lambda}\right) \leq 1$$

holds if and only if

$$\frac{|x_n|}{\lambda} \leq 1 \text{ for every } n \in \mathbb{N}.$$

Indeed, if $|x_n|/\lambda > 1$ for some n, then one term of the series equals $+\infty$, making the entire sum infinite. Thus the above condition is equivalent to

$$\sup_n |x_n| \leq \lambda.$$

Consequently,

$$\ell^{\Phi_\infty} = \{x = (x_n): \sup_n |x_n| < \infty\} = \ell_\infty.$$

Furthermore, the Luxemburg norm becomes

$$\| x \|_{\Phi_\infty} = \inf \left\{\lambda > 0: \sup_n |x_n| \leq \lambda\right\} = \sup_n |x_n| = \| x \|_\infty.$$

Therefore,

$$\ell^{\Phi_\infty} = \ell_\infty, \qquad \| x \|_{\Phi_\infty} = \| x \|_\infty.$$

Remark 2

Recall that

$$\Phi_p(t) = t^p, \qquad p \geq 1.$$

For every $t \geq 0$,

$$\lim_{p\to\infty} t^p = \begin{cases} 0, & 0 \leq t < 1, \\ 1, & t = 1, \\ +\infty, & t > 1. \end{cases}$$

Thus,

$$\Phi_p(t) \longrightarrow \Phi_\infty(t) \ \ (p \to \infty)$$

pointwise on $[0,\infty)$. This provides a natural interpretation of ℓ_∞ as the limiting member of the scale of spaces ℓ^p within the Orlicz framework.

Motivated by this observation, we introduce the class

$$S_\infty = \{\Phi: \ell^\Phi = \ell_\infty\},$$

consisting of all Young functions whose associated Orlicz sequence space coincides with ℓ_∞ .

2.3. Some Properties of Young Functions

The following lemma gives a useful property of the slopes of a convex Young function.

Lemma 3 Let Φ be a Young function. Then

$$\frac{\Phi(w)-\Phi(v)}{w-v} \geq \frac{\Phi(w)-\Phi(u)}{w-u} \geq \frac{\Phi(v)-\Phi(u)}{v-u},$$

for all $0 \leq u < v < w < \infty$.

Proof Since Φ is convex,

$$\Phi(\lambda u + (1-\lambda)w) \leq \lambda\Phi(u) + (1-\lambda)\Phi(w)$$

for every $\lambda \in [0,1]$.

Choose

$$\lambda = \frac{w-v}{w-u},$$

so that

$$v = \lambda u + (1 - \lambda)w.$$

Then

$$\Phi(v) \le \frac{w - v}{w - u}\Phi(u) + \frac{v - u}{w - u}\Phi(w).$$

Rearranging this inequality yields

$$\frac{\Phi(v) - \Phi(u)}{v - u} \le \frac{\Phi(w) - \Phi(u)}{w - u}.$$

Similarly,

$$\frac{\Phi(w) - \Phi(u)}{w - u} \le \frac{\Phi(w) - \Phi(v)}{w - v}.$$

Combining these inequalities proves the result.

Taking $u = 0$ and using $\Phi(0) = 0$, we obtain

$$\frac{\Phi(w) - \Phi(v)}{w - v} \ge \frac{\Phi(w)}{w} \ge \frac{\Phi(v)}{v}, \qquad 0 < v < w < \infty.$$

Consequently, the function

$$t \mapsto \frac{\Phi(t)}{t}$$

is nondecreasing on $(0, \infty)$.

To compare the growth of Young functions near the origin, we introduce the following notion.

Definition 4 Let $f, g\colon (0, \infty) \to (0, \infty)$ be two nonnegative functions.

We say that f and g are comparable near the origin, and write

$$f(t) \asymp g(t) \quad (t \to 0^+),$$

if there exist constants $c, C > 0$ and $\delta > 0$ such that

$$c\, g(t) \leq f(t) \leq C\, g(t), 0 < t < \delta.$$

Similarly, f and g are said to be comparable at infinity, written

$$f(t) \asymp g(t)(t \to \infty),$$

if there exist constants $c, C > 0$ and $M > 0$ such that

$$c\, g(t) \leq f(t) \leq C\, g(t), \qquad t > M.$$

The relation $f(t) \asymp g(t)$ means that the two functions have the same growth rate, up to multiplicative constants. In particular, if

$$\lim_{t \to 0^+} \frac{f(t)}{g(t)} = L, \qquad 0 < L < \infty,$$

then

$$f(t) \asymp g(t) \quad (t \to 0^+).$$

Likewise, if

$$\lim_{t \to \infty} \frac{f(t)}{g(t)} = L, \qquad 0 < L < \infty,$$

then

$$f(t) \asymp g(t) \quad (t \to \infty).$$

This notion will be used throughout the sequel to compare the local growth behavior of Young functions near the origin and at infinity.

Comparison of Growth Rates

To compare the behavior of functions near the origin, we use the following notion.

Definition 5 Let $f, g: (0,\infty) \to (0,\infty)$ be nonnegative functions. We say that f and g are comparable near the origin, and write

$$f(t) \asymp g(t)(t \to 0^+),$$

if there exist constants $c, C > 0$ and $\delta > 0$ such that

$$c\,g(t) \leq f(t) \leq C\,g(t)$$

for all $0 < t < \delta$. This means that f and g have the same growth rate up to multiplicative constants.

Proposition 6 Let Φ be a Young function. Suppose there exists $\alpha \geq 1$ such that

$$\lim_{t\to 0^+} \frac{\Phi(t)}{t^\alpha} > 0.$$

Then there exist constants $C_1, C_2 > 0$ such that

$$C_1 t^\alpha \leq \Phi(t) \leq C_2 t, 0 < t \leq 1.$$

Proof Since

$$\lim_{t\to 0^+} \frac{\Phi(t)}{t^\alpha} > 0,$$

there exists a constant $C_1 > 0$ such that

$$\Phi(t) \geq C_1 t^\alpha, \qquad 0 < t \leq 1.$$

This gives a lower bound for Φ.

On the other hand, since $t \mapsto \Phi(t)/t$ is nondecreasing,

$$\frac{\Phi(t)}{t} \leq \frac{\Phi(1)}{1}, \qquad 0 < t \leq 1.$$

Hence, setting $C_2 = \Phi(1)$,

$$\Phi(t) \le C_2 t, \qquad 0 < t \le 1.$$

Combining the two estimates yields

$$C_1 t^\alpha \le \Phi(t) \le C_2 t, \qquad 0 < t \le 1.$$

This proposition shows that a Young function satisfying the above condition has power-type growth near the origin.

Typical examples include:

- $\Phi(t) = t^p$ for $p \ge 1$;
- $\Phi(t) = t\log(1+t)$, for which

$$\Phi(t) \sim t^2 (t \to 0^+).$$

These examples illustrate common growth behaviors encountered in Orlicz space theory.

Young Functions Associated with ℓ_∞

Recall the extended-valued Young function

$$\Phi_\infty(t) = \begin{cases} 0, & 0 \le t < 1, \\ 1, & t = 1, \\ +\infty, & t > 1. \end{cases}$$

Define

$$S_\infty = \{\Phi : \ell^\Phi = \ell^\infty\}.$$

The following result describes the behavior of such functions near the origin.

Proposition 7 Let $\Phi \in S_\infty$. Then, for every $\alpha \geq 1$,

$$\lim_{t \to 0^+} \frac{\Phi(t)}{t^\alpha} = 0.$$

Proof Since $\Phi \in S_\infty$, there exists $\delta > 0$ such that

$$\Phi(t) = 0, \qquad 0 \leq t < \delta.$$

Therefore,

$$\frac{\Phi(t)}{t^\alpha} = 0, \qquad 0 < t < \delta,$$

for every $\alpha \geq 1$. Hence,

$$\lim_{t \to 0^+} \frac{\Phi(t)}{t^\alpha} = 0.$$

The proposition shows that every Young function generating the space ℓ_∞ must vanish identically in a neighborhood of the origin. Consequently, its growth near zero is strictly slower than any positive power t^α, $\alpha \geq 1$.

The study of Young functions and their associated Orlicz spaces originates from the pioneering work of Władysław Orlicz and was further developed by Rao and Ren (Rao and Ren, 1991). These contributions established a powerful framework extending the classical ℓ^p and L^p spaces by allowing more general convex growth conditions.

3. Main Results

Equivalence of Two Norms On Sequence Spaces

A vector space may admit more than one norm. It is therefore important to compare different norms defined on the same space.

Definition 8 Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two norms on a vector space X. These norms are said to be equivalent, denoted by $\|\cdot\|_a \sim \|\cdot\|_b$, if there exists a constant $C \geq 1$such that

$$\frac{1}{C} \| x \|_a \leq \| x \|_b \leq C \| x \|_a, x \in X.$$

Equivalent norms induce the same topology and therefore have the same notions of convergence and completeness (Conway, 1990; Rudin, 1991).

The next result shows that whenever a Young function behaves like a power function near the origin, the corresponding Luxemburg norm is equivalent to the classical ℓ^β-norm.

Theorem 9 Let $\beta \geq 1$. Assume that there exist constants $c, C > 0$ such that

$$c\, t^\beta \leq \Phi(t) \leq C\, t^\beta, 0 \leq t \leq 1.$$

Then there exists a constant $M \geq 1$ such that

$$\frac{1}{M} \| x \|_{\ell^\beta} \leq \| x \|_\Phi \leq M \| x \|_{\ell^\beta}, \qquad x \in \ell^\beta.$$

Proof Let $x \in \ell^\beta$ and put

$$y = \frac{x}{\| x \|_{\ell^\beta}}.$$

Then $\| y \|_{\ell^\beta} = 1$. Since $0 \leq | y_k | \leq 1$, we obtain

$$c \mid y_k \mid^\beta \leq \Phi(\mid y_k \mid) \leq C \mid y_k \mid^\beta.$$

Summing over k gives

$$c \leq \sum_{k=1}^{\infty} \Phi(|\, y_k \,|) \leq C.$$

Using the definition of the Luxemburg norm and the monotonicity of Φ, it follows that

$$\frac{1}{C} \leq \| \, y \, \|_{\Phi} \leq \frac{1}{c}.$$

Since

$$y = \frac{x}{\| \, x \, \|_{\ell^{\beta}}},$$

we obtain

$$\frac{1}{C} \; \| \, x \, \|_{\ell^{\beta}} \leq \| \, x \, \|_{\Phi} \leq \frac{1}{c} \; \| \, x \, \|_{\ell^{\beta}}.$$

Setting

$$M = \max \left\{ C, \frac{1}{c} \right\},$$

the desired inequalities follow.

Corollary 10 Suppose that there exist constants $c, C > 0$ such that

$$c\, t^{\beta} \leq \Phi(t) \leq C\, t^{\beta}, \qquad 0 \leq t \leq 1.$$

Then

$$\ell^{\Phi} = \ell^{\beta}.$$

Proof By the preceding theorem, the norms $\|\cdot\|_{\Phi}$ and $\|\cdot\|_{\ell^{\beta}}$ are equivalent. Hence a sequence belongs to ℓ^{Φ} if and only if it belongs to ℓ^{β}. Therefore, $\ell^{\Phi} = \ell^{\beta}$.

The corollary shows that the behavior of a Young function near the origin completely determines the associated discrete Orlicz sequence space. In particular, if Φ has power-type growth of order β, then the corresponding Orlicz space coincides with the classical space ℓ^{β}.

Completeness of Discrete Orlicz Spaces

An important question is whether the normed space $(\ell^{\Phi}, \|\cdot\|_{\Phi})$ is complete. Recall that a normed space is complete if every Cauchy sequence converges to a limit belonging to the same space. A complete normed space is called a Banach space (Rudin, 1991).

The following theorem establishes the completeness of ℓ^{Φ} under the same power-growth assumption.

Theorem 11 Let $\beta \geq 1$. Suppose that there exist constants $c, C > 0$ such that

$$c\,t^{\beta} \leq \Phi(t) \leq C\,t^{\beta}, 0 \leq t \leq 1.$$

Then the normed space $(\ell^{\Phi}, \|\cdot\|_{\Phi})$ is complete.

Proof By the previous theorem, there exists a constant $M \geq 1$such that

$$\frac{1}{M} \parallel x \parallel_{\ell^{\beta}} \leq \parallel x \parallel_{\Phi} \leq M \parallel x \parallel_{\ell^{\beta}}, x \in \ell^{\Phi} = \ell^{\beta}.$$

Let $(x^{(n)})$ be a Cauchy sequence in $(\ell^{\Phi}, \|\cdot\|_{\Phi})$. Then

$$\parallel x^{(n)} - x^{(m)} \parallel_{\ell^{\beta}} \leq M \parallel x^{(n)} - x^{(m)} \parallel_{\Phi},$$

which shows that $\{x^{(n)}\}$ is a Cauchy sequence in ℓ^{β}. Since $\beta \geq 1$, the space ℓ^{β} is a Banach space. Therefore, there exists an element $x \in \ell^{\beta}$ such that

$$x^{(n)} \longrightarrow x \text{ in } \ell^{\beta}.$$

Using the norm equivalence again,

$$\parallel x^{(n)} - x \parallel_{\Phi} \leq M \parallel x^{(n)} - x \parallel_{\ell^{\beta}} \longrightarrow 0.$$

Thus $x^{(n)} \to x$ in ℓ^{Φ}. Since $\ell^{\Phi} = \ell^{\beta}$, the limit x belongs to ℓ^{Φ}. Hence every Cauchy sequence in ℓ^{Φ} converges in ℓ^{Φ}. Therefore, $(\ell^{\Phi}, \|\cdot\|_{\Phi})$ is a Banach space.

This result shows that whenever a Young function behaves like t^{β} near the origin, the corresponding discrete Orlicz space inherits all the fundamental topological properties of the classical Banach space ℓ^{β}. In particular, ℓ^{Φ} and ℓ^{β} coincide as sets and have equivalent norms.

Inclusion Properties of Discrete Orlicz Sequence Spaces

We begin by recalling the classical inclusion property of the spaces ℓ^{p}.

Let $1 \leq p < q \leq \infty$. Then $\ell^{p} \subset \ell^{q}$. Moreover, for every $x \in \ell^{p}$,

$$\| x \|_{\ell^{q}} \leq \| x \|_{\ell^{p}}.$$

Thus, stronger summability implies weaker summability. In particular, the spaces ℓ^{p} become larger as the exponent p increases.

For $\beta \geq 1$, the space ℓ^{β} is a Banach space endowed with the norm

$$\| x \|_{\ell^{\beta}} = \left(\sum_{n=1}^{\infty} | x_n |^{\beta} \right)^{1/\beta}.$$

The inclusion relations among the spaces ℓ^{β} are completely determined by the exponent β.

We now investigate analogous inclusion properties for discrete Orlicz sequence spaces.

Theorem 12 Let $1 \leq \beta_1 < \beta_2$. For $i = 1,2$, let Φ_i be Young functions satisfying

$$c_i t^{\beta_i} \leq \Phi_i(t) \leq C_i t^{\beta_i}, 0 \leq t \leq 1,$$

for some positive constants $c_i$and C_i. Then $\ell^{\Phi_1} \subset \ell^{\Phi_2}$. Furthermore, the inclusion is continuous.

Proof By the norm-equivalence theorem, we have

$$\ell^{\Phi_i} = \ell^{\beta_i},$$

with equivalent norms. Since $1 \leq \beta_1 < \beta_2$, the classical inclusion property of ℓ^p spaces gives $\ell^{\beta_1} \subset \ell^{\beta_2}$. Therefore,

$$\ell^{\Phi_1} = \ell^{\beta_1} \subset \ell^{\beta_2} = \ell^{\Phi_2}.$$

This proves the inclusion. Moreover, using the equivalence of norms, there exists a constant $M > 0$ such that

$$\| x \|_{\Phi_2} \leq M \| x \|_{\Phi_1}, \qquad x \in \ell^{\Phi_1}.$$

Hence the embedding $\ell^{\Phi_1} \mapsto \ell^{\Phi_2}$ is continuous.

A Family of Orlicz Sequence Spaces

Let $\{\Phi_i\}_{i \in I}$ be a family of Young functions. Assume that for every $i \in I$ there exists an exponent $\beta_i \geq 1$ and positive constants c_i, C_i such that

$$c_i t^{\beta_i} \leq \Phi_i(t) \leq C_i t^{\beta_i}, \qquad 0 \leq t \leq 1.$$

In other words, each Young function Φ_i has power-type growth of order β_i near the origin.
By the norm-equivalence theorem,

$$\ell^{\Phi_i} = \ell^{\beta_i}, \qquad i \in I.$$

Consequently, the inclusion properties of the corresponding Orlicz sequence spaces are completely determined by the exponents β_i.

Indeed, if $\beta_i \leq \beta_j$, then the classical inclusion relation yields

$$\ell^{\beta_i} \subset \ell^{\beta_j}.$$

Hence,

$$\ell^{\Phi_i} \subset \ell^{\Phi_j}.$$

Furthermore, this embedding is continuous. Therefore,

$$\ell^{\Phi_i} \hookrightarrow \ell^{\Phi_j} \ (\beta_i \leq \beta_j).$$

Thus, among Orlicz sequence spaces generated by Young functions with power-type behavior near the origin, the inclusion structure is completely governed by the corresponding growth exponents.

5. Conclusion

Discrete Orlicz sequence spaces provide a natural generalization of the classical l^p-spaces through Young functions. This chapter investigates discrete Orlicz sequence spaces generated by Young functions and established their fundamental connections with the classical sequence spaces l^p. The Luxemburg norm provides a natural norm structure for Orlicz sequence spaces and establishes a unified framework in which the classical spaces $l^p (1 \leq p < \infty)$and l^∞arise from appropriate Young functions.

The results obtained show that when a Young function has power-type growth near the origin, the corresponding Orlicz sequence space coincides with a classical l^β-space and possesses equivalent norm and completeness properties. Moreover, the inclusion relations between such spaces are determined by the associated growth exponents, highlighting the central role of the local behavior of Young functions in the structure of discrete Orlicz sequence spaces.

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CHAPTER 6

λ −PARAMETRELİ GENUINE q −BERNSTEIN – DURRMEYER OPERATÖR DİZİSİ İÇİN VORONOVSKAJA TİP YAKLAŞIM TEOREMİ

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Giriş

λ-Bernstein operatörleri, $x \in [0,1]$ için $b_{k,i}^{\lambda}(x)$, $\lambda \in [-1,1]$ şekil parametresiyle birlikte Bezier temel fonksiyonları

$$\begin{cases} b_{k,0}^{\lambda}(x) = \mathrm{b}_{k,0}(x) - \dfrac{\lambda}{k+1}\mathrm{b}_{k+1,1}(x), \\ b_{k,i}^{\lambda}(x) = \mathrm{b}_{k,i}(x) + \dfrac{\lambda}{k+1}\left(\mathrm{b}_{k+1,i}(x) - \mathrm{b}_{k+1,i+1}(x)\right), \quad i = 1, \dots, k-1, \\ b_{k,k}^{\lambda}(x) = \mathrm{b}_{k,k}(x) + \dfrac{\lambda}{k+1}\mathrm{b}_{k+1,k}(x) \end{cases}$$

olmak üzere

$$B_k^{\lambda}(g,x) = \sum_{i=0}^{k} g\left(\frac{i}{k}\right) b_{k,i}^{\lambda}(x)$$

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şeklinde tanımlanmıştır (Zhou et al, 2025). Bu tip çalışmalarla ilgilenen araştırmacılar (Cai et al., 2018), (Cai et al., 2021), (Lin et al., 2024), (Su et al., 2024) makalelerinden faydalanabilirler.

Lupaş, Bernstein polinomlarının q-analoğunu literatüre kazandırmıştır (Lupaş, 1987). Ardından Phillips q-Bernstein polinomlarının bir başka genellemesini vermiştir (Phillips, 1997). Ardından, birçok araştırmacı bu tip operatörleri daha da geliştirerek çeşitli q −Bernstein operatörlerinin yaklaşım özelliklerini araştırmışlardır. Araştırmacılar bu tür çalışmaların bazılarını (Gupta & Heping, 2008), (Yüksel & Dinlemez, 2015), (Dinlemez et al., 2015), (Cai et al., 2025) makalelerinde bulabilirler.

q −Analiz ile ilgili temel tanım ve kavramlar De Sole ve Kac tarafından verilmiştir (De Sole & Kac,2005).

$n \in \mathbb{N}$ doğal sayısının $q-$ genelleştirmesi

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1 \end{cases}$$

ile, ayrıca n! ve binom katsayılarının q −genelleştirmesi

$$[n]_q! = [1]_q[2]_q[3]_q \dots [n]_q$$

ve

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

biçiminde tanımlıdır.

$(1-x)^n$ ifadesinin q −genelleştirmesi

$$(1 \ominus x)^n = (1-x)(1-qx) \dots (1-q^n x)$$

şeklinde ve q −Bernstein taban fonksiyonu

$$p_{n,k}^{q}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1 \ominus x)^{n-k}$$

biçiminde tanımlıdır.

$-1 \le \lambda \le 1$ ve $0 < \mathrm{q} \le 1$ için (λ, q) –Bernstein temel fonksiyonları

$$\begin{cases} p_{n,0}^{\lambda,q}(x) = p_{n,0}^{q}(x) - \dfrac{\lambda}{[n+1]_q} p_{n+1,1}^{q}(x), \\ p_{n,k}^{\lambda,q}(x) = p_{n,k}^{q}(x) + \dfrac{\lambda}{[n+1]_q} \left(p_{n+1,k}^{q}(x) - p_{n+1,k+1}^{q}(x) \right), k = 1, \dots, n-1, \\ p_{n,n}^{\lambda,q}(x) = p_{n,n}^{q}(x) + \dfrac{\lambda}{[n+1]_q} p_{n+1,n}^{q}(x) \end{cases}$$

biçiminde tanımlı olmak üzere $g \in C[0,1]$ fonksiyonu için, genuine $q -$ Bernstein Durrmeyer operatör dizisinin λ- parametrik genelleştirmesi

$$U_n^{\lambda,q}(g(r), x) = g(0) p_{n,0}^{\lambda,q}(x) + g(1) p_{n,n}^{\lambda,q}(x)$$
$$+ [n-1]_q \sum_{k=1}^{\mathrm{n}-1} p_{n,k}^{\lambda,q}(x) \int_0^1 p_{n-2,k-1}^{q}(r) g(r) d_q r, \qquad (1)$$

şeklinde tanımlanır.

Bu operatörler $q = 1$ ve $\lambda = 0$ için $U_n^{0,1}$ genuine Bernstein operatörlerine indirgenir (Chen, 1987).

(1) operatör dizisinin asimptotik yaklaşımını incelemek için Voronovskaya tip teorem verilecektir. Bu çeşit çalışmalar (Dinlemez & Yüksel, 2016), (Altun & Dinlemez Kantar, 2024), (Torun et al., 2022), (Dinlemez Kantar & Ergelen, 2019) makalelerinde bulunabilir. Son olarak elde edilen teorik sonuçların grafik analiziyle incelemesi yapılacaktır.

Şimdi, Voronovskaya tip teoremi elde edebilmek için gerekli olan lemmalar verilecektir.

Lemma 1 in ispatı "Genuine q −Bernstein – Durrmeyer Operatör Dizisinin Parametrik Genelleştirmesi" bölümünde $e_j(r) = r^j$ olmak üzere $j = 0,1,2$ için açıkça yapıldığından bu ispata benzer şekilde $j = 3,4$ için ispatlar okuyucuya bırakılmıştır.

Lemma 1. $j = 0,1,2,3,4$ doğal sayıları olmak üzere $e_j(r) = r^j$ test fonksiyonlarına karşılık (1) de tanımlanan $U_n^{\lambda,q}$ operatörleri için aşağıdaki eşitlikler elde edilir:

i) $U_n^{\lambda,q}(e_0; x) = 1,$

ii) $$U_n^{\lambda,q}(e_1; x) = x + \frac{\lambda}{[n]_q}\left\{x - x^{n+1} - \frac{1}{q}(1 - 2x(1 \ominus x)^n - x^{n+1})\right\}$$
$$+\frac{\lambda}{q[n]_q[n+1]_q}\{1 - (1 \ominus x)^{n+1} - x^{n+1}\},$$

iii) $$U_n^{\lambda,q}(e_2; x) = \frac{[2]_q}{[n+1]_q}x + \frac{q^2[n-1]_q}{[n+1]_q}x^2 + \frac{\lambda(q^2-1)}{[n+1]_q}(x^2 - x^{n+1})$$
$$+\frac{[2]_q\lambda}{[n]_q[n+1]_q}(x - x^{n+1}),$$

iv) $$U_n^{\lambda,q}(e_3; x) = \frac{q^6[n-2]_q[n-1]_q}{[n+1]_q[n+2]_q}x^3 + \frac{([2]_q q^3+[4]_q q^2)[n-1]_q}{[n+1]_q[n+2]_q}x^2$$
$$+\frac{[2]_q[3]_q}{[n+1]_q[n+2]_q}x + \frac{\lambda}{[n+1]_q}\left(\left\{\frac{[2]_q[3]_q}{[n]_q[n+2]_q}x - x^{n+1}\right)\right.$$
$$+\frac{[2]_q q^3+[4]_q q^2}{[n+2]_q}(x^2 - x^{n+1}) + \frac{q^6[n-1]_q}{[n+2]_q}(x^3 - x^{n+1})$$
$$\left.-\frac{[3]_q}{[n+2]_q}(x^2 - x^{n+1}) - \frac{q^3[n-1]_q}{[n+2]_q}(x^3 - x^{n+1})\right\},$$

v) $$U_n^{\lambda,q}(e_4; x) = \frac{[2]_q[3]_q[4]_q}{[n+1]_q[n+2]_q[n+3]_q}x$$
$$+\frac{([2]_q[3]_q q^4+([2]_q q^3+[4]_q q^2)[5]_q)[n-1]_q}{[n+1]_q[n+2]_q[n+3]_q}x^2$$

$$+\frac{\left([2]_q q^8+[4]_q q^7+[6]_q q^6\right)[n-2]_q[n-1]_q}{[n+1]_q[n+2]_q[n+3]_q}x^3$$

$$+\frac{q^{12}[n-3]_q[n-2]_q[n-1]_q}{[n+1]_q[n+2]_q[n+3]_q}x^4$$

$$+\frac{\left([2]_q[3]_q q^4+\left([2]_q q^3+[4]_q q^2\right)[5]_q\right)[n-1]_q}{[n+2]_q[n+3]_q}\left(x^2-x^{n+1}\right)$$

$$+\frac{\left([2]_q q^8+[4]_q q^7+[6]_q q^6\right)[n-1]_q}{[n+2]_q[n+3]_q}\left(x^3-x^{n+2}\right)$$

$$+\frac{q^{12}[n-2]_q[n-1]_q}{[n+2]_q[n+3]_q}\left(x^4-x^{n+3}\right)$$

$$-\frac{[3]_q[4]_q}{[n+2]_q[n+3]_q}\left(x^2-x^n\right)$$

$$-\frac{\left([3]_q q^4+q^3[5]_q\right)[n-1]_q}{[n+2]_q}\left(x^3-x^{n+2}\right)$$

$$-\frac{q^8[n-2]_q[n-1]_q}{[n+2]_q[n+3]_q}\left(x^4-x^{n+2}\right)\Big\}.$$

Lemma 2. $j=1,2,4$ doğal sayıları olmak üzere $\varphi_j(r)=(r-x)^j$ moment fonksiyonlarına karşılık $U_n^{\lambda,q}$ operatörleri için aşağıdaki eşitlikler elde edilir:

i) $U_n^{\lambda,q}(\varphi_1;x)=\frac{\lambda(q-1)x(1-x^n)}{q[n]_q}+\frac{\lambda\{1-(1\ominus x)^{n+1}-x^{n+1}\}}{q[n]_q[n+1]_q}$,

ii) $U_n^{\lambda,q}(\varphi_2;x)=\frac{[2]_q x(1-x)}{[n+1]_q}+\lambda\left\{\frac{[2]_q x(1-x^n)}{[n]_q[n+1]_q}-\frac{\left(1-(1\ominus x)^{n+1}-x^{n+1}\right)}{q[n]_q[n+1]_q}\right.$

$$\left.+\frac{(q^2-1)x^2\left(1-x^{n-1}\right)}{[n+1]_q}-\frac{2(q-1)x(1-x^n)}{q[n]_q}\right\},$$

iii) $U_n^{\lambda,q}(\varphi_4;x)=\left\{\frac{q^{12}[n-3]_q[n-2]_q[n-1]_q}{[n+1]_q[n+2]_q[n+3]_q}-4\frac{q^6[n-2]_q[n-1]_q}{[n+1]_q[n+2]_q}\right.$

$$\left.+6\frac{q^2[n-1]_q}{[n+1]_q}-3\right\}x^4$$

$$+\left\{\frac{([2]_q q^8+[4]_q q^7+[6]_q q^6)[n-2]_q[n-1]_q}{[n+1]_q[n+2]_q[n+3]_q}\right.$$

$$\left.-\frac{4([2]_q q^3+[4]_q q^2)[n-1]_q}{[n+1]_q[n+2]_q}+\frac{6[2]_q}{[n+1]_q}\right\}x^3$$

$$+\left\{\frac{([2]_q[3]_q q^4+([2]_q q^3+[4]_q q^2)[5]_q)[n-1]_q}{[n+1]_q[n+2]_q[n+3]_q}-\frac{4[2]_q[3]_q}{[n+1]_q[n+2]_q}\right\}x^2$$

$$+\frac{[2]_q[3]_q[4]_q}{[n+1]_q[n+2]_q[n+3]_q}x+\frac{\lambda}{[n+1]_q}\left\{\frac{[n-2]_q[n-1]_q(q^{12}-q^8)x^4(1-x^{n-3})}{[n+2]_q[n+3]_q}\right.$$

$$-\frac{4(q^6-q^3)[n-1]_q x^4(1-x^{n-2})}{[n+2]_q}+6(q^2-1)x^4(1-x^{n-1})$$

$$-\frac{4(q-1)[n+1]_q x^4(1-x^n)}{q[n]_q}+\frac{6[2]_q x^3(1-x^n)}{[n]_q}$$

$$-\frac{4([2]_q q^3+[4]_q q^2-[3]_q)x^3(1-x^{n-1})}{[n+2]_q}-\frac{4[2]_q[3]_q x^2(1-x^n)}{[n]_q[n+2]_q}$$

$$-\frac{4x^3(1-(1\ominus x)^{n+1}-x^{n+1})}{q[n]_q}+\frac{([2]_q[3]_q q^4-[3]_q[4]_q)x^2(1-x^{n-1})}{[n+2]_q[n+3]_q}$$

$$\left.+\frac{[2]_q[3]_q[4]_q x(1-x^{n+1})}{[n]_q[n+2]_q[n+3]_q}\right\}.$$

İspat. $U_n^{\lambda,q}$ operatör dizisinin lineerliğini ve Lemma 1 in sonuçları kullanılarak ve gerekli düzenlemeler yapılarak Lemma 2 nin ispatı tamamlanmış olur.

Lemma 3. $(q_n)\subset(0,1)$ dizisi için $n\to\infty$ iken $q_n\to 1$ ve $n\to\infty$ iken $q_n^n\to\beta$ olsun. Bu durumda aşağıdaki eşitlikler elde edilir:

i) $\lim\limits_{n\to\infty}[n]_{q_n}U_n^{\lambda,q_n}(\varphi_1;x)=0,$

ii) $\lim\limits_{n\to\infty}[n]_{q_n}U_n^{\lambda,q_n}(\varphi_2;x)=2x(1-x),$

iii) $\lim\limits_{n\to\infty}[n]_{q_n}^2U_n^{\lambda,q_n}(\varphi_4;x)=12x^2\,(x-1)^2.$

İspat. i) ve ii) nin ispatı “Genuine q −Bernstein – Durrmeyer Operatör Dizisinin Parametrik Genelleştirmesi” bölümünde ispatlanmıştır.

iii) iii nin ispatında Lemma 3 ün iii eşitliği kullanılırsa,

$$\lim_{n\to\infty}[n]_{q_n}^2 U_n^{\lambda,q_n}(\varphi_4;x)=\lim_{n\to\infty}[n]_{q_n}^2\left[\left\{\frac{{q_n}^{12}[n-3]_{q_n}[n-2]_{q_n}[n-1]_{q_n}}{[n+1]_{q_n}[n+2]_{q_n}[n+3]_{q_n}}\right.\right.$$

$$\left.-\frac{4{q_n}^6[n-2]_{q_n}[n-1]_{q_n}}{[n+1]_{q_n}[n+2]_{q_n}}+\frac{6{q_n}^2[n-1]_{q_n}}{[n+1]_{q_n}}-3\right\}x^4$$

$$+\left\{\frac{\left([2]_{q_n}{q_n}^8+[4]_{q_n}{q_n}^7+[6]_{q_n}{q_n}^6\right)[n-2]_{q_n}[n-1]_{q_n}}{[n+1]_{q_n}[n+2]_{q_n}[n+3]_{q_n}}\right.$$

$$\left.-\frac{4\left([2]_{q_n}{q_n}^3+[4]_{q_n}{q_n}^2\right)[n-1]_{q_n}}{q_n[n+2]_{q_n}}+\frac{6[2]_{q_n}}{[n+1]_{q_n}}\right\}x^3$$

$$+\left\{\frac{\left([2]_{q_n}[3]_{q_n}{q_n}^4+\left([2]_{q_n}{q_n}^3+[4]_{q_n}{q_n}^2\right)[5]_{q_n}\right)[n-1]_{q_n}}{[n+1]_{q_n}[n+2]_{q_n}[n+3]_{q_n}}\right.$$

$$\left.-\frac{4[2]_{q_n}[3]_{q_n}}{[n+1]_{q_n}[n+2]_{q_n}}\right\}x^2+\frac{[2]_{q_n}[3]_{q_n}[4]_{q_n}}{[n+1]_{q_n}[n+2]_{q_n}[n+3]_{q_n}}x$$

$$+\frac{\lambda}{[n+1]_{q_n}}\left\{\frac{[n-2]_{q_n}[n-1]_{q_n}\left({q_n}^{12}-{q_n}^8\right)x^4\left(1-x^{n-3}\right)}{[n+2]_{q_n}[n+3]_{q_n}}\right.$$

$$-\frac{4\left({q_n}^6-{q_n}^3\right)[n-1]_{q_n}x^4\left(1-x^{n-2}\right)}{[n+2]_{q_n}}$$

$$+6\left({q_n}^2-1\right)x^4\left(1-x^{n-1}\right)$$

$$+\frac{[4]_{q_n}[n-1]_{q_n}{q_n}^3\left(1-q_n+{q_n}^3[3]_{q_n}\right)x^3\left(1-x^{n-2}\right)}{[n+2]_{q_n}[n+3]_{q_n}}$$

$$-\frac{4(q_n-1)[n+1]_{q_n}x^4(1-x^n)}{q_n[n]_{q_n}}+\frac{6[2]_{q_n}x^3(1-x^n)}{[n]_{q_n}}$$

$$-\frac{4\left([2]_{q_n}{q_n}^3+[4]_{q_n}{q_n}^2-[3]_{q_n}\right)x^3\left(1-x^{n-1}\right)}{[n+2]_{q_n}}$$

$$-\frac{4x^3\left(1-(1\ominus x)^{n+1}-x^{n+1}\right)}{q_n[n]_{q_n}}-\frac{4[2]_{q_n}[3]_{q_n}x^2(1-x^n)}{[n]_{q_n}[n+2]_{q_n}}$$

$$+\frac{\left([2]_{q_n}{q_n}^3+[4]_{q_n}{q_n}^2\right)[5]_{q_n}x^2\left(1-x^{n-1}\right)}{[n+2]_{q_n}[n+3]_{q_n}}$$

$$+\frac{\left([2]_{q_n}[3]_{q_n}{q_n}^4-[3]_{q_n}[4]_{q_n}\right)x^2\left(1-x^{n-1}\right)}{[n+2]_{q_n}[n+3]_{q_n}}$$

$$\left.\left.+\frac{[2]_{q_n}[3]_{q_n}[4]_{q_n}x(1-x^n)}{[n]_{q_n}[n+2]_{q_n}[n+3]_{q_n}}\right\}\right]$$

elde edilir.

Yukarıdaki limitler ayrı ayrı aşağıdaki gibi hesaplanır:

$$L_1=\lim_{n\to\infty}[n]_{q_n}^2\left\{\frac{{q_n}^{12}[n-3]_{q_n}[n-2]_{q_n}[n-1]_{q_n}}{[n+1]_{q_n}[n+2]_{q_n}[n+3]_{q_n}}-4\frac{{q_n}^6[n-2]_{q_n}[n-1]_{q_n}}{[n+1]_{q_n}[n+2]_{q_n}}\right.$$

$$\left.+\frac{6{q_n}^2[n-1]_{q_n}}{[n+1]_{q_n}}-3\right\}x^4.$$

Daha sonra parantezin içinin paydaları eşitlenip pay kısmında

$$[n-1]_{q_n}=\frac{[n+1]_{q_n}-[2]_{q_n}}{{q_n}^2},$$

$$[n-2]_{q_n}=\frac{[n+2]_{q_n}-[4]_{q_n}}{{q_n}^4},$$

$$[n-3]_{q_n}=\frac{[n+3]_{q_n}-[6]_{q_n}}{{q_n}^2},$$

yazılır ve $[n]_q=\begin{cases}\frac{1-q^n}{1-q}, & q\neq 1\\ n, & q=1\end{cases}$ tanımı kullanılırsa

$L_1=12x^4$

bulunur. Benzer şekilde

$$L_2=\lim_{n\to\infty}[n]_{q_n}^2\left\{\frac{\left([2]_{q_n}{q_n}^8+[4]_{q_n}{q_n}^7+[6]_{q_n}{q_n}^6\right)[n-2]_{q_n}[n-1]_{q_n}}{[n+1]_{q_n}[n+2]_{q_n}[n+3]_{q_n}}\right.$$

$$-\frac{4\left([2]_{q_n}{q_n}^3+[4]_{q_n}{q_n}^2\right)[n-1]_{q_n}}{[n+1]_{q_n}[n+2]_{q_n}}$$

$$\left.+\frac{6[2]_{q_n}}{[n+1]_{q_n}}\right\}x^3=-24x^3,$$

$$L_3 =$$

$$\lim_{n\to\infty}[n]_{q_n}^2\left\{\frac{\left([2]_{q_n}[3]_{q_n}{q_n}^4+\left([2]_{q_n}{q_n}^3+[4]_{q_n}{q_n}^2\right)[5]_{q_n}\right)[n-1]_{q_n}}{[n+1]_{q_n}[n+2]_{q_n}[n+3]_{q_n}}-\frac{4[2]_{q_n}[3]_{q_n}}{[n+1]_{q_n}[n+2]_{q_n}}\right\}x^2$$

$$=12x^2,$$

$$L_4=\lim_{n\to\infty}[n]_{q_n}^2\left\{\frac{[2]_{q_n}[3]_{q_n}[4]_{q_n}}{[n+1]_{q_n}[n+2]_{q_n}[n+3]_{q_n}}x\right.$$

$$+\frac{\lambda}{[n+1]_{q_n}}\left(\frac{[n-2]_{q_n}[n-1]_{q_n}\left({q_n}^{12}-{q_n}^8\right)x^4\left(1-x^{n-3}\right)}{[n+2]_{q_n}[n+3]_{q_n}}\right.$$

$$-\frac{4\left({q_n}^6-{q_n}^3\right)[n-1]_{q_n}x^4\left(1-x^{n-2}\right)}{[n+2]_{q_n}}+6\left({q_n}^2-1\right)x^4\left(1-x^{n-1}\right)$$

$$+\frac{[4]_{q_n}[n-1]_{q_n}{q_n}^3\left(1-q_n+{q_n}^3[3]_{q_n}\right)x^3\left(1-x^{n-2}\right)}{[n+2]_{q_n}[n+3]_{q_n}}$$

$$-\frac{4\left(q_n-1\right)[n+1]_{q_n}x^4\left(1-x^n\right)}{q_n[n]_{q_n}}+\frac{6[2]_{q_n}x^3\left(1-x^n\right)}{[n]_{q_n}}$$

$$-\frac{4\left([2]_{q_n}{q_n}^3+[4]_{q_n}{q_n}^2-[3]_{q_n}\right)x^3\left(1-x^{n-1}\right)}{[n+2]_{q_n}}$$

$$-\frac{4x^3\left(1-(1\ominus x)^{n+1}-x^{n+1}\right)}{q_n[n]_{q_n}}-\frac{4[2]_{q_n}[3]_{q_n}x^2\left(1-x^n\right)}{[n]_{q_n}[n+2]_{q_n}}$$

$$+\frac{\left([2]_{q_n}{q_n}^3+[4]_{q_n}{q_n}^2\right)[5]_{q_n}x^2\left(1-x^{n-1}\right)}{[n+2]_{q_n}[n+3]_{q_n}}$$

$$+\frac{\left([2]_{q_n}[3]_{q_n}{q_n}^4-[3]_{q_n}[4]_{q_n}\right)x^2\left(1-x^{n-1}\right)}{[n+2]_{q_n}[n+3]_{q_n}}$$

$$\left.\left.+\frac{[2]_{q_n}[3]_{q_n}[4]_{q_n}x\left(1-x^n\right)}{[n]_{q_n}[n+2]_{q_n}[n+3]_{q_n}}\right)\right\}=0$$

elde edilir. Böylece ispat tamamlanmış olur.

Teorem 2. $(q_n)\subset(0,1)$ dizisi için $n\to\infty$ iken $q_n\to\beta$ olsun. $g\in C^2[0,1]$ için aşağıdaki limit elde edilir:

$$\lim_{n\to\infty}[n]_{q_n}\left(U_n^{\lambda,q_n}(g,x)-g(x)\right)=x(1-x)g''(x).$$

İspat. g fonksiyonunun x noktasındaki Taylor açılımı,

$r\to x$ iken $\varepsilon(r,x)\to 0$ olmak üzere,

$$g(t)=g(x)+g'(x)(r-x)+\frac{g''(x)}{2}(r-x)^2+\varepsilon(r,x)(r-x)^2$$

dir. Bu eşitliğe U_n^{λ,q_n} operatörü uygulanırsa operatör dizisinin lineerliğinden

$$U_n^{\lambda,q_n}(g,x)-g(x)=g'(x)U_n^{\lambda,q_n}(r-x,x)$$

$$+\frac{g''(x)}{2}U_n^{\lambda,q_n}((r-x)^2,x)+U_n^{\lambda,q_n}(\varepsilon(r,x)(r-x)^2,x)$$

yazılır.

Her iki tarafı $[n]_{q_n}$ ile çarpılır ve limit alınırsa,

$$\lim_{n\to\infty}[n]_{q_n}\left(U_n^{\lambda,q_n}(g,x)-g(x)\right)=g'(x)\lim_{n\to\infty}[n]_{q_n}U_n^{\lambda,q_n}(r-x,x)$$

$$+\frac{g''(x)}{2}\lim_{n\to\infty}[n]_{q_n}U_n^{\lambda,q_n}((r-x)^2,x)$$

$$+\lim_{n\to\infty}[n]_{q_n}U_n^{\lambda,q_n}(\varepsilon(r,x)(r-x)^2,x) \qquad (2)$$

elde edilir. Sağ taraftaki son terim için Cauchy- Schwarz eşitsizliği

uygulanırsa,

$$\lim_{n\to\infty}[n]_{q_n}\left(U_n^{\lambda,q_n}(\varepsilon(r,x)(r-x)^2,x)\right)$$

$$\leq\sqrt{\lim_{n\to\infty}U_n^{\lambda,q_n}(\varepsilon(r,x),x)}\sqrt{\lim_{n\to\infty}[n]_{q_n}^2U_n^{\lambda,q_n}((r-x)^4,x)}$$

(3)

bulunur. Lemma 3 ten dolayı (2) eşitliğindeki limitler sırasıyla

$$\lim_{n\to\infty}[n]_{q_n}U_n^{\lambda,q_n}(r-x,x)=0,$$

$$\lim_{n\to\infty}[n]_{q_n}U_n^{\lambda,q_n}((r-x)^2,x)=2x(1-x),$$

$\lim_{n\to\infty}U_n^{\lambda,q_n}(\varepsilon(r,x),x)=0$ ve $\lim_{n\to\infty}[n]_{q_n}^2U_n^{\lambda,q_n}(\varphi_4;x)=12x^2\,(x-1)^2$
sonlu olduğundan (3) eşitsizliğinden

$\lim_{n\to\infty}[n]_{q_n}\left(U_n^{\lambda,,q_n}(\varepsilon(r,x)(r-x)^2,x)\right)=0$ hesaplanır. Böylece

$$\lim_{n\to\infty}[n]_{q_n}\left(U_n^{\lambda,q_n}(g,x)-g(x)\right)=x(1-x)g''(x)$$

elde edilir. Böylece ispat tamamlanmış olur.

Şimdi (1) operatörü kullanılarak,

$$\widehat{U}_n^{\lambda,q}(g;x)=U_n^{\lambda,q}(g(r);x)-g\left(x+\sigma_n^{\lambda,q}(x)\right)+g(x) \tag{4}$$

yardımcı operatörü tanımlansın. Burada

$$\sigma_n^{\lambda,q}(x)=\frac{\lambda}{[n]_q}\left\{x-x^{n+1}-\frac{1}{q}(1-2x(1\ominus x)^n-x^{n+1})\right\}$$
$$+\frac{\lambda}{q[n]_q[n+1]_q}\{1-(1\ominus x)^{n+1}-x^{n+1}\}$$

dır. Lemma 4 ün ispatı açık olduğundan verilmeyecektir.

Lemma 4. $\widehat{U}_n^{\lambda,q}$ operatörleri için aşağıdaki özellikler sağlanır.

i) $\widehat{U}_n^{\lambda,q}(1;x)=1,$

ii) $\widehat{U}_n^{\lambda,q}(r-x;x)=0.$

Teorem 3. $\widehat{U}_n^{\lambda,q}$ operatör dizisi her $x\in[0,1]$ ve her

$g \in C^2[0,1]$ için

$$\left|\widehat{U}_n^{\lambda,q}(g,x) - g(x)\right| \leq \frac{\left(\sqrt{\varrho_n^{\lambda,q}} + \left(\sigma_n^{\lambda,q}(x)\right)^2\right)}{2} \|g''\|$$

eşitsizliği vardır. Burada $\varrho_n^{\lambda,q} = \frac{[2]_q}{4[n+1]_q} + \frac{8}{q[n]_q}|\lambda|$ dır.

$g \in C^2[0,1]$ fonksiyonunun x noktasındaki Taylor açılımı

$$g(r) = g(x) + g'(x)(r-x) + \int_x^r (r-u)g''(u)du \tag{5}$$

şeklindedir. (5) eşitliğinin her iki yanına $\widehat{U}_n^{\lambda,q}$ operatör dizisini uygulayıp, Lemma 4 kullanılırsa,

$$\widehat{U}_n^{\lambda,q}(g(t);x) - g(x) = \widehat{U}_n^{\lambda,q}\left(\int_x^r (r-u)g''(u)du; x\right)$$

olur. (4) ve Lemma 2 nin (ii) eşitliğinden

$$\left|\widehat{U}_n^{\lambda,q}(g(r);x) - g(x)\right| \leq U_n^{\lambda,q}\left(\left|\int_x^r (r-u)g''(u)du\right|; x\right)$$

$$+ \int_x^{x+\sigma_n^{\lambda,q}(x)} \left|x + \sigma_n^{\lambda,q}(x) - u\right| |g''(u)|du,$$

$$\leq \|g''\| \left(U_n^{\lambda,q}\left(\frac{(r-u)^2}{2}; x\right) + \frac{\left(\sigma_n^{\lambda,q}(x)\right)^2}{2} \right)$$

$$\leq \frac{\|g''\|}{2}\left(\sqrt{\varrho_n^{\lambda,q}} + \left(\sigma_n^{\lambda,q}(x)\right)^2\right)$$

eşitsizliği elde edilir.

$\delta > 0$ için Peetre κ-fonksiyoneli

$$\kappa_2(g;\delta) \coloneqq \inf_{h \in C^2[0,1]} \{\|g-h\| + \delta\|h''\|\} \tag{6}$$

olarak tanımlanır. De Vore ve Lorentz (1993) den

$$\kappa_2(g;\delta) \le L\omega_2\left(g;\sqrt{\delta}\right) \tag{7}$$

olacak biçimde bir $L > 0$ sabit vardır. Burada $\omega_2\left(f;\sqrt{\delta}\right)$ fonksiyonu g nin ikinci dereceden süreklilik modülüdür ve

$$\omega_2\left(g;\sqrt{\delta}\right) := \sup_{0<k\le\sqrt{\delta}} \sup_{x,x+k,x+2k\in[0,1]} |g(x+2k) - 2g(x+k) + g(x)| \tag{8}$$

şeklinde tanımlanır.

Teorem 2. $g \in C[0,1]$ olsun. $x \in [0,1]$ için $U_n^{\lambda,q}$ operatörleri $L > 0$ pozitif bir sabit olmak üzere aşağıdaki eşitsizliği sağlar:

$$\left|U_n^{\lambda,q}(g(t);x) - g(x)\right| \le L\omega_2\left(g;\sqrt{\frac{\sqrt{\varrho_n^{\lambda,q}}+\left(\sigma_n^{\lambda,q}(x)\right)^2}{8}}\right)$$

$$+\omega\left(g;\left|\sigma_n^{\lambda,q}(x)\right|\right).$$

İspat. $g \in C[0,1]$ olsun. $h \in C^2[0,1]$ için

$$\begin{aligned}\left|U_n^{\lambda,q}(g(r);x) - g(x)\right| &\le \left|\widehat{U}_n^{\lambda,q}(g(r)-h(r);x) - (g-h)(x)\right| \\ &\quad + \left|\widehat{U}_n^{\lambda,q}(h(r);x) - h(x)\right| \\ &\quad + \left|g\left(x+\sigma_n^{\lambda,q}(x)\right) - g(x)\right| \\ &\le 4\|g-h\| + \omega\left(g;\left|\sigma_n^{\lambda,q}(x)\right|\right) \\ &\quad + \frac{\|h''\|}{2}\left(\sqrt{\varrho_n^{\lambda,q}} + \left(\sigma_n^{\lambda,q}(x)\right)^2\right)\end{aligned} \tag{9}$$

olur. (9) eşitsizliğinin sağ tarafında her $h \in C^2[0,1]$ için infimum alınırsa,

$$\left|U_n^{\lambda,q}(g(r);x)-(x)\right|$$
$$\leq 4\kappa_2\left(g;\frac{\sqrt{\varrho_n^{\lambda,q}+\left(\sigma_n^{\lambda,q}(x)\right)^2}}{8}\right)+\omega\left(g;\left|\sigma_n^{\lambda,q}(x)\right|\right)$$

eşitsizliği elde edilir.

Grafiksel Sonuçlar

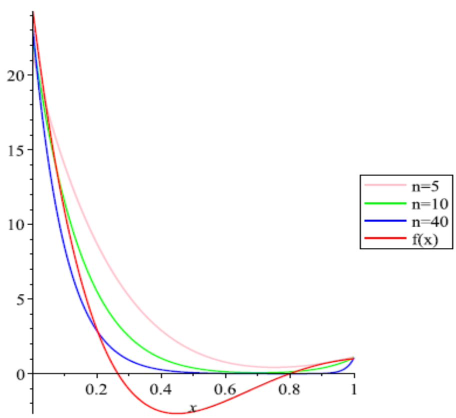
Çalışmada elde edilen teorik sonuçları desteklemek amacıyla, $U_n^{\lambda,q}(f(r);x)$ operatörler dizisinin

$$f(x)=\left(-3\left(x-\frac{4}{5}\right)^3+4\left(x-\frac{4}{5}\right)^2+3x-\frac{12}{5}\right)\left(\frac{5}{2}-x\right)^3\ln\left(\frac{5}{2}-x\right) \quad (10)$$

fonksiyonuna yaklaşımının grafiksel sonuçları gösterilecektir.

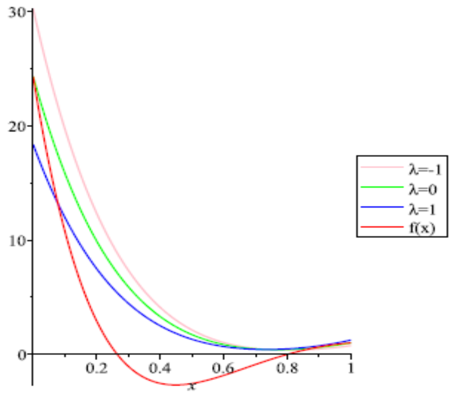
Örnek. $x\in[0,1]$ olmak üzere, ilk olarak (10) da verilen f fonksiyonu için aşağıdaki grafik $U_n^{1/2,0.9}(f;x)$ operatör dizisinin $n=5,10,40$ değerleri için $f(x)$ fonksiyonuna yakınsamasını göstermektedir (Şekil 1).

Şekil 1. $U_5^{\lambda,0.9}(f;x)$ operatörlerinin $f(x)$ fonksiyonuna yakınsaması



İkinci olarak, aşağıdaki grafik $n = 5$ ve $q = 0.9$ olmak üzere $\lambda = -1,0,1$ üç farklı değerleri için $U_5^{\lambda,0.9}(f;x)$ operatörlerinin $f(x)$ fonksiyonuna yakınsamasını göstermektedir (Şekil 2).

Şekil 2. $U_5^{\lambda,0.9}(f;x)$ operatörlerinin $f(x)$ fonksiyonuna yakınsaması



Burada $n > 1$ doğal sayı olmak üzere, $[0,1]$ üzerinde $U_n^{\lambda,q}(f;x)$ operatör dizisinin n arttıkça $f \in C[0,1]$ fonksiyonuna doğru eğilim gösterdiği gözlemlenmektedir.

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CHAPTER 7

GENUINE q − BERNSTEIN – DURRMEYER OPERATÖR DİZİSİNİN λ − PARAMETRİK GENELLEŞTİRMESİ ÜZERİNE

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Giriş

Yaklaşım teorisi kapsamında, sınırlı kapalı bir aralıkta, sürekli fonksiyonlara düzgün yakınsayan, lineer pozitif operatör dizileriyle yaklaşım problemleri incelenmektedir.

Weierstrass, [a, b] kapalı aralık üzerinde tanımlanan her bir sürekli fonksiyona düzgün yakınsayan bir polinom serisinin varlığını (Weierstrass, 1885) de kanıtlamıştır.

Bu kanıtın uzunluğu ve karmaşıklığı nedeniyle, birçok matematikçi Weierstrass Yaklaşım Teoremi üzerinde çalışarak, daha basit ve daha erişilebilir bir kanıt sağlamak için uygun diziler geliştirmişlerdir.

Bernstein (1912), [0, 1] üzerinde tanımlanan herhangi sürekli bir g fonksiyonuna yaklaşabilmek için

$$B_k(g,x) = \sum_{i=0}^{k} g\left(\frac{i}{k}\right) b_{k,i}(x), \qquad x \in [0,1]$$

şeklinde tanımlanan ve Bernstein polinomları adı verilen bir polinom serisi formüle etmiştir. Burada, $k \in \mathbb{N}$ olmak üzere $b_{k,i}(x) = \binom{k}{i} x^k (1-x)^{k-i}$ dir.

$x \in [0,1]$ için, $\lambda \in [-1,1]$ şekil parametresiyle birlikte $b_{k,i}^{\lambda}(x)$ Bezier temel fonksiyonları

$$\begin{cases} b_{k,0}^{\lambda}(x) = \mathrm{b}_{k,0}(x) - \dfrac{\lambda}{k+1}\mathrm{b}_{k+1,1}(x), \\ b_{k,i}^{\lambda}(x) = \mathrm{b}_{k,i}(x) + \dfrac{\lambda}{k+1}\left(\mathrm{b}_{k+1,i}(x) - \mathrm{b}_{k+1,i+1}(x)\right), \quad i = 1, \dots, k-1, \\ b_{k,k}^{\lambda}(x) = \mathrm{b}_{k,k}(x) + \dfrac{\lambda}{k+1}\mathrm{b}_{k+1,k}(x) \end{cases}$$

olmak üzere, λ-Bernstein operatörlerini

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$$B_k^{\lambda}(g,x)=\sum_{i=0}^{k} g\left(\frac{i}{k}\right) b_{k,i}^{\lambda}(x)$$

ile tanımladılar.

Lupaş (1987), Bernstein polinomlarının q-analoğunu literatüre kazandırmıştır. Ardından Phillips (1997), $q-$Bernstein polinomlarının bir başka genellemesini vermiştir. Birçok araştırmacı bu tip operatörleri daha da geliştirerek çeşitli $q-$Bernstein operatörlerinin yaklaşım özelliklerini vermişlerdir (Khan et al, 2021) (Sabancıgil, 2023) (Gupta V. , 2008) (Gupta & Heping, 2008) (Dinlemez & Yüksel, 2016). $q-$analiz ile ilgili temel tanım ve kavramlar için (De Sole & Kac, 2005) çalışmasına bakılabilir.

$n \in \mathbb{N}$ doğal sayısının $q-$ genelleştirmesi $[n]_q$ ile gösterilmek üzere

$$[n]_q=\begin{cases}\dfrac{1-q^n}{1-q}, & q\neq 1\\ n, & q=1\end{cases}$$

şeklinde, ayrıca $[n]_q!$ İle gösterilen $q-$faktöriyel ve $\begin{bmatrix} n\\ k\end{bmatrix}_q$ ile gösterilen $q-$binom katsayıları sırasıyla

$$[n]_q!=[1]_q[2]_q[3]_q\ldots[n]_q$$

ve

$$\begin{bmatrix} n\\ k\end{bmatrix}_q=\frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

biçiminde tanımlanmaktadır.

$(1-x)^n$ ifadesinin $q-$genelleştirmesi

$$(1\ominus x)^n=(1-x)(1-qx)\ldots(1-q^n x)$$

eşitliği ile verilmektedir, ayrıca $q-$Bernstein taban fonksiyonu

$$p_{n,k}^{q}(x)=\begin{bmatrix} n\\ k\end{bmatrix}_q x^k(1\ominus x)^{n-k}$$

biçiminde tanımlıdır.

$-1\leq\lambda\leq 1$ ve $0<\mathrm{q}\leq 1$ için $(\lambda,q)-$Bernstein temel fonksiyonları

$$\begin{cases} p_{n,0}^{\lambda,q}(x)=p_{n,0}^{q}(x)-\dfrac{\lambda}{[n+1]_q}p_{n+1,1}^{q}(x),\\ p_{n,k}^{\lambda,q}(x)=p_{n,k}^{q}(x)+\dfrac{\lambda}{[n+1]_q}\left(p_{n+1,k}^{q}(x)-p_{n+1,k+1}^{q}(x)\right), k=1,\ldots,n-1,\\ p_{n,n}^{\lambda,q}(x)=p_{n,n}^{q}(x)+\dfrac{\lambda}{[n+1]_q}p_{n+1,n}^{q}(x)\end{cases}$$

biçiminde tanımlı olmak üzere, $f \in C[0,1]$ fonksiyonu için, $q-$genuine Bernstein operatörlerinin λ- parametrik genelleştirmesi

$$U_n^{\lambda,q}(f(r),x) = f(0)p_{n,0}^{\lambda,q}(x) + f(1)p_{n,n}^{\lambda,q}(x)$$

$$+[n-1]_q \sum_{k=1}^{n-1} p_{n,k}^{\lambda,q}(x) \int_0^1 p_{n-2,k-1}^{q}(r) f(r) d_q r \qquad (1)$$

şeklinde tanımlanır.

Bu operatörler $q=1$ ve $\lambda=0$ için $U_n^{0,1}$ genuine Bernstein operatörlerine indirgenir (Chen, 1987).

Şimdi bu çalışmada elde edilecek Teorem için gerekli olan lemmalar verilecektir.

Lemma 1. $j=0,1,2$ olmak üzere $e_j(r)=r^j$ test fonksiyonlarına karşılık, (1) de tanımlı $U_n^{\lambda,q}$ operatörleri için aşağıdaki eşitlikler elde edilir:

i) $U_n^{\lambda,q}(e_0;x)=1,$

ii) $U_n^{\lambda,q}(e_1;x) = x + \frac{\lambda}{[n]_q}\left\{x - x^{n+1} - \frac{1}{q}(1 - 2x(1\ominus x)^n - x^{n+1})\right\}$

$$+\frac{\lambda}{q[n]_q[n+1]_q}\{1-(1\ominus x)^{n+1} - x^{n+1}\},$$

iii) $U_n^{\lambda,q}(e_2;x) = \frac{[2]_q}{[n+1]_q}x + \frac{q^2[n-1]_q}{[n+1]_q}x^2 + \frac{\lambda(q^2-1)}{[n+1]_q}(x^2 - x^{n+1})$

$$+\frac{[2]_q\lambda}{[n]_q[n+1]_q}(x - x^{n+1}).$$

İspat. İspatı yapabilmek için beta fonksiyonu yardımıyla ilk önce aşağıdaki eşitlikler

$$I_0 = \int_0^1 p_{n-2,k-1}^{q}(r) d_q r = \int_0^1 \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_q r^{k-1}(1\ominus r)^{n-1-k} d_q r$$

$$= \frac{[n-2]_q!}{[k-1]_q!\,[n-k-1]_q!}\int_0^1 r^{k-1}(1\ominus r)^{n-1-k} d_q r = \frac{1}{[n-1]_q}, \qquad (2)$$

$$I_1 = \int_0^1 p_{n-2,k-1}^{q}(r)\, r\, d_q r = \int_0^1 \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_q r^{k}(1\ominus r)^{n-1-k} d_q r$$

$$= \frac{[k]_q}{[n-1]_q[n]_q}, \qquad (3)$$

$$I_2 = \int_0^1 p_{n-2,k-1}^{q}(r)\, r^2 d_q r = \int_0^1 \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_q r^{k+1}(1\ominus r)^{n-1-k} d_q r$$

$$= \frac{[k]_q[k+1]_q}{[n-1]_q[n]_q[n+1]_q}, \qquad (4)$$

elde edilmiştir.

i) (1) operatör dizisinde $f(r) = e_0(r)$ alınarak (2) de verilen eşitlik kullanılırsa,

$$U_n^{\lambda,q}(e_0;x) = p_{n,0}^{\lambda,q}(x) + p_{n,n}^{\lambda,q}(x) + [n-1]_q \sum_{k=1}^{n-1} p_{n,k}^{\lambda,q}(x) I_0$$

$$= \sum_{k=0}^{n} p_{n,k}^{q}(x) - \frac{\lambda}{[n+1]_q} \sum_{k=0}^{n-1} p_{n+1,k+1}^{q}(x)$$

$$+ \frac{\lambda}{[n+1]_q} \sum_{k=1}^{n} p_{n+1,k}^{q}(x)$$

$$= 1$$

elde edilir.

ii) Yine (1) operatör dizisinde $f(r) = e_1(r)$ alınarak (3) eşitliği kullanılırsa,

$$U_n^{\lambda,q}(r,x) = p_{n,n}^{\lambda,q}(x) + [n-1]_q \sum_{k=1}^{n-1} p_{n,k}^{\lambda,q}(x) I_1$$

$$= \sum_{k=0}^{n} p_{n,k}^{\lambda,q}(x) \frac{[k]_q}{[n]_q}$$

$$= \sum_{k=1}^{n-1} \left\{ p_{n,k}^{q}(x) + \frac{\lambda}{[n+1]_q} \left(p_{n+1,k}^{q}(x) - p_{n+1,k+1}^{q}(x) \right) \right\} \frac{[k]_q}{[n]_q} + p_{n,n}^{q}(x)$$

$$+ \frac{\lambda}{[n+1]_q} p_{n+1,n}^{q}(x)$$

$$= x + \frac{\lambda}{[n+1]_q [n]_q} \sum_{k=1}^{n} \frac{[n+1]_q!}{[k-1]_q! \, [n-k+1]_q!} x^k (1 \ominus x)^{n+1-k}$$

$$- \frac{\lambda}{[n+1]_q [n]_q} \sum_{k=1}^{n-1} \frac{[n+1]_q! \, [k]_q}{[k+1]_q! \, [n-k]_q!} x^{k+1} (1 \ominus x)^{n-k}$$

elde edilir. Daha sonra eşitliğin son toplamında $[k]_q = \frac{[k+1]_q - 1}{q}$ yerine yazılırsa,

$$U_n^{\lambda,q}(r,x) = x + \frac{\lambda x}{[n]_q} \sum_{k=0}^{n-1} \frac{[n]_q!}{[k]_q! \, [n-k]_q!} x^k (1 \ominus x)^{n-k}$$

$$- \frac{\lambda x}{q[n]_q} \sum_{k=1}^{n-1} \frac{[n]_q!}{[k]_q! \, [n-k]_q!} x^k (1 \ominus x)^{n-k}$$

$$+ \frac{\lambda x}{q[n]_q} \sum_{k=1}^{n-1} \frac{[n]_q!}{[k+1]_q! \, [n-k]_q!} x^k (1 \ominus x)^{n-k}$$

$$= x + \frac{\lambda(\mathrm{x} - x^{n+1})(q-1)}{q[n]_q} + \frac{\lambda(1 - x^{n+1} - (1 \ominus x)^{n+1})}{[n+1]_q [n]_q}$$

elde edilir.

iii) Tekrar (1) operatör dizisinde $f(r) = e_2(r)$ alınarak (4) eşitliği kullanılırsa,

$$U_n^{\lambda,q}(r^2,x) = p_{n,n}^{\lambda,q}(x) + [n-1]_q \sum_{k=1}^{n-1} p_{n,k}^{\lambda,q}(x) I_2$$

$$U_n^{\lambda,q}(r^2,x) = \sum_{k=0}^{n} p_{n,k}^{\lambda,q}(x) \frac{[k]_q[k+1]_q}{[n]_q[n+1]_q}$$

$$= \frac{1}{[n+1]_q[n]_q}\left\{\sum_{k=1}^{n} p_{n,k}^{q}(x)[k]_q[k+1]_q + \frac{\lambda}{[n+1]_q}\sum_{k=1}^{n} p_{n+1,k}^{q}(x)[k]_q[k+1]_q\right.$$

$$\left. - \frac{\lambda}{[n+1]_q}\sum_{k=1}^{n-1} p_{n+1,k+1}^{q}(x)[k]_q[k+1]_q\right\}$$

$$= \frac{1}{[n+1]_q[n]_q}\left\{[n]_q\sum_{k=1}^{n} p_{n,k}^{q}(x)\frac{[k]_q}{[n]_q} + q[n]_q^2\sum_{k=1}^{n} p_{n,k}^{q}(x)\frac{[k]_q^2}{[n]_q^2}\right.$$

$$+\frac{\lambda}{[n+1]_q}\sum_{k=1}^{n}\frac{[n+1]_q!\,[k]_q[k+1]_q}{[k]_q!\,[n+1-k]_q!}x^k(1\ominus x)^{n+1-k}$$

$$\left. -\frac{\lambda}{[n+1]_q}\sum_{k=1}^{n-1}\frac{[n+1]_q!\,[k]_q[k+1]_q}{[k+1]_q!\,[n-k]_q!}x^{k+1}(1\ominus x)^{n-k}\right\}$$

elde edilir. Son eşitliğin sağ tarafındaki üçüncü serisinde

$$[k+1]_q = [2]_q + q^2[k-1]_q$$

yerine yazılmasıyla ve gerekli sadeleştirilmeler yapılarak;

$$U_n^{\lambda,q}(r^2,x) = \frac{1}{[n+1]_q[n]_q}\left\{[2]_q[n]_q x + q^2[n-1]_q[n]_q x^2\right.$$

$$+\frac{\lambda x}{[n+1]_q[n]_q}\sum_{k=0}^{n-1}\frac{[n]_q!}{[k]_q!\,[n-k]_q!}x^k(1\ominus x)^{n-k}$$

$$+\frac{\lambda q x}{[n+1]_q[n]_q}\sum_{k=0}^{n-1}\frac{[n]_q!}{[k]_q!\,[n-k]_q!}x^k(1\ominus x)^{n-k}$$

$$+\frac{\lambda q^2 x^2}{[n+1]_q}\sum_{k=0}^{n-2}\frac{[n-1]_q!}{[k]_q!\,[n-1-k]_q!}x^k(1\ominus x)^{n-1-k}$$

$$-\frac{\lambda x^2}{[n+1]_q}\sum_{k=0}^{n-2}\frac{[n-1]_q!}{[k]_q!\,[n-1-k]_q!}x^k(1\ominus x)^{n-1-k}$$

$$U_n^{\lambda,q}(r^2,x)=\frac{q^2[n-1]_q}{[n+1]_q}x^2+\frac{[2]_q x+\lambda(x^2-x^{n+1})(q^2-1)}{[n+1]_q}$$

$$+\frac{\lambda[2]_q(x-x^{n+1})}{[n+1]_q[n]_q}.$$

hesaplanmış olur.

Aşağıda verilen Lemma 2 nin ispatı açık olduğundan araştırmacıya bırakılmıştır.

Lemma 2. $(q_n)\subset(0,1)$ dizisi $n\to\infty$ için $q_n\to 1$ ve $q_n^n\to\beta$ olsun. $j=0,1,2$ için

$$\lim_{n\to\infty}(U_n^{\lambda,q_n}(e_j,x)-x^j)=0$$

dır.

Lemma 3. $j=1,2$ olmak üzere $\varphi_j(r)=(r-x)^j$ moment fonksiyonlarına karşılık U_n^{λ,q_n} operatörleri için aşağıdaki eşitlikler elde edilir:

i) $$U_n^{\lambda,q_n}(\varphi_1;x)=\frac{\lambda}{[n]_{q_n}}\left\{\left(1-\frac{1}{q_n}\right)x(1-x^n)+\frac{2x(1\ominus x)^n}{q_n}\right\}$$

$$+\frac{\lambda}{q_n[n]_{q_n}[n+1]_{q_n}}\{1-(1\ominus x)^{n+1}-x^{n+1}\},$$

ii) $$U_n^{\lambda,q_n}(\varphi_2;x)=\frac{{q_n}^2[n-1]_{q_n}}{[n+1]_{q_n}}x^2+\frac{[2]_{q_n}}{[n+1]_{q_n}}x+\frac{\lambda({q_n}^2-1)}{[n+1]_{q_n}}x^2(1-x^{n-1})+\frac{[2]_{q_n}\lambda}{[n]_{q_n}[n+1]_{q_n}}x(1-x^n)$$

$$-2x\left\{x+\frac{\lambda}{[n]_{q_n}}\left[\left(1-\frac{1}{q_n}\right)x(1-x^n)+\frac{2x(1\ominus x)^n}{q_n}\right]\right.$$

$$\left.+\frac{\lambda}{q_n[n]_{q_n}[n+1]_{q_n}}[1-(1\ominus x)^{n+1}-x^{n+1}]\right\}+x^2$$

$$=\frac{{q_n}^2[n-1]_{q_n}-[n+1]_{q_n}}{[n+1]_{q_n}}x^2+\frac{[2]_{q_n}}{[n+1]_{q_n}}x+\frac{\lambda(q^2-1)}{[n+1]_{q_n}}x^2(1-x^{n-1})$$

$$+\frac{[2]_{q_n}\lambda}{[n]_{q_n}[n+1]_{q_n}}x(1-x^n)\ -2x\frac{\lambda}{[n]_{q_n}}\left[\left(1-\frac{1}{q_n}\right)x(1-x^n)+\frac{2x(1\ominus x)^n}{q_n}\right]$$

$$-2x\frac{\lambda}{q_n[n]_{q_n}[n+1]_{q_n}}\{1-(1\ominus x)^{n+1}-x^{n+1}\}$$

$$=\frac{[2]_{q_n}x(1-x)}{[n+1]_{q_n}}+\lambda\left\{\frac{[2]_{q_n}x(1-x^n)-2x(1-(1\ominus x)^{n+1}-x^{n+1})}{[n]_{q_n}[n+1]_{q_n}}\right.$$

$$\left.+\frac{({q_n}^2-1)x^2(1-x^{n-1})}{[n+1]_{q_n}}-2x\frac{\lambda}{[n]_{q_n}}\left[\left(1-\frac{1}{q_n}\right)x(1-x^n)+\frac{2x(1\ominus x)^n}{q_n}\right]\right\}.$$

İspat. i) ve ii) nin ispatları, $U_n^{\lambda,q}$ operatör dizisinin lineerliği ve Lemma 1 in sonuçları kullanılarak tamamlanır.

Lemma 3. $(q_n) \subset (0,1)$ dizisi için, $n \to \infty$ iken $q_n \to 1$ ve $n \to \infty$ için $q_n^n \to \beta$ olsun. Bu durumda aşağıdaki limitler elde edilir:

i) $\lim\limits_{n\to\infty} [n]_{q_n} U_n^{\lambda,q_n}(\varphi_1; x) = 0,$

ii) $\lim\limits_{n\to\infty} [n]_{q_n} U_n^{\lambda,q_n}(\varphi_2; x) = 2x(1-x).$

İspat.

i) $\lim\limits_{n\to\infty} [n]_{q_n} U_n^{\lambda,q_n}(\varphi_1; x) = \lim\limits_{n\to\infty} [n]_{q_n} \left\{ \left[\frac{\lambda}{[n]_{q_n}} \left(1 - \frac{1}{q_n}\right) x(1-x^n) + \frac{2x(1\ominus x)^n}{q_n} \right] = 0, \right.$

ii)
$$\lim_{n\to\infty} [n]_{q_n} U_n^{\lambda,q_n}(\varphi_2; x) = \lim_{n\to\infty} [n]_{q_n} \frac{[2]_{q_n} x(1-x)}{[n+1]_{q_n}} + \lambda \lim_{n\to\infty} \left\{ \frac{[2]_{q_n}(x-x^{n+1}) - 2(x - x(1\ominus x)^{n+1} - x^{n+2})}{[n]_{q_n}[n+1]_{q_n}} \right.$$
$$\left. + \frac{(q_n^2-1)(x^2-x^{n+1})}{[n+1]_{q_n}} - 2x\frac{\lambda}{[n]_{q_n}} \left[\left(1 - \frac{1}{q_n}\right)(x - x^{n+1}) + \frac{2x(1\ominus x)^n}{q_n} \right] \right\}$$
$$= 2x(1-x).$$

Yaklaşım Özelliği

$U_n^{\lambda,q}$operatörleri için Korovkin tipi yaklaşım teoreminin koşulları Lemma 2 den sağlanır. Burada önce süreklilik modülünü kullanarak yakınsaklık oranını veren teorem verilecektir.

Birinci süreklilik modülünü $h \in C[0,1]$ fonksiyonu için

$$\omega(h;\delta) := max\{h(x+k) - h(x),\ \ x, x+k \in [0,1], 0 < k \leq \delta\} \tag{5}$$

olarak verilsin.

Birinci süreklilik modülü

$$|h(x+k) - h(x)| \leq \omega(h;\delta)\left(1 + \frac{|r-x|}{\delta}\right) \tag{6}$$

eşitsizliğini sağlar.

Teorem 1. $h \in C[0,1]$ olsun. $\lambda \in [-1,1]$ ve $q \in (0,1)$ sayıları için

$\varrho_n^{\lambda,q} = \frac{[2]_q}{4[n+1]_q} + \frac{8}{q[n]_q}|\lambda|$ olmak üzere,

$$\left|U_n^{\lambda,q}(h;x) - h(x)\right| \leq 2\omega\left(h; \sqrt{\varrho_n^{\lambda,q}}\right) \tag{7}$$

eşitsizliği vardır.

İspat. İlk olarak (1) de tanımlanan $U_n^{\lambda,q}$ operatörünün lineerliğini kullanacağız, daha sonra eşitliğin her iki tarafının mutlak değerini alacağız. Son olarak (6) da verilen eşitsizliği kullanırsak

$$
\begin{aligned}
\left|U_n^{\lambda,q}(h;x)-h(x)\right| &= \left|U_n^{\lambda,q}(h(r)-h(x);x)\right| \\
&\leq U_n^{\lambda,q}(|h(t)-h(x)|;x) \\
&\leq U_n^{\lambda,q}\left(\omega(h;\delta)\left(1+\tfrac{|r-x|}{\delta}\right);x\right) \\
&= \omega(h;\delta)\left(1+\tfrac{1}{\delta}U_n^{\lambda,q}(|r-x|;x)\right)
\end{aligned}
$$

olur. Bundan sonra

$U_n^{\lambda,q}(|r-x|;x)$ ifadesine Cauchy Schwarz eşitsizliği uygulanırsa,

$$
\left|U_n^{\lambda,q}(h;x)-h(x)\right| \leq \omega(h;\delta)\left(1+\frac{1}{\delta}\sqrt{U_n^{\lambda,q}((r-x)^2;x)}\right)
$$

elde edilir. Lemma 3 te elde edilen ii) sonucu kullanılırsa,

$$
\begin{aligned}
U_n^{\lambda,q_n}((r-x)^2;x) &= \frac{[2]_q x(1-x)}{[n+1]_q} + \lambda\left\{\frac{[2]_q x(1-x^n) - 2x(1-(1\ominus x)^{n+1} - x^{n+1})}{[n]_q[n+1]_q}\right. \\
&\quad \left. + \frac{(q^2-1)x^2(1-x^{n-1})}{[n+1]_q} - 2x\frac{\lambda}{[n]_q}\left[\left(1-\frac{1}{q}\right)x(1-x^n) + \frac{2x(1\ominus x)^n}{q}\right]\right\} \\
&\leq \frac{[2]_q}{4[n+1]_q} + \frac{8}{q[n]_q}|\lambda|
\end{aligned}
$$

eşitsizliği elde edilir. Burada $\varrho_n^{\lambda,q} = \frac{[2]_q}{4[n+1]_q} + \frac{8}{q[n]_q}|\lambda|$ seçilirse;

$$
\left|U_n^{\lambda,q}(h;x)-h(x)\right| \leq 2\omega\left(h;\sqrt{\varrho_n^{\lambda,q}}\right)
$$

eşitsizliği sağlanır.

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