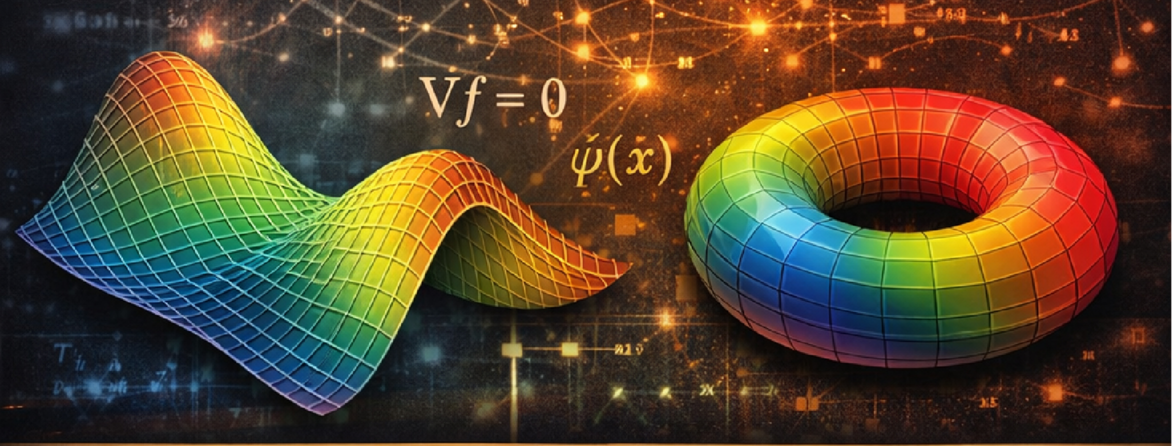

Advanced Studies in

Differential Geometry, Functional Analysis and Algebraic Systems

EDITOR: PROF. DR. ŞÜKRAN KONCA

BİDGE

BİDGE Yayınları

Advanced Studies in Differential Geometry, Functional Analysis and Algebraic Systems

Editor: PROF. DR. ŞÜKRAN KONCA

ISBN: 978-625-372-993-6

1st Edition
Page Layout By: Gözde YÜCEL
Publication Date: 2025-12-25
BİDGE Yayınları

Certificate No: 71374

www.bidgeyayinlari.com.tr - bidgeyayinlari@gmail.com
Krc Bilişim Ticaret ve Organizasyon Ltd. Şti.
Güzeltepe Mahallesi Abidin Daver Sokak Sefer Apartmanı No: 7/9 Çankaya / Ankara

FOREWORD

As one of the most fundamental and universal sciences in human history, mathematics continues to be one of the key drivers of scientific progress with both its theoretical depth and practical power.

This academic book, titled *Advanced Studies in Differential Geometry, Functional Analysis and Algebraic Systems*, brings together a collection of studies that reflect the diversity of modern mathematical research. The chapters span differential geometry, functional analysis and algebraic structures. Although each topic stands on its own, they collectively illustrate how different branches of mathematics often intersect and enrich one another. The aim of this book is to offer readers a clear and accessible overview of these contemporary themes. Whether the focus is on geometric models, analytical methods or algebraic systems, the chapters highlight both foundational ideas and advanced techniques.

We hope this book serves as a helpful resource for students and researchers interested in exploring the breadth and unity of today's mathematical landscape.

As editor, I would like to thank all our authors who contributed to the scientific content of this work and all stakeholders who contributed to the publication process.

Prof. Dr. Şükran KONCA

Izmir Bakırçay University

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CHAPTER 1

SOME FUNCTIONAL ANALYTICAL ASPECTS OF ALMOST CONVERGENCE METHOD

MAHMUT KARAKUŞ[1]

1. Introduction and Preliminaries

The multiplier form of a series $\sum_k x_k$ in a normed space X associated with an arbitrary real or complex sequence $a = (a_k)$ is given as $\sum_k a_k x_k$ and is important to understand the behaviors of the series $\sum_k x_k$ in X, (Karakuş & Başar, 2020a). For simplicity in notation, here and after, the summation without limits runs from 1 to ∞. A series $\sum_k x_k$ in a Banach space X is weakly unconditionally Cauchy (wuC) or unconditional convergent (uc) series if and only if $\sum_k a_k x_k$ is convergent for every null or bounded sequence $a = (a_k)$. Let us recall, a series $\sum_k x_k$ in a Banach space X is said to be unconditionally convergent (uc) or unconditionally Cauchy (uC) if the series $\sum_k x_{\pi(k)}$ converges or a Cauchy series for every permutation π of elements of $\mathbb{N}$, the set of positive integers. It is called weakly unconditionally Cauchy (wuC) if for every permutation π of elements of $\mathbb{N}$, the sequence $(\sum_{k=1}^{n} x_{\pi(k)})$ is a

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weakly Cauchy sequence or alternatively, $\sum_k x_k$ is wuC if and only if $\sum_k |x^*(x_k)| < \infty$ for all $x^* \in X^*$, the space of all bounded linear functionals defined on X. It is well known that every wuC series in a Banach space X is uc if and only if X contains no copy of c_0, the space of null sequences; (Diestel, 1984) and (Albiac & Kalton, 2006).

One of the most significant applications on theorem of Hahn-Banach rises the concept of Banach limits. These are non-negative, normalized, and shift-invariant linear functionals defined on ℓ_∞, (Karakuş, 2025b). Banach limits generalize the ordinary limit and have numerous applications in various mathematical fields, (Eberlein, 1950; Lorentz, 1948; Semenov & Sukochev, 2010; Semenov et al., 2019). In their research paper on functional characteristics and extreme points of the set of Banach limits on ℓ_∞, Semenov et al. provide a thorough introduction to recent results and developments in the theory of Banach limits and almost convergence, (Semenov et al., 2019). Banach limits effectively extend the limit functional on the space of convergent sequences, c, to ℓ_∞. An important result in this area is due to Lorentz (Lorentz, 1948), who, in 1948, presented an effective characterization of almost convergence by using Banach limits. Additionally, Eberlein introduced the concept of the Banach-Hausdorff limit, emphasizing the invariance of Banach limits under regular Hausdorff transformations, (Eberlein, 1950). The reader can refer to (Boos, 2000; Başar, 2022) and (Mursaleen, 2014) for the recent results and related topics in summability.

Quite recently, the authors investigated some new problems related to f_λ-convergence which is a generalization of almost convergence, (Karakuş & Başar, 2019; 2020b). The authors established some results on unconditionally convergence and weakly unconditionally Cauchy series in (Karakuş & Başar, 2022b). The authors also obtained some new characterizations related to the classical properties of a normed space such as completeness, reflexivity, Schur property, Grothendieck property, and the property of containing a copy of the space c_0, by means of the f_λ-convergence and invariant means, in (Karakuş & Başar, 2022a; 2024). By employing the concept of invariant summability, the author establishes a version of Hahn–Schur type theorem and proves several functional-analytic results concerning the multipliers of operator-valued series, (Karakuş, 2025a; 2025b).

By ω, we denote the space of all real or complex valued sequences and any vector subspace of ω is also called as a sequence space. The sequence spaces ℓ_∞, c and c_0 of bounded, convergent and null sequences are Banach spaces, with $\| x \|_\infty = \sup_{k\in\mathbb{N}}|x_k|$. By bs and cs, we also denote the Banach spaces of all sequences $x = (x_k)$ such that the series $\sum_k x_k$ is bounded and convergent, respectively, with $\| x \|_{bs} = \sup_{n\in\mathbb{N}}|\sum_{k=1}^{n} x_k|$; (Başar, 2022).

Let X and Y be two normed spaces. By $\omega(X)$, we denote the space of all X-valued sequences. By $\ell_\infty(X)$, $c(X)$, $c_0(X)$, $cs(X)$ and $bs(X)$, we also denote the spaces of all X-valued bounded, convergent, null sequences, and convergent sums and bounded sums in a real normed space X, respectively, (Karakuş, 2019). $\phi(X)$ is also the space of X-valued finitely non-zero sequences. If $\mathcal{V}$ is a vector space of X-valued sequences equipped with a locally convex Hausdorff topology, then the definition of K space is similar to scalar case, that is, $\mathcal{V}$ is a K space if the maps $x = (x_k) \mapsto x_k$ from $\mathcal{V}$ into X are continuous for all $k \in \mathbb{N}$. If $x \in X$, then by $e^k \otimes x$, we denote the sequence whose only non-zero term is x in the k^{th} place for all $k \in \mathbb{N}$. By $B(X{:}Y)$, we denote the space of all bounded and linear operators defined from X into Y. If $\mathcal{V}$ is a space of X-valued sequences such that $\phi(X) \subset \mathcal{V}$, it is said that the series $\sum_k T_k$ is $\mathcal{V}$-multiplier convergent or $\mathcal{V}$-multiplier Cauchy if the series $\sum_k T_k x_k$

converges or is a Cauchy series, i.e., the partial sums of the series $\sum_k T_k x_k$ form a norm Cauchy sequence in Y for all $(x_k) \in \mathcal{V}$, (Karakuş & Başar, 2020a).

The shift operator P is defined on ω by $(Px)_n = x_{n+1}$ for all $n \in \mathbb{N}$. A Banach limit L is defined on ℓ_∞ as a nonnegative linear functional such that $L(Px) = L(x)$ and $L(e) = 1$, where $e = (1,1,1,\dots)$; (Banach, 1978). A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent to the generalized limit $l \in \mathbb{C}$ if all Banach limits of x are l, and is denoted by $f - \lim x_k = l$. The reader can refer to (Boos, 2000; Başar, 2022), for details. Lorentz proved that a sequence $(x_k) \in \ell_\infty$ is almost convergent to the point $l \in \mathbb{C}$ if and only if

$$\lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = l$$

holds uniformly in $n \in \mathbb{N}$, (Lorentz, 1948).

By f, we denote the space of all scalar valued almost convergent sequences. It is well-known that a convergent sequence is almost convergent such that its ordinary and generalized limits are equal, (Karakuş & Başar, 2019). For the following definitions regarding vector valued almost or weakly almost convergence of a sequence and almost sum or weakly almost sum of a series in a normed space, we refer to (Aizpuru, Armario & Pérez-Fernández, 2008) and (Aizpuru et al., 2014).

Definition 1.1 A sequence $x = (x_k)$ in a real normed space X is said to be almost convergent or weakly almost convergent to $x_0 \in X$ which is called the almost limit or weakly almost limit of x, and is denoted by $f - lim x_k = x_0$ or $wf - lim x_k = x_0$, if

$$\lim_{m\to\infty}\left\|\sum_{k=n}^{n+m}\frac{x_k}{m+1}-x_0\right\|=0$$

or

$$\lim_{m\to\infty}\left|\sum_{k=n}^{n+m}\frac{x^*(x_k)}{m+1}-x^*(x_0)\right|=0$$

holds uniformly in $n\in\mathbb{N}$, for every $x^*\in X^*$, (Aizpuru, Armario & Pérez-Fernández, 2008).

By $f(X)$ and $wf(X)$, we denote the space of all almost convergent and weakly almost convergent X-valued sequences. So, every convergent sequence is almost convergent, every weakly almost convergent sequence is bounded and every almost convergent sequence is weakly almost convergent, that is, the following inclusions hold:

$$c(X)\subset f(X)\subset wf(X)\subset \ell_\infty(X).$$

Definition 1.2 A series $\sum_k x_k$ in a real normed space X is said to be almost convergent or weakly almost convergent to $x_0\in X$ which is called the almost sum or weakly almost sum of the series $\sum_k x_k$, and is denoted by $f-\sum_k x_k = x_0$ or $wf-\sum_k x_k = x_0$, if

$$\lim_{m\to\infty}\left\|\sum_{k=n}^{n+m}\frac{s_k}{m+1}-x_0\right\|=0$$

or

$$\lim_{m\to\infty}\left|\sum_{k=n}^{n+m}\frac{x^*(s_k)}{m+1}-x^*(x_0)\right|=0$$

holds uniformly in $n \in \mathbb{N}$, for every $x^* \in X^*$, respectively, where $s_k = \sum_{j=1}^{k} x_j$ for all $k \in \mathbb{N}$.

By $fs(X)$ and $wfs(X)$, we denote the space of all X-valued sequences $x = (x_k)$ such that the series $\sum_k x_k$ is almost convergent and is weakly almost convergent. Therefore, the inclusion relations $cs(X) \subset fs(X) \subset wfs(X) \subset bs(X)$ hold. Besides, by some easy calculations, $x = (x_k) \in fs(X)$ with $x_0 \in X$ if and only if

$$\lim_{m\to\infty}\left[\sum_{k=1}^{n} x_k + \frac{1}{m+1}\sum_{k=1}^{m} (m-k+1)x_{n+k}\right] = x_0,$$

uniformly in $n \in \mathbb{N}$ and $x = (x_k) \in wfs(X)$ with $x_0 \in X$ if and only if

$$\lim_{m\to\infty}\left[\sum_{k=1}^{n} x^*(x_k) + \frac{1}{m+1}\sum_{k=1}^{m} (m-k+1)x^*(x_{n+k})\right] = x^*(x_0),$$

uniformly in $n \in \mathbb{N}$ for all $x^* \in X^*$, (Aizpuru, Armario & Pérez-Fernández, 2008).

Prior to giving the required definitions and main results, we present the following lemma which states a well-known result of characterization of a wuC series in a normed space X.

Lemma 1.3 In a normed space X, a formal series $\sum_n x_n$ is a wuC series if and only if there exists a positive real H such that

$$H = \sup_{n\in\mathbb{N}}\{\|\textstyle\sum_{k=1}^{n} a_k x_k\| : |a_k| \le 1, k \in \{1,2,\dots,n\} \subset \mathbb{N}\}. \qquad (1)$$

Regarding a formal series $\sum_n x_n$ in a Banach space X is uc (respectively wuC) series if and only if for any $(t_n) \in \ell_\infty$ (respectively for any $(t_n) \in c_0$), $\sum_n t_n x_n$ converges, that is, $\sum_n x_n$ is an ℓ_∞-(respectively a c_0-) multiplier convergent series, (Diestel, 1984).

2. Results on $M_f^{\infty}(\sum_k T_k)$

We give the definitions of almost convergence and weakly almost convergence with association of an operator valued series in the vector valued multiplier spaces, and obtain some results on the characterizations of $c_0(X)$- and $\ell_\infty(X)$-multiplier convergent (Cauchy) series. Firstly, we introduce the almost convergence in a vector valued multiplier space and the summing operator $\mathcal{S}$ associated with an operator valued series.

Definition 2.1 Let X and Y be two normed spaces, and $T_k \in B(X:Y)$ for all $k \in \mathbb{N}$. The almost convergence in a vector valued multiplier space $M_f^{\infty}(\sum_k T_k)$ associated to the operator valued series $\sum_k T_k$ is defined by

$$M_f^{\infty}(\sum_k T_k) := \{x = (x_k) \in \ell_\infty(X) : f - \sum_k T_k x_k \text{ exists}\} \tag{2}$$

endowed with the sup norm and the summing operator $\mathcal{S}$ is also given as

$$\begin{array}{cccc} \mathcal{S} : & M_f^{\infty}(\sum_k T_k) & \longrightarrow & Y \\ & x = (x_k) & \longmapsto & \mathcal{S}(x) = f - \sum_k T_k x_k. \end{array} \tag{3}$$

It can be easily checked that the inclusions

$$\phi(X) \subseteq M_f^{\infty}(\sum_k T_k) \subseteq \ell_\infty(X) \tag{4}$$

hold, (Karakuş & Başar, 2020a).

Theorem 2.2 Let X and Y be any two Banach spaces, and $T_k \in B(X:Y)$ for all $k \in \mathbb{N}$. Then, the series $\sum_k T_k$ is $c_0(X)$-multiplier convergent if and only if $M_f^{\infty}(\sum_k T_k)$ is a Banach space, (Karakuş & Başar, 2020a).

Proof. Let us suppose that the series $\sum_k T_k$ is $c_0(X)$-multiplier convergent. Then, there exists a positive real H such that

$$H = \sup_{n\in\mathbb{N}} \left\{ \left\| \sum_{k=1}^{n} T_k x_k \right\| : \| x_k \| \leq 1, k \in \{1,2,\dots,n\} \subset \mathbb{N} \right\}$$

from (1).

If (x_k^m) is a Cauchy sequence in $M_f^\infty(\sum_k T_k)$ then we have $x^0 = (x_k^0) \in \ell_\infty(X)$ such that $x^m \to x^0$, as $m \to \infty$, since $\ell_\infty(X)$ is a Banach space (recall that X is a Banach space) and the relation (4) holds. Now, we prove that $x^0 \in M_f^\infty(\sum_k T_k)$. Let us define $y_m = f - \sum_k T_k x_k^m$ for all $m \in \mathbb{N}$. Now, for every $\epsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that $\| x^p - x^q \| < \epsilon/(3H)$ for all $p,q \geq m_0$. So, if $p,q \geq m_0$ are fixed, then there exists $m \in \mathbb{N}$ such that the following inequalities hold;

$$\left\| y_p - \left[\sum_{k=1}^{n} T_k x_k^p + \frac{1}{m+1} \sum_{k=1}^{m} (m-k+1) T_{n+k} x_{n+k}^p \right] \right\| < \frac{\epsilon}{3}, \quad (5)$$

$$\left\| y_q - \left[\sum_{k=1}^{n} T_k x_k^q + \frac{1}{m+1} \sum_{k=1}^{m} (m-k+1) T_{n+k} x_{n+k}^q \right] \right\| < \frac{\epsilon}{3}, \quad (6)$$

$$\left\| \sum_{k=1}^{n} T_k (x_k^p - x_k^q) + \sum_{k=1}^{m} \frac{m-k+1}{m+1} T_{n+k} (x_{n+k}^p - x_{n+k}^q) \right\| < \frac{\epsilon}{3}, \quad (7)$$

uniformly in $n \in \mathbb{N}$. So, for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\| y_p - y_q \| \leq (5) + (6) + (7) < \epsilon$$

for all $p,q \geq n_0$. Since Y is also a Banach space, there exists a $y_0 \in Y$ such that $y_m \to y_0$, as $m \to \infty$. Let us show that $f - \sum T_k x_k^0 = y_0$. We see for every $\epsilon > 0$ and fix j that $\| x^j - x^0 \| < \epsilon/(3H)$ and

$$\| y_j - y_0 \| < \frac{\epsilon}{3}. \quad (8)$$

Therefore, there exists $m_0 \in \mathbb{N}$ such that

$$\left\| y_j - \left[\sum_{k=1}^{n} T_k x_k^j + \frac{1}{m+1} \sum_{k=1}^{m} (m-k+1) T_{n+k} x_{n+k}^j \right] \right\| < \frac{\epsilon}{3}, \quad (9)$$

uniformly in $n \in \mathbb{N}$, for all $m \geq m_0$. By taking account $y_j = f - \sum_k T_k x_k^j$ for all $j \in \mathbb{N}$, we have from Lemma 5.3 that

$$\left[\sum_{k=1}^{n} T_k \frac{(x_k^j - x_k^0)}{\|x^j - x^0\|} + \sum_{k=1}^{m} \frac{m-k+1}{m+1} T_{n+k} \frac{(x_{n+k}^j - x_{n+k}^0)}{\|x^j - x^0\|} \right] \leq H, \quad (10)$$

since $\sum_k T_k$ is a $c_0(X)$-multiplier convergent series. So, for every $\epsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that

$$\left\| y_0 - \left[\sum_{k=1}^{n} T_k x_k^0 + \sum_{k=1}^{m} \frac{m-k+1}{m+1} T_{n+k} x_{n+k}^0 \right] \right\| \leq (8) + (9)$$

$$+ \left\| \sum_{k=1}^{n} T_k (x_k^j - x_k^0) + \sum_{k=1}^{m} \frac{m-k+1}{m+1} T_{n+k} (x_{n+k}^j - x_{n+k}^0) \right\| <$$

$$< \frac{2\epsilon}{3} + \| x^j - x^0 \| \cdot (10) \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3H} \cdot H = \epsilon,$$

uniformly in $n \in \mathbb{N}$, for all $m \geq m_0$. Hence, $x^0 = (x_k^0) \in M_f(\sum_k x_k)$.

Conversely, let us suppose that the space $M_f^\infty(\sum_k T_k)$ is complete and take $x = (x_k) \in c_0(X)$. Then, we have $c_0(X) \subseteq M_f^\infty(\sum_k T_k)$ since the space $M_f^\infty(\sum_k T_k)$ is closed and $\phi(X) \subset M_f^\infty(\sum_k T_k)$. Therefore, the series $\sum_k T_k x_k$ is almost convergent for all $x = (x_k) \in c_0(X)$. From the monotonicity of $c_0(X)$, we have that the series $\sum_k T_k x_k$ is subseries almost convergent, and so is weakly subseries almost convergent. As a consequence of Orlicz-Pettis theorem, $\sum_k T_k x_k$ is subseries norm convergent. This completes the proof.

Corollary 2.3 Let X and Y be Banach spaces, and $T_k \in B(X:Y)$ for all $k \in \mathbb{N}$. Then, the series $\sum_k T_k$ *is* $c_0(X)$-multiplier convergent if and only if the inclusion $c_0(X) \subseteq M_f^\infty(\sum_k T_k)$ holds, (Karakuş & Başar, 2020a).

Let X be a Banach space, Y be any normed space and $T_k \in B(X:Y)$ for all $k \in \mathbb{N}$. Consider the space $CM^\infty(\sum_k T_k)$ defined by

$$CM^\infty(\sum_k T_k) := \left\{x = (x_k) \in \ell_\infty(X): \sum_k T_k x_k - \text{is Cauchy}\right\}.$$

Proposition 2.4 Let X be a Banach space, Y be a normed space and $T_k \in B(X:Y)$ for all $k \in \mathbb{N}$. Then, the following holds, (Karakuş & Başar, 2020a):

$$CM_f^\infty(\sum_k T_k) = M_f^\infty(\sum_k T_k) \cap CM^\infty(\sum_k T_k) = M^\infty(\sum_k T_k).$$

Proof. If $x = (x_k) \in M^\infty(\sum_k T_k)$, therefore, one can see that $x = (x_k) \in M_f^\infty(\sum_k T_k) \cap CM^\infty(\sum_k T_k)$. This leads us to the inclusion $M^\infty(\sum_k T_k) \subseteq CM_f^\infty(\sum_k T_k)$.

Let us suppose that $x = (x_k) \in CM_f^\infty(\sum_k T_k)$. So, $\sum_k T_k x_k$ is almost convergent and also is a Cauchy series. Therefore, $\sum_k T_k x_k$ converges from Theorem 4.1 of (Aizpuru et al., 2014). This completes the proof.

Theorem 2.5 Let X be a Banach space, Y be any normed space and $T_k \in B(X:Y)$ for all $k \in \mathbb{N}$. Then, Y is a Banach space if and only if $M_f^\infty(\sum_k T_k)$ is a Banach space for every $c_0(X)$-multiplier Cauchy series, (Karakuş & Başar, 2020a).

Theorem 2.6 Let X and Y be two normed spaces, and $T_k \in B(X:Y)$ for all $k \in \mathbb{N}$. Then, the summing operator $\mathcal{S}$ defined by (3) is continuous if and only if the series $\sum_k T_k$ is $c_0(X)$-multiplier Cauchy, (Karakuş & Başar, 2020a).

Proof. Let us suppose that $\mathcal{S}$ is continuous and define the set G by

$$G := \{\|\sum_{k=1}^{n} T_k x_k\| : \| x_k \| \leq 1, k \in \{1,2,\ldots,n\} \subset \mathbb{N}\}. \tag{11}$$

Since the inclusion $\phi(X) \subset M_f^{\infty}(\sum_k T_k)$ holds, the series $\sum_k T_k$ is $c_0(X)$-multiplier Cauchy from the inequality $H = \sup_{n\in\mathbb{N}} G \leq \| \mathcal{S} \|$.

Conversely, let us suppose that $\sum_k T_k$ is $c_0(X)$-multiplier Cauchy series. Therefore, the set G defined by (11) is bounded and so, $H = \sup_{n\in\mathbb{N}} G$. If $x = (x_k) \in M_f^{\infty}(\sum_k T_k)$, then the proof follows from the inequality

$$\| \mathcal{S}(x) \| = \|f - \sum_k T_k x_k\| \leq H \| x \|.$$

Theorem 2.7 Let X be any normed space, Y be a Banach space and $T_k \in B(X:Y)$ for all $k \in \mathbb{N}$. Then the series $\sum_k T_k$ is $\ell_\infty(X)$-multiplier convergent if and only if the summing operator $\mathcal{S}$ defined by (3) is compact (weakly compact), (Karakuş & Başar, 2020a).

Proof. Let us suppose that $\mathcal{S}$ is compact. If $x = (x_k) \in \ell_\infty(X)$, then the set

$$H := \left\{\sum_{i\in\sigma} e^i \otimes x_i : \sigma \text{ finite and } \| x_i \| \leq 1\right\} \subset M_f^{\infty}(\sum_k T_k)$$

is bounded. By the hypothesis,

$$\mathcal{S}(H) := \left\{f - \sum_{k\in\sigma} T_k x_k : \sigma \text{ finite and } \| x_k \| \leq 1\right\}$$

is relatively compact. Therefore, the series $\sum_k T_k x_k$ is subseries norm almost convergent, and so is weakly subseries almost convergent. Further, by a consequence of Orlicz-Pettis theorem, the series $\sum_k T_k x_k$ is subseries norm convergent, that is, the series $\sum_k T_k$ is $\ell_\infty(X)$-multiplier convergent.

Conversely, suppose that the series $\sum_k T_k$ is $\ell_\infty(X)$-multiplier convergent. Let us define $\mathcal{S}_n^f$ by

$$\begin{array}{cccl} \mathcal{S}_n^f & : & M_f^\infty(\sum_k T_k) & \longrightarrow & Y \\ & & x = (x_k) & \longmapsto & \mathcal{S}_n^f(x) = f - \sum_{k=1}^n T_k x_k \end{array}$$

for every $n \in \mathbb{N}$. It is sufficient to prove that $\lim_{n\to\infty} \| \mathcal{S}_n^f - \mathcal{S} \| = 0$. Since $\sum_k T_k$ is $\ell_\infty(X)$-multiplier convergent, then the series $\sum_k T_k x_k$ is uniformly almost convergent for $\| x_k \| \leq 1$. Therefore,

$$\lim_{n\to\infty} \left\| \mathcal{S}_n^f - \mathcal{S} \right\| = \lim_{n\to\infty} \left\| \left(f - \sum_{k=1}^{n} T_k x_k \right) - \left(f - \sum_{k=1}^{\infty} T_k x_k \right) \right\|$$

$$= \lim_{n\to\infty} \left\| f - \sum_{k=n+1}^{\infty} T_k x_k \right\| = 0.$$

3. Results on $M_{wf}^\infty(\sum_k T_k)$

From the study (Karakuş & Başar, 2020a), we may give the multiplier space of weak almost convergence associated to the series $\sum_k T_k$ and give the corresponding results similar to the previous theorems and corollaries. Since their proofs are very similar to the proofs of the above results, in order to avoid the repetition of the similar statements we will give them without proof.

Definition 3.1 Let X and Y be normed spaces, and $T_k \in B(X:Y)$ for all $k \in \mathbb{N}$. The vector valued multiplier space $M_{wf}^\infty(\sum_k T_k)$ of weakly almost convergence associated to the operator series $\sum_k T_k$ is defined by

$$M_{wf}^{\infty}(\sum_k T_k) := \left\{ x = (x_k) \in \ell_\infty(X) : wf - \sum_k T_k x_k \text{ exists} \right\}$$

and the summing operator $\mathcal{S}$ is also given by

$$\begin{array}{llll} \mathcal{S} & : & M_{wf}^{\infty}(\sum_k T_k) & \longrightarrow & Y \\ & & x = (x_k) & \longmapsto & \mathcal{S}(x) = wf - \sum_k T_k x_k. \end{array} \tag{12}$$

Since the inclusion $M_f^{\infty}(\sum_k T_k) \subseteq M_{wf}^{\infty}(\sum_k T_k)$ clearly holds, we have the following, (Karakuş & Başar, 2020a):

$$\phi(X) \subseteq M_f^{\infty}(\sum_k T_k) \subseteq M_{wf}^{\infty}(\sum_k T_k) \subseteq \ell_\infty(X). \tag{13}$$

Theorem 3.2 Let X and Y be any given Banach spaces, and $T_k \in B(X:Y)$ for every $k \in \mathbb{N}$. Then, the series $\sum_k T_k$ is $c_0(X)$-multiplier convergent if and only if $M_{wf}^{\infty}(\sum_k T_k)$ is a Banach space, (Karakuş & Başar, 2020a).

Corollary 3.3 Let X and Y be Banach spaces, and $T_k \in B(X:Y)$ for all $k \in \mathbb{N}$. Then, the series $\sum_k T_k$ is $c_0(X)$-multiplier convergent if and only if the inclusion $c_0(X) \subseteq M_{wf}^{\infty}(\sum_k T_k)$ holds, (Karakuş & Başar, 2020a).

Remark 3.4 Let X and Y be any given Banach spaces, $T_k \in B(X:Y)$ for all $k \in \mathbb{N}$ and the series $\sum_k T_k$ be a $c_0(X)$-multiplier convergent. Then, the series $\sum_k y^*(T_k x_k)$ is convergent for all $x = (x_k) \in c_0(X)$ and for all $y^* \in Y^*$, that is, the series is weakly convergent. It is known that $x = (x_k) \in M_f^{\infty}(\sum_k T_k)$, and so $x = (x_k) \in M_{wf}^{\infty}(\sum_k T_k)$. This means that there exists $y_0 \in Y$ with $wf - \sum_k T_k x_k = y_0$ such that

$$\sum_k y^*(T_k x_k) = f - \sum_k y^*(T_k x_k) = y^*(y_0).$$

Therefore, the inclusion $M_f^\infty(\sum_k T_k) \subseteq M_w^\infty(\sum_k T_k)$ holds. However, we have no an idea on the sufficient conditions for the reverse inclusion. By using the similar technique, the sum $f-\sum_k (T_k x_k)$ exists for $x=(x_k)\in c_0(X)$. Hence, for $y^*\in Y$, the sum $f-\sum_k y^*(T_k x_k)$ exists. It is also known that the series $\sum_k T_k x_k$ is Cesàro convergent, and so is weakly Cesàro convergent. If there exists $y_0\in Y$ such that $wC-\sum_k T_k x_k = y_0$, then we have

$$C-\sum_k y^*(T_k x_k) = f-\sum_k y^*(T_k x_k) = y^*(y_0).$$

Therefore, the inclusion $M_C^\infty(\sum_k T_k) \subseteq M_{wf}^\infty(\sum_k T_k)$ also holds. However, we have no an idea on the sufficient conditions for the reverse inclusion, (Karakuş & Başar, 2020a).

Theorem 3.5 Let X be a Banach space, Y be any normed space and $T_k\in B(X:Y)$ for all $k\in\mathbb{N}$. Y is a Banach space if and only if the space $M_{wf}^\infty(\sum_k T_k)$ is Banach for every $c_0(X)$-multiplier Cauchy series, (Karakuş & Başar, 2020a).

Theorem 3.6 Let X and Y be normed spaces, and $T_k\in B(X:Y)$ for all $k\in\mathbb{N}$. Then, the summing operator $\mathcal{S}$ defined by (12) is continuous if and only if the series $\sum_k T_k$ is $c_0(X)$-multiplier Cauchy, (Karakuş & Başar, 2020a).

Proof. Let us suppose that the summing operator $\mathcal{S}$ is continuous and consider the set G given by (11). Then, the desired result follows from the inequality

$$\sup_{n\in\mathbb{N}} G = |wf-\textstyle\sum_k T_k x_k| \leq \| \mathcal{S} \|,$$

since the inclusion $\phi\subset S_{wf}(\sum_k T_k)$ holds.

Conversely, if $\sum_k T_k$ is $c_0(X)$-multiplier Cauchy series, then the set G is bounded and so $H = \sup_{n\in\mathbb{N}} G$. If $x = (x_k) \in M_{wf}^{\infty}(\sum_k T_k)$, then the proof follows from the inequality

$$\| \mathcal{S}(x) \| = |f - \sum_k y^*(T_k x_k)| \leq H \| x \|$$

for every $y^* \in Y$.

Theorem 3.7 Let X be any normed space, Y be a Banach space and $T_k \in B(X:Y)$ for all $k \in \mathbb{N}$. Then, the series $\sum_k T_k$ is $\ell_\infty(X)$-multiplier convergent if and only if the summing operator $\mathcal{S}$ defined by (12) is compact (weakly compact), (Karakuş & Başar, 2020a).

Remark 3.8 Let σ and τ be two linear topologies on the vector space X such that τ is linked to σ. If a Cauchy sequence $x = (x_k) \subseteq X$ is convergent to x_0 in (X, σ), then it is convergent to x_0 in (X, τ), (Swartz, 2009)

Proposition 3.9 Let X and Y be normed spaces. *If* $\sum_k T_k$ is $\ell_\infty(X)$-multiplier Cauchy, then $M_{wf}^{\infty}(\sum_k T_k) = M_f^{\infty}(\sum_k T_k)$, (Karakuş & Başar, 2020a).

Proof. Let X and Y be normed spaces, and $x = (x_k) \in M_{wf}^{\infty}(\sum_k T_k)$. From hypothesis, the partial sums of the series $\sum_k T_k x_k$ form a Cauchy sequence in Y which is also weakly almost convergent to, say $y \in Y$. So, it is almost convergent to $y \in Y$ with the norm topology since weak topology is linked to norm topology from Remark 3.8. Therefore, $x = (x_k) \in M_f^{\infty}(\sum_k T_k)$.

References

Aizpuru, A., Armario, R. & Pérez-Fernández, F. J. (2008). Almost summability and unconditionally Cauchy series. *Bull. Belg. Math. Soc. Simon Stevin*, 15, 635-644.

Aizpuru, A., Armario, R., García-Pacheco, F. J. & Pérez-Fernández, F. J. (2014). Vector-valued almost convergence and classical properties in normed spaces. *Proc. Indian Acad. Sci. Math.,* 124(1), 93-108.

Albiac, F. & Kalton, N. J. (2006). *Topics in Banach Space Theory.* New York: Springer.

Banach, S. (1978). *Théorie des Opérations Linéaires*. New York: Chelsea Publishing Company.

Başar, F. (2022). *Summability Theory and Its Applications*. New York: CRC Press-Taylor & Francis Group.

Boos, J. (2000). *Classical and Modern Methods in Summability*. New York: Oxford University Press.

Diestel, J. (1984). *Sequences and Series in Banach spaces*. New York: Springer-Verlag.

Eberlein, W.F. (1950). Banach-Hausdorff limits. *Proc. Amer. Math. Soc.,* 1(5), 662-665.

Karakuş, M. (2019). On certain vector valued multiplier spaces and series of operators. *J. Math. Anal.,* 10(2), 1-11.

Karakuş, M. & Başar, F. (2019). A generalization of almost convergence, completeness of some normed spaces with wuC series and a version of Orlicz-Pettis theorem. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 113(4), 3461-3475.

Karakuş, M. & Başar, F. (2020a). Operator valued series, almost summability of vector valued multipliers and (weak) compactness of summing operator. *J. Math. Anal. Appl.,* 484(1), 1-16.

Karakuş, M. & Başar, F. (2020b). Vector valued multiplier spaces of f_λ-summability, completeness through $c_0(X)$-multiplier convergence and continuity and compactness of summing operators. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 114:169, 1-17.

Karakuş, M. & Başar, F. (2022a). On some classical properties of normed spaces via generalized vector valued almost convergence. *Math. Slovaca*, 72 (6), 1551-1566.

Karakuş, M. & Başar, F. (2022b). Characterizations of Unconditionally Convergent and Weakly Unconditionally Cauchy Series via w_p^R-Summability, Orlicz-Pettis Type Theorems and Compact Summing Operator. *Filomat***,** 36 (18), 6347-6358.

Karakuş, M. & Başar, F. (2024). Vector valued closed subspaces and characterizations of normed spaces through σ-summability. *Indian J. Math.,* 66 (1), 85-105.

Karakuş, M. (2025a). Vector valued multipliers of invariant means and compact summimg operators. *Journal of the Institute of Science and Technology,* 15(1), 298-307.

Karakuş, M. (2025b). Spaces of multiplier σ-convergent vector valued sequences and uniform σ-summability. *AIMS Mathematics,* 10(3), 5095-5109.

Lorentz, G. G. (1948). A contribution to the theory of divergent sequences. *Acta Math.***,** 80, 167-190.

Semenov, E.M. & Sukochev, F.A. (2010). Invariant Banach limits and applications. *J. Funct. Anal.,* 259 (6), 1517-1541.

Semenov, E.M., Sukochev, F.A. & Usachev, A.S. (2019). The main classes of invariant Banach limits. *Izv. Math.,* 83 (1), 124-150.

Swartz, C. (2009). *Multiplier Convergent Series*. Singapore:World Scientific Publishing.

CHAPTER 2

ON $\boldsymbol{\beta}_{\mathbb{D}}$-DUALS OF $\mathcal{Z}$-SPACES OF SEQUENCES WITH BICOMPLEX TERMS

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Abstract

This study examines the β-dual spaces of certain Z-spaces defined on sequences with bicomplex terms. The notions of Z-spaces and duality frameworks developed by E. Malkowsky and E. Savaş for scalar sequence spaces are extended to the bicomplex setting. By employing functional analytic methods presented in A. Wilansky's *Summability through Functional Analysis*, the β-dual structures of classical sequence spaces are systematically characterized. Furthermore, the structural properties of the β-dual spaces of Cesàro-type sequence spaces are analyzed using techniques developed by P. N. Ng and P. Y. Lee. In the bicomplex context, new β-dual spaces are introduced through idempotent decomposition and the fundamental properties of hyperbolic numbers, and the essential characteristics of these spaces are rigorously established.

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INTRODUCTION

Bicomplex numbers were introduced in (Segre, 1892), and the spaces generated by these numbers have subsequently been studied within the framework of functional analysis. The theoretical foundations and fundamental properties of bicomplex spaces were presented in detail in (Luna-Elizarraras & et al., 2015). The β-dual structures of classical sequence spaces defined over scalar fields play a central role in summability theory and functional analysis. This approach has been extended to bicomplex sequence spaces by means of idempotent decomposition and the intrinsic properties of hyperbolic numbers. Within this context, the bicomplex counterparts of classical $\mathcal{Z}$-spaces and the structural properties of their associated $\beta_{\mathbb{D}}$-dual spaces are investigated. In addition, the duality properties of sequence spaces defined via paranorms have been examined in (Altay & Başar, 2006) and (Maddox, 1968), and the topological structures of these spaces, particularly with respect to continuity and convergence, have been analyzed in (Altay & Başar, 2007) and (Maddox, 1969). For a broader theoretical perspective, the reader is referred to (Luna–Elizarraras & et al., 2015) and (Toksoy & Sağır 2024).

1.Bicomplex Numbers and Their Properties

Let $\mathbb{C}$ be the set of complex numbers with i as the imaginary unit, and let j be another imaginary unit satisfying the conditions $i \neq j, ij = ji = \xi,\ i^2 = j^2 = -1$. In this case, the set represented by

$$\mathbb{BC} = \{z_1 + jz_2 : z_1, z_2 \in \mathbb{C}\}$$

is the set of all bicomplex numbers and each element of this set is called a bicomplex number. Furthermore, there is the equality $\xi^2 = (ij)^2 = 1$. Therefore, bicomplex numbers are "complex numbers with complex coefficients," which explains the name "bicomplex".

$\mathbb{BC}$ is a commutative ring with unity $1_{\mathbb{BC}} = 1 + j \cdot 0 = 1$ (Luna–Elizarraras & et al., 2015).

In addition, when $z_2 = 0$ in $z = z_1 + jz_2$, that is, $z = z_1$, the set of these numbers is represented by $\mathbb{C}(i)$. If the coefficients z_1 and z_2 are real numbers, that is, $z = x + jy$ with $x, y \in \mathbb{R}$, then the set of these numbers is represented by $\mathbb{C}(j)$. The sets $\mathbb{C}(i)$ and $\mathbb{C}(j)$ are isomorphic fields.

We take into account the bicomplex numbers $e_1 = (1 + ij)/2$ and $e_2 = (1 - ij)/2$. It can be easily seen that $e_1 \cdot e_2 = e_2 \cdot e_1 = 0$ and there are also equations $(e_1)^n = e_1$ and $(e_2)^n = e_2$ with $n \in \mathbb{N}$. For any $u = u_1 + ju_2 \in \mathbb{BC}$, we have

$$u = (u_1 - iu_2)e_1 + (u_1 + iu_2)e_2 = \delta_1 e_1 + \delta_2 e_2$$

with $\delta_1 = (u_1 - iu_2)$ and $\delta_2 = (u_1 + iu_2)$ in $\mathbb{C}(i)$. This is named as $\mathbb{C}(i)$-idempotent representation of the bicomplex number u (Işık & Duyar, 2023).

The set of hyperbolic numbers is described by

$$\mathbb{D} = \{g + \xi h : g, h \in \mathbb{R}, \xi = ij\},$$

where ξ is a hyperbolic imaginary unit with $\xi^2 = 1$. The following subsets $\mathbb{D}^+$ and $\mathbb{D}^+ \backslash \{0\}$ of $\mathbb{D}$ are called as

non-negative and positive hyperbolic numbers, respectively:

$$\mathbb{D}^+ = \{g + \xi h : g^2 - h^2 \geq 0, \qquad g \geq 0\},$$

$$\mathbb{D}^{+\backslash}\{0\} = \{g + \xi h : g^2 - h^2 \geq 0, g > 0\}.$$

Similarly, non-positive and negative hyperbolic numbers are defined as follows:

$$\mathbb{D}^- = \{g + \xi h : g^2 - h^2 \geq 0, g \leq 0\}$$

and

$$\mathbb{D}^- \backslash \{0\} = \{g + \xi h : g^2 - h^2 \geq 0, g < 0\}.$$

Let $\pi, \rho \in \mathbb{D}^+$. If $\pi - \rho \in \mathbb{D}^+$, then we write $\pi \succcurlyeq \rho$ or $\rho \preccurlyeq \pi$, and say that π is $\mathbb{D}$-greater than ρ or $\mathbb{D}$-equal to ρ, or that ρ is $\mathbb{D}$-less than π or $\mathbb{D}$-equal to π. If $\pi - \rho \in \mathbb{D}^+ \backslash \{0\}$, then we write $\pi \succ \rho$ or $\rho \prec \pi$, and say that π is $\mathbb{D}$-greater than ρ, or that ρ is $\mathbb{D}$-less than π. If $\pi = \pi_1 e_1 + \pi_2 e_2$ and $\rho = \rho_1 e_1 + \rho_2 e_2$ with real numbers π_1, π_2, ρ_1 and ρ_2, we can write $\rho \preccurlyeq \pi \Leftrightarrow \rho_1 \leq \pi_1$ and $\rho_2 \leq \pi_2$ (or $\rho \prec \pi \Leftrightarrow \rho_1 < \pi_1$ and $\rho_2 < \pi_2$). If π is a (strictly) positive hyperbolic number, then it is inversible and its inverse is also positive. Additionally, if $\pi \succ \theta = 0e_1 + 0e_2$ and $\pi \prec \rho$, then $\rho^{-1} \succ \theta = 0e_1 + 0e_2$ and $\rho^{-1} \prec \pi^{-1}$ (Luna–Elizarraras & et al., 2015).

Similarly, along with the coefficients in $\mathbb{C}(j)$, there is also a representation of the bicomplex number u with respect to e_1 and e_2.

As a result, any bicomplex number has an idempotent representation with its coefficients in $\mathbb{C}(i)$ or $\mathbb{C}(j)$, that is,

$$u = \delta_1 e_1 + \delta_2 e_2 = \rho_1 e_1 + \rho_2 e_2,$$

where $\delta_1, \delta_2 \in \mathbb{C}(i)$ and $\rho_1, \rho_2 \in \mathbb{C}(j)$.

If a function $|.|_\xi$ from $\mathbb{BC}$ to $\mathbb{D}^+$ is defined as $|u|_\xi = |u_1| e_1 + |u_2| e_2$ for each $u = u_1 e_1 + u_2 e_2 \in \mathbb{BC}$ and provides the following properties, then it is called as a $\mathbb{D}$-norm or a hyperbolic-valued norm:

a) Since $|u_1| \geq 0$ and $|u_2| \geq 0$ for a $u = (u_1 e_1 + u_2 e_2) \in \mathbb{BC}$,

$$|u|_\xi = |u_1| e_1 + |u_2| e_2 \succcurlyeq 0e_1 + 0e_2 = \theta.$$

b) $|u|_\xi = |u_1| e_1 + |u_2| e_2 = \theta = 0e_1 + 0e_2$ if and only if $|u_1| = 0$ and $|u_2| = 0$, and so

$$u = 0e_1 + 0e_2 = \theta.$$

c) $|\lambda u|_\xi = (|\lambda_1| e_1 + |\lambda_2| e_2)(|u_1| e_1 + |u_2| e_2) = |\lambda|_\xi |u|_\xi$ for $\lambda \in \mathbb{D}$.

d) $|u+v|_\xi \preccurlyeq |u|_\xi + |v|_\xi$ for $u = u_1e_1 + u_2e_2, v = v_1e_1 + v_2e_2 \in \mathbb{BC}$ (Luna–Elizarraras & et al., 2015).

If there is a $\rho \in \mathbb{D}$ such that $\pi \preccurlyeq \rho(\rho \preccurlyeq \pi)$ for all $\pi \in G$, then it is said that a subset $G \subset \mathbb{D}$ is a $\mathbb{D}$-bounded from above(below). This number $\rho \in \mathbb{D}$ is called a $\mathbb{D}$-upper($\mathbb{D}$-lower) boundary of G. If $G \subset \mathbb{D}$ is a $\mathbb{D}$-bounded set from above, then we describe the its $\mathbb{D}$-supremum showed by $\sup_{\mathbb{D}} G$, the smallest upper bound of G, and its $\mathbb{D}$-infimum showed by $\inf_{\mathbb{D}} G$, largest lower bound of G. Let $G \subset \mathbb{D}$ be a subset, let the sets G_1 and G_2 be defined by

$$G_1 = \{\pi_1 : \pi_1e_1 + \pi_2e_2 \in G\}$$

and

$$G_2 = \{\pi_2 : \pi_1e_1 + \pi_2e_2 \in G\}.$$

If G is a $\mathbb{D}$-bounded set from above(below), then the $\sup_{\mathbb{D}} G$ ($\inf_{\mathbb{D}} G$) can be computed by the formula

$$\sup_{\mathbb{D}} G = \sup G_1 e_1 + \sup G_2 e_2 \ (\inf_{\mathbb{D}} G = \inf G_1 e_1 + \inf G_2 e_2).$$

If G and H are two $\mathbb{D}$-bounded set from above, then so is $G + H$ and

$$\sup_{\mathbb{D}}(G+H) = \sup_{\mathbb{D}} G + \sup_{\mathbb{D}} H.$$

If two subsets $G \subset \mathbb{D}^+$ and $H \subset \mathbb{D}^+$ are $\mathbb{D}$-bounded from above, then so is $G \cdot H$ and

$$\sup_{\mathbb{D}}(G \cdot H) = \sup_{\mathbb{D}} G \cdot \sup_{\mathbb{D}} H.$$

For the $\mathbb{D}$-bounded subsets from below of $\mathbb{D}$, the last two equations are still true when $\inf_{\mathbb{D}}$ is written instead of $\sup_{\mathbb{D}}$ (Işık & Duyar, 2023).

2.Sequence Spaces with Bicomplex Terms

Definition 2.1. Let $\mathcal{W}$ be the set of all sequences with the terms in $\mathbb{C}(i)$. The set defined as

$$w_{\mathbb{BC}} = \{\mathcal{w} = (\mathcal{w}(s)) | \mathcal{w}_1, \mathcal{w}_2 \in \mathcal{W}\}$$

with $\mathcal{w}(s) = \mathcal{w}_1(s)e_1 + \mathcal{w}_2(s)e_2$ for all $s \in \mathbb{N}$ is the set of sequences with bicomplex terms written according to idempotent representation. This set is a commutative group under the $\mathbb{D}$-addition operation with $\oplus: w_{\mathbb{BC}} \times w_{\mathbb{BC}} \to w_{\mathbb{BC}}$, $\mathcal{w} \oplus x = (\mathcal{w}(s) + x(s))$, where

$$\mathcal{w}(s) + x(s) = \sum_{i=1}^{2} (\mathcal{w}_i(s) + x_i(s)) e_i$$

with the unit element $\theta = 0e_1 + 0e_2$, and the additive inverse

$$-\mathcal{w} = (-\mathcal{w}(n)) = (-\mathcal{w}_1(s)e_1 - \mathcal{w}_2(s)e_2)$$

of each $\mathcal{w} = (\mathcal{w}(s))_{s \in \mathbb{N}} \in w_{\mathbb{BC}}$.

Furthermore, let the $\mathbb{D}$-scalar multiplication $\odot: \mathbb{C}(i) \times w_{\mathbb{BC}} \to w_{\mathbb{BC}}$ with $z \odot \mathcal{w} = (z\mathcal{w}(s))$ be defined as

$$z \odot \mathcal{w}(s) = \sum_{i=1}^{2} (z \cdot \mathcal{w}_i(s)) e_i.$$

In this case, $w_{\mathbb{BC}}$ is a linear space over the field $\mathbb{C}(i)$.

Now, let the $\mathbb{D}$-vector product $w_{\mathbb{BC}} \times w_{\mathbb{BC}} \to w_{\mathbb{BC}}$ be defined as $\mathcal{w} \otimes x = (\mathcal{w}(s) \cdot x(s))$ with

$$\mathcal{w}(s) \cdot x(s) = \sum_{i=1}^{2} \mathcal{w}_i(s) x_i(s) e_i.$$

Under this operation, the $w_{\mathbb{BC}}$ becomes a commutative and unitary algebra over the field $\mathbb{C}(i)$.

Definition 2.2. Let $\mathcal{w} = (\mathcal{w}(s))$ with $\mathcal{w}(s) = \mathcal{w}_1(s)e_1 + \mathcal{w}(s)_2 e_2$ and let $\infty_{\mathbb{D}} = \infty e_1 + \infty e_2$. Then, the set $\Phi_{\mathbb{BC}}$ defined by

$$\Phi_{\mathbb{BC}} = \left\{\mathcal{w} \in w_{\mathbb{BC}}: \sup_{s \in \mathbb{N}}{}_{\mathbb{D}} |\mathcal{w}(s)|_{\xi} \prec \infty_{\mathbb{D}}\right\}$$

$$= \sum_{i=1}^{2} \left\{\mathcal{w}_i \in \mathcal{W}: \sup_{s \in \mathbb{N}} |\mathcal{w}_i(s)| < \infty\right\} e_i. \qquad \textbf{(2.1)}$$

is the set of $\mathbb{D}$-bounded sequences with bicomplex terms and each of its elements is called a $\mathbb{D}$-bounded sequence.

For all $w, x \in \Phi_{\mathbb{BC}}$ and $z \in \mathbb{C}(i)$, since $\sup_{\mathbb{D}}|w \oplus x|_\xi \prec \infty_{\mathbb{D}}$ and $\sup_{\mathbb{D}}|z \odot w|_\xi \prec \infty_{\mathbb{D}}$, we have $w \oplus x \in \Phi_{\mathbb{BC}}$ and $z \odot w \in \Phi_{\mathbb{BC}}$, therefore, the $\Phi_{\mathbb{BC}}$ is a subspace of the set $w_{\mathbb{BC}}$. Also, according to the (2.1),

$$\Phi_{\mathbb{BC}} = \ell_\infty e_1 + \ell_\infty e_2 \quad \textbf{(2.2)}$$

is written, where ℓ_∞ is the space of the well-known bounded sequences of scalars.

Definition 2.3. Let $\big(w(s)\big)$ be a bicomplex sequence with $w(s) = w_1(s)e_1 + w_2(s)e_2$ for every $s \in \mathbb{N}$. If, given any $\varepsilon = \varepsilon_1 e_1 + \varepsilon_2 e_2 \in \mathbb{D}^+\backslash\{0\}$, there exists at least one $s_0 \in \mathbb{N}$ such that $|w(s) - w_0|_\xi \prec \varepsilon$ for all $s > s_0$, then the sequence $\big(w(s)\big)$ is said to be $\mathbb{D}$-convergent to $w_0 = w_{01}e_1 + w_{02}e_2 \in \mathbb{BC}$ with respect to the hyperbolic-valued norm.

Again, if, given any $\varepsilon \in \mathbb{D}^+\backslash\{0\}$, there exists at least one $s_0 \in \mathbb{N}$ such that $|w(s) - w(t)|_\xi \prec \varepsilon$ for every $s, t > s_0$, then the sequence $\big(w(s)\big)$ is said to be $\mathbb{D}$-Cauchy with respect to the hyperbolic-valued norm (Luna–Elizarraras & et al., 2015).

Since $|w(s) - w_0|_\xi \prec \varepsilon$ if and only if $|w_1(s) - w_{01}| < \varepsilon_1 \wedge |w_2(s) - w_{02}| < \varepsilon_2$, it is observed that the $\mathbb{D}$-convergence of the sequence $\big(w(s)\big)$ depends on the convergence of the sequences $\big(w_1(s)\big)$ and $\big(w_2(s)\big)$ in $\mathbb{C}(i)$. If $\big(w_1(s)\big)$ or $\big(w_2(s)\big)$ diverges in $\mathbb{C}(i)$, then the sequence $\big(w(s)\big)$ is also $\mathbb{D}$-diverges in $w_{\mathbb{BC}}$. Now, let the set $C_{\mathbb{BC}}$ be defined by

$$C_{\mathbb{BC}} = \Big\{w \in w_{\mathbb{BC}}: l \in \mathbb{BC}, \lim_{s\to\infty}{}_{\mathbb{D}}|w(s) - l|_\xi = \theta\Big\}.$$

If $w(s) = w_1(s)e_1 + w_2(s)e_2$ for all $s \in \mathbb{N}$ and $l = l_1e_1 + l_2e_2$, then we can write

$$C_{\mathbb{BC}} = \sum_{i=1}^{2} \left\{ w_i \in \mathcal{W}: l_i \in \mathbb{C}, \lim_{s\to\infty} |w_i(s) - l_i| = 0 \right\} e_i. \quad \textbf{(2.3)}$$

Let us take any $w, x \in C_{\mathbb{BC}}$ and $z \in \mathbb{C}(i)$. Since $w \oplus x \in C_{\mathbb{BC}}$ and $z \odot w \in C_{\mathbb{BC}}$, $C_{\mathbb{BC}}$ is a complex subspace of $w_{\mathbb{BC}}$. If we use (2.3) and the space C of convergent sequences with complex terms, then we can write

$$C_{\mathbb{BC}} = Ce_1 + Ce_2 \,. \quad \textbf{(2.4)}$$

The space $C_{\mathbb{BC}}$ is the space of $\mathbb{D}$-convergent sequences with bicomplex terms.

Similarly, the set

$$C_{\mathbb{BC}}^{0} = \left\{ w \in w_{\mathbb{BC}}: \lim_{s\to\infty}{}_{\mathbb{D}} |w(s)|_{\xi} = \theta \right\}$$

is defined. If C_0 represents the set of sequences with complex terms that converge to zero, then

$$C_{\mathbb{BC}}^{0} = C_0e_1 + C_0e_2. \quad \textbf{(2.5)}$$

can be written. The space $C_{\mathbb{BC}}^{0}$ is the space of bicomplex sequences that converge to the zero $\theta \in \mathbb{BC}$, say $\mathbb{D}$-zero.

Let $1 \leq p < \infty$ and let $w(s) = w_1(s)e_1 + w_2(s)e_2$ for each $s \in \mathbb{N}$. Then, the following equality is observed

$$\mathcal{L}_{\mathbb{BC}}^{p} = \left\{ w \in w_{\mathbb{BC}}:\ \sum_{s=0}^{\infty} \left(|w(s)|_{\xi}\right)^{p} \prec \infty_{\mathbb{D}} \right\}$$

$$= \sum_{i=1}^{2} \{ w_i \in \mathcal{W}: \sum_{s=0}^{\infty} |w_i(s)|^{p} < \infty \} e_i.$$

Here, if ℓ_p is the set of p-absolutely summable sequences with complex terms, then we can write

$$\mathcal{L}_{\mathbb{BC}}^{p} = \ell_p e_1 + \ell_p e_2 \quad \textbf{(2.6)}$$

and $\mathcal{L}_{\mathbb{BC}}^{p}$ is the set of p-absolutely $\mathbb{D}$-summable sequences with bicomplex terms.

For $1 \leq p < \infty$, let $w, x \in \mathcal{L}_{\mathbb{BC}}^{p}$ and $w(s) = w_1(s)e_1 + w_2(s)e_2$, $x(s) = x_1(s)e_1 + x_2(s)e_2$ for every s$\in \mathbb{N}$. Then, we can write $\sum_{s=0}^{\infty}\left(|w(s)|_{\xi}\right)^{p} \prec \infty_{\mathbb{D}}$ and $\sum_{s=0}^{\infty}\left(|x(s)|_{\xi}\right)^{p} \prec \infty_{\mathbb{D}}$. This implies that $\sum_{s=0}^{\infty}|w_i(s)|^{p} < \infty$ and $\sum_{s=0}^{\infty}|x_i(s)|^{p} < \infty$ for $i = 1,2$, then we have

$$\sum_{s=0}^{\infty}\left(|(w \oplus x)(s)|_{\xi}\right)^{p} = \sum_{s=0}^{\infty}\left(\left|\sum_{i=1}^{2}\left(w_i(s) + x_i(s)\right)e_i\right|_{\xi}\right)^{p}$$

$$= \sum_{s=0}^{\infty}\left(\sum_{i=1}^{2}|w_i(s) + x_i(s)|e_i\right)^{p}$$

$$= \sum_{s=0}^{\infty}\left(\sum_{i=1}^{2}|w_i(s) + x_i(s)|^{p}e_i\right)$$

$$= \sum_{i=1}^{2}\left(\sum_{s=0}^{\infty}|w_i(s) + x_i(s)|^{p}\right)e_i$$

$$\leq \sum_{i=1}^{2}\left(\sum_{s=0}^{\infty}(|w_i(s)| + |x_i(s)|)^{p}\right)e_i$$

$$\leq \sum_{i=1}^{2}\left(\sum_{s=0}^{\infty} 2^{p-1}(|w_i(s)|^{p} + |x_i(s)|^{p})\right)e_i$$

$$= 2^{p-1}\sum_{i=1}^{2}\begin{pmatrix}(\sum_{s=0}^{\infty}|w_i(s)|^{p})e_i \\ +(\sum_{s=0}^{\infty}|x_i(s)|^{p})e_i\end{pmatrix}$$

$$= 2^{p-1}\begin{Bmatrix}(\sum_{s=0}^{\infty}|w_1(s)|^{p} + \sum_{s=0}^{\infty}|x_1(s)|^{p})e_1 \\ +(\sum_{s=0}^{\infty}|w_2(s)|^{p} + \sum_{s=0}^{\infty}|x_2(s)|^{p})e_2\end{Bmatrix} \prec \infty_{\mathbb{D}}.$$

Furthermore, for any $z \in \mathbb{C}(i)$, we can write

$$\sum_{s=0}^{\infty}\left(|(z \odot w)(s)|_{\xi}\right)^{p} = \sum_{s=0}^{\infty}\left(\sum_{i=1}^{2}|z||w_i(s)|e_i\right)^{p}$$

$$= |z|^{p}\sum_{s=0}^{\infty}\left(\sum_{i=1}^{2}|w_i(s)|^{p}e_i\right)$$

$$= |z|^{p}\sum_{i=1}^{2}\left(\left(\sum_{s=0}^{\infty}|w_i(s)|^{p}\right)e_i\right) \prec \infty_{\mathbb{D}}.$$

Thus, it is shown that the set $\mathcal{L}_{\mathbb{BC}}^{p}$ is a subspace of $w_{\mathbb{BC}}$.

The space of convergent series with complex terms is denoted by $\mathcal{CS}$. For every $s \in \mathbb{N}$, let $\tau(s) = \tau_1(s)e_1 + \tau_2(s)e_2$. Then

$$\mathcal{CS}_{\mathbb{BC}} = \{\tau : (\tau(k)) = \left(\sum_{s=0}^{k} \tau(s)\right) \in C_{\mathbb{BC}}\}$$

$$= \left\{\tau : (\tau(k)) = \left(\left(\sum_{s=0}^{k} \tau_1(s)\right)e_1 + \left(\sum_{s=0}^{k} \tau_2(s)\right)e_2\right) \in C_{\mathbb{BC}}\right\}$$

$$= \{\tau_1 \in \mathcal{W} : (\tau_1(k)) = \left(\sum_{s=0}^{k} \tau_1(s)\right) \in C\}e_1$$

$$+\{\tau_2 \in \mathcal{W} : (\tau_2(k)) = \left(\sum_{s=0}^{k} \tau_2(s)\right) \in C\}e_2$$

and in this case, it is written

$$\mathcal{CS}_{\mathbb{BC}} = \mathcal{CS}e_1 + \mathcal{CS}e_2. \qquad \textbf{(2.7)}$$

Furthermore, due to its closure under $\mathbb{D}$-addition and $\mathbb{D}$-scalar multiplication, the set $\mathcal{CS}_{\mathbb{BC}}$ is a subspace of the space $w_{\mathbb{BC}}$. Thus, $\mathcal{CS}_{\mathbb{BC}}$ is the space of the $\mathbb{D}$-convergent series with bicomplex terms.

The space of bounded series with complex terms is denoted by $\mathcal{BS}$. For every $s \in \mathbb{N}$, let $w(s) = w_1(s)e_1 + w_2(s)e_2$. The $\mathbb{D}$-bounded series space of bicomplex terms is defined as

$$\mathcal{BS}_{\mathbb{BC}} = \left\{w \in w_{\mathbb{BC}} : \sup_{k\in\mathbb{N}}{}_{\mathbb{D}} \left|\sum_{s=0}^{k} w(s)\right|_{\xi} \prec \infty_{\mathbb{D}}\right\}.$$

If some properties of $\mathbb{BC}$ are used, we have

$$\mathcal{BS}_{\mathbb{BC}} = \left\{w_1 \in \mathcal{W} : \sup_{k\in\mathbb{N}} \left|\sum_{s=0}^{k} w_1(s)\right| < \infty\right\} e_1$$

$$+\left\{w_2 \in \mathcal{W} : \sup_{k\in\mathbb{N}} \left|\sum_{s=0}^{k} w_2(s)\right| < \infty\right\} e_2.$$

In this case, it is obviously written

$$\mathcal{BS}_{\mathbb{BC}} = \mathcal{BS}e_1 + \mathcal{BS}e_2. \qquad \textbf{(2.8)}$$

Now, let $w = (w(s))$ and $x = (x(s))$ with $w(s) = w_1(s)e_1 + w_2(s)e_2$ and $x(s) = x_1(s)e_1 + x_2(s)e_2$ for all $s \in \mathbb{N}$ be two elements of $\mathcal{BS}_{\mathbb{BC}}$. Then

$$\sup_{k\in\mathbb{N}}{}_{\mathbb{D}} \left|\sum_{s=0}^{k} w(s)\right|_{\xi} \prec \infty_{\mathbb{D}} \text{ and } \sup_{k\in\mathbb{N}}{}_{\mathbb{D}} \left|\sum_{s=0}^{k} x(s)\right|_{\xi} \prec \infty_{\mathbb{D}}.$$

This implies that

$$\sup_{k\in\mathbb{N}}\left|\sum_{s=0}^{k}\mathcal{w}_i(s)\right| < \infty \wedge \sup_{k\in\mathbb{N}}\left|\sum_{s=0}^{k}x_i(s)\right| < \infty$$

for $i = 1,2$. Also, we have

$$\sup_{k\in\mathbb{N}}{}_{\mathbb{D}}\left|\sum_{s=0}^{k}\mathcal{w}(s) \oplus x(s)\right|_{\xi} =$$

$$\sup_{k\in\mathbb{N}}{}_{\mathbb{D}}\left\{\begin{matrix}\left(\left|\sum_{s=0}^{k}\mathcal{w}_1(s) + x_1(s)\right|\right)e_1\\ +\left(\left|\sum_{s=0}^{k}\mathcal{w}_2(s) + x_2(s)\right|\right)e_2\end{matrix}\right\}$$

$$\preccurlyeq \sup_{k\in\mathbb{N}}{}_{\mathbb{D}}\left\{\begin{matrix}\left(\left|\sum_{s=0}^{k}\mathcal{w}_1(s)\right| + \left|\sum_{s=0}^{k}x_1(s)\right|\right)e_1\\ +\left(\left|\sum_{s=0}^{k}\mathcal{w}_2(s)\right| + \left|\sum_{s=0}^{k}x_2(s)\right|\right)e_2\end{matrix}\right\}$$

$$= \left\{\sup_{k\in\mathbb{N}}\left(\left|\sum_{s=0}^{k}\mathcal{w}_1(s)\right| + \left|\sum_{s=0}^{k}x_1(s)\right|\right)\right\}e_1$$

$$+\left\{\sup_{k\in\mathbb{N}}\left(\left|\sum_{s=0}^{k}\mathcal{w}_2(s)\right| + \left|\sum_{s=0}^{k}x_2(s)\right|\right)\right\}e_2,$$

and, for any $z \in \mathbb{C}(i)$,

$$\sup_{k\in\mathbb{N}}{}_{\mathbb{D}}\left|\sum_{s=0}^{k}(z \odot \mathcal{w})(s)\right|_{\xi} = \sup_{k\in\mathbb{N}}{}_{\mathbb{D}}\left\{\left(\left|\sum_{s=0}^{k}z\mathcal{w}_1(s)\right|\right)e_1 + \left(\left|\sum_{s=0}^{k}z\mathcal{w}_2(s)\right|\right)e_2\right\}$$

$$= |z|\left(\left\{\sup_{k\in\mathbb{N}}\left(\left|\sum_{s=0}^{k}\mathcal{w}_1(s)\right|\right)\right\}e_1 + \left\{\sup_{k\in\mathbb{N}}\left(\left|\sum_{s=0}^{k}\mathcal{w}_2(s)\right|\right)\right\}e_2\right).$$

Thus, it is seen that the set $\mathcal{BS}_{\mathbb{BC}}$ is closed under $\mathbb{D}$-addition and $\mathbb{D}$-scalar multiplication, and therefore, it is a subspace of the space $w_{\mathbb{BC}}$.

The space of sequences of bounded variation with complex terms is denoted by $\mathcal{BV}$. Let $\tau = (\tau(s))$ with $\tau(s) = \tau_1(s)e_1 + \tau_2(s)e_2$ for every $s \in \mathbb{N}$. The set called the $\mathbb{D}$-bounded variation series space with bicomplex terms is represented as

$$\mathcal{BV}_{\mathbb{BC}} = \left\{\tau: \sum_{s=1}^{\infty}|\tau(s) - \tau(s-1)|_{\xi} \prec \infty_{\mathbb{D}}\right\}$$

$$= \{\tau_1 \in \mathcal{W} : \sum_{s=1}^{\infty} |\tau_1(s) - \tau_1(s-1)| < \infty\} e_1$$

$$+ \{\tau_2 \in \mathcal{W} : \sum_{s=1}^{\infty} |\tau_2(s) - \tau_2(s-1)| < \infty\} e_2.$$

Therefore, it is clear that

$$\mathcal{BV}_{\mathbb{BC}} = \mathcal{BV} e_1 + \mathcal{BV} e_2 \qquad \textbf{(2.9)}$$

can be written.

For proving that $\mathcal{BV}_{\mathbb{BC}}$ is a subspace of $w_{\mathbb{BC}}$, if it is taken two arbitrary sequences $w, x \in \mathcal{BV}_{\mathbb{BC}}$, this explicitly states that their membership in $\mathcal{BV}_{\mathbb{BC}}$ implies for their complex components with $\sum_{s=1}^{\infty} |w_i(s) - w_i(s-1)| < \infty$ and $\sum_{s=1}^{\infty} |x_i(s) - x_i(s-1)| < \infty$ for $i = 1,2$. The next steps in a full proof would involve showing that the sum of two sequences in $\mathcal{BV}_{\mathbb{BC}}$ is also in $\mathcal{BV}_{\mathbb{BC}}$ (closure under $\mathbb{D}$-addition) and that a scalar multiple of a sequence in $\mathcal{BV}_{\mathbb{BC}}$ is also in $\mathcal{BV}_{\mathbb{BC}}$ (closure under $\mathbb{D}$-scalar multiplication). These proofs would follow a similar pattern to the $\mathcal{L}_{\mathbb{BC}}^{p}$ and $\mathcal{BS}_{\mathbb{BC}}$ proofs, utilizing the component-wise nature of the bicomplex operations and standard inequalities from real/complex analysis.

The fundamental concept is that sequence spaces with bicomplex terms can be expressed as sums of idempotent representations that take the sequence spaces with complex terms as components. This means that whether you are working with convergence, boundedness, summability, or bounded variation in the bicomplex space, the behavior of these sequences can be decomposed into two independent, parallel behaviors of their complex components. The idempotent basis (e_1, e_2) allows for this elegant decomposition. It effectively simplifies the analysis of bicomplex sequence spaces by transforming them into a well-understood framework for complex sequence spaces.

Lemma 2.1. For every $s \in \mathbb{N}$, let $w(s) = w_1(s)e_1 + w_2(s)e_2$. The sequence $(w(s))$ is a $\mathbb{D}$-Cauchy sequence in the space $w_{\mathbb{BC}}$ if and only if the sequences $(\mathcal{w}_1(s))$ and $(\mathcal{w}_2(s))$ are Cauchy sequences in the field $\mathbb{C}(i)$ (Işık & Duyar, 2023).

Definition 2.4. A transformation $\|\cdot\|_{\mathbb{BC}}$ from any linear space $\mathcal{X} \subset w_{\mathbb{BC}}$ to the set $\mathbb{D}^+$ is called a $\mathbb{D}$-valued $\mathbb{D}$-norm (or simply a $\mathbb{D}$-norm) if it satisfies the following properties:

a) For every $\mathcal{w} \in w_{\mathbb{BC}}$, $\|\mathcal{w}\|_{\mathbb{BC}} = \theta \Leftrightarrow \mathcal{w} = \theta$,

b) For any $\lambda \in \mathbb{D}^+$ and $\mathcal{w} \in w_{\mathbb{BC}}$, $\|\lambda \otimes \mathcal{w}\|_{\mathbb{BC}} = |\lambda|_\xi \otimes \|\mathcal{w}\|_{\mathbb{BC}}$,

c) For all $\mathcal{w}, x \in w_{\mathbb{BC}}$, $\|\mathcal{w} \oplus x\|_{\mathbb{BC}} \preccurlyeq \|\mathcal{w}\|_{\mathbb{BC}} \oplus \|x\|_{\mathbb{BC}}$($\mathbb{D}$-triangle inequality).

If every $\mathbb{D}$-Cauchy sequence in $\mathcal{X}$ is $\mathbb{D}$-convergent with respect to $\mathbb{D}$-norm, then it is called a $\mathbb{D}$-Banach space.

Lemma 2.2. If $(\mathcal{X}_1, \|\cdot\|_1)$ and $(\mathcal{X}_2, \|\cdot\|_2)$ are two normed subspaces of $\mathcal{W}$, then the transformation $\|\cdot\|_{\mathbb{BC}}$ defined from the space $\mathcal{X} = \mathcal{X}_1 e_1 + \mathcal{X}_2 e_2$ to $\mathbb{D}^+$ with $\|x\|_{\mathbb{BC}} = \|x_1\|_1 e_1 + \|x_2\|_2 e_2$ for all $x = (x(s)) = (x_1(s)e_1 + x_2(s)e_2) \in \mathcal{X}$ is a $\mathbb{D}$-norm and consequently, $(\mathcal{X}, \|\cdot\|_{\mathbb{BC}})$ is a $\mathbb{D}$-normed space.

Proof. It's straightforward to show that the $\mathbb{D}$-norm axioms are satisfied by the construction presented in Lemma 1.2. The theorem essentially guarantees that this structure always produces a valid $\mathbb{D}$-norm.

Lemma 2.3. Let $(\mathcal{X}_1, \|\cdot\|_1)$, $(\mathcal{X}_2, \|\cdot\|_2)$ and $(\mathcal{X}, \|\cdot\|_{\mathbb{BC}})$ be given as in Lemma 1.2. The space $(\mathcal{X}, \|\cdot\|_{\mathbb{BC}})$ is a $\mathbb{D}$-Banach space if and only if the spaces $(\mathcal{X}_1, \|\cdot\|_1)$ and $(\mathcal{X}_2, \|\cdot\|_2)$ are Banach spaces.

Proof. Assume that $\mathcal{X}$ is a $\mathbb{D}$-Banach Space, then every $\mathbb{D}$-Cauchy sequence in $\mathcal{X}$ converges (in the $\mathbb{D}$-norm) to an element within $\mathcal{X}$. Let $(x_i(s))$ be one each Cauchy sequences in $\mathcal{X}_i$ for $i = 1,2$. Given any $\varepsilon > 0$, we can write $\varepsilon = \varepsilon e_1 + \varepsilon e_2 \in \mathbb{D}^+$. According to the hypothesis, there exists an $r \in \mathbb{N}$ such that

$$\|x(s) - x(t)\|_{\mathbb{BC}} = \sum_{i=1}^{2} \|x_i(s) - x_i(t)\|_i e_i \prec \varepsilon$$

for $s, t > r$. Thus, $(x(s))$ is a $\mathbb{D}$-Cauchy sequence in $\mathcal{X}$ and since $\mathcal{X}$ is a $\mathbb{D}$-Banach space, this sequence $(x(s))$ must $\mathbb{D}$-converge to some element $\alpha = \alpha_1 e_1 + \alpha_2 e_2 \in \mathcal{X}$. This means that $\|x - \alpha\|_{\mathbb{BC}} \to \theta$ and so

$$\|x_1 - \alpha_1\|_1 \to 0 \wedge \|x_2 - \alpha_2\|_2 \to 0. \qquad \textbf{(2.10)}$$

This shows that the Cauchy sequences $(x_1(s))$ and $(x_2(s))$ converge to $\alpha_1 \in \mathcal{X}_1$ and $\alpha_2 \in \mathcal{X}_2$ respectively. This proves that $(\mathcal{X}_1, \|\cdot\|_1)$ and $(\mathcal{X}_2, \|\cdot\|_2)$ are Banach spaces.

Conversely, assume $\mathcal{X}_1$ and $\mathcal{X}_2$ are Banach Spaces, then every Cauchy sequence in $\mathcal{X}_i$ converges in $\mathcal{X}_i$ for $i = 1,2$. Now, let $(x(s))$ be a $\mathbb{D}$-Cauchy sequence in $\mathcal{X}$. Then, given any $\varepsilon = \varepsilon_1 e_1 + \varepsilon_2 e_2 \succ \theta$, there exists an $r \in \mathbb{N}$ such that $\|x(s) - x(t)\|_{\mathbb{BC}} \prec \varepsilon$ for all $s, t > r$. Thus, we have $\|x_1(s) - x_1(t)\|_1 < \varepsilon_1$ and $\|x_2(s) - x_2(t)\|_2 < \varepsilon_2$ for all $s, t > r$. Hence $(x_i(s))$ are Cauchy sequences in $\mathcal{X}_i$ for $i = 1,2$. Since $\mathcal{X}_1$ and $\mathcal{X}_2$ are one each Banach spaces, these Cauchy sequences must converge to some $\alpha_1 \in \mathcal{X}_1$ and $\alpha_2 \in \mathcal{X}_2$, respectively. Let $\alpha = \alpha_1 e_1 + \alpha_2 e_2 \in \mathcal{X}$. It is easily seen that arbitrary $\mathbb{D}$-Cauchy sequence $(x(s))$ in $\mathcal{X}$ converges to $\alpha \in \mathcal{X}$. Thus, $(\mathcal{X}, \|\cdot\|_{\mathbb{BC}})$ is a $\mathbb{D}$-Banach space.

If Lemma 2.2 and Lemma 2.3 are applied, then the spaces $\Phi_{\mathbb{BC}}$, $C_{\mathbb{BC}}$ and $C_{\mathbb{BC}}^0$ are $\mathbb{D}$-Banach spaces with respect to the $\mathbb{D}$-norm defined as $\|x\|_{\mathbb{BC}}^{\infty} = \sup_{\mathbb{D}, s\in\mathbb{N}} |x(s)|_{\xi}$, the space $\mathcal{L}_{\mathbb{BC}}^{p}$, $1 \leq p < \infty$ is a $\mathbb{D}$-Banach space with respect to the $\mathbb{D}$-norm defined by

$$\|x\|_{\mathbb{BC}}^{p} = \left(\sum_{s=0}^{\infty} \left(|x(s)|_{\xi}\right)^{p}\right)^{\frac{1}{p}},$$

the spaces $\mathcal{BS}_{\mathbb{BC}}$ and $\mathcal{CS}_{\mathbb{BC}}$ are $\mathbb{D}$-Banach spaces with respect to the $\mathbb{D}$-norm defined as

$$\|x\|_{\mathbb{BC}}^{bs} = \|x\|_{\mathbb{BC}}^{cs} = \sup_{m\in\mathbb{N}}{}_{\mathbb{D}} \left|\sum_{s=0}^{m} x(s)\right|_{\xi},$$

and the space $\mathcal{BV}_{\mathbb{BC}}$ is a $\mathbb{D}$-Banach space with respect to the $\mathbb{D}$-norm defined by

$$\|x\|_{\mathbb{BC}}^{bv} = \sup_{m\in\mathbb{N}}{}_{\mathbb{D}} \left(\sum_{s=0}^{m} |x(s) - x(s-1)|_{\xi}\right), \text{ with } x(-1) = \theta$$

3.Multiplier and β-Dual Spaces of Sequence Spaces with Bicomplex Terms

Definition 3.1. Assume that two subspaces $\mathcal{X} = \mathcal{X}_1 e_1 + \mathcal{X}_2 e_2 \subset w_{\mathbb{BC}}$ and $\mathcal{Y} = \mathcal{Y}_1 e_1 + \mathcal{Y}_2 e_2 \subset w_{\mathbb{BC}}$. Let $w = (w(s))$ and $x = (x(s))$ be two sequences in $w_{\mathbb{BC}}$ with $w(s) = w_1(s)e_1 + w_2(s)e_2$ and $x(s) = x_1(s)e_1 + x_2(s)e_2$ for each $s \in \mathbb{N}$. Then, the set defined by

$$\mathcal{M}_{\mathbb{BC}}(\mathcal{X},\mathcal{Y}) = \{w \in w_{\mathbb{BC}}: \forall x \in \mathcal{X}, w \otimes x \in \mathcal{Y}\}$$

can be expressed according to the idempotent representation as follows

$$\begin{aligned}\mathcal{M}_{\mathbb{BC}}(\mathcal{X},\mathcal{Y}) &= \{w_1: \forall x_1 \in \mathcal{X}_1,\ w_1 x_1 \in \mathcal{Y}_1\}e_1 \\ &\quad + \{w_2: \forall x_2 \in \mathcal{X}_2, w_2 x_2 \in \mathcal{Y}_2\}e_2 \\ &= \mathcal{M}(\mathcal{X}_1,\mathcal{Y}_1)e_1 + \mathcal{M}(\mathcal{X}_2,\mathcal{Y}_2)e_2. \qquad \textbf{(3.1)}\end{aligned}$$

This set is called the $\mathbb{D}$-multiplier space of $\mathcal{X}$ and $\mathcal{Y}$. If we specifically take $\mathcal{Y} = \mathcal{CS}_{\mathbb{BC}}$, then we can write

$$\mathcal{M}_{\mathbb{BC}}(\mathcal{X},\mathcal{CS}_{\mathbb{BC}}) = \mathcal{M}(\mathcal{X}_1,\mathcal{CS})e_1 + \mathcal{M}(\mathcal{X}_2,\mathcal{CS})e_2 \qquad \textbf{(3.2)}$$

This particular set is called the $\beta_{\mathbb{D}}$-dual of $\mathcal{X}$. Furthermore, the $\beta_{\mathbb{D}}$-dual of the space $\mathcal{X}$ is, by virtue of (3.2), the linear combination of the β-duals of $\mathcal{X}_1 \subset \mathcal{W}$ and $\mathcal{X}_2 \subset \mathcal{W}$, with respect to idempotent bases. Thus, it is written

$$\mathcal{X}^{\beta_{\mathbb{D}}} = \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, \mathcal{CS}_{\mathbb{BC}}) = \mathcal{X}_1^{\beta} e_1 + \mathcal{X}_2^{\beta} e_2.$$

Definition 3.2. For every $s, t \in \mathbb{N}$, let $a(s,t) = a_1(s,t)e_1 + a_2(s,t)e_2$ with $a(s,t) \in \mathbb{BC}$. In this case, $A_{\mathbb{BC}} = \big(a(s,t)\big)$ is called as a infinite matrix (or a double sequence) with bicomplex terms. Furthermore, using the infinite matrices $A_1 = \big(a_1(s,t)\big)$ and $A_2 = \big(a_2(s,t)\big)$ with complex terms, we can write

$$\begin{aligned} A_{\mathbb{BC}} &= \big(a(s,t)\big) = (a_1(s,t)e_1 + a_2(s,t)e_2) \\ &= \big(a_1(s,t)\big)e_1 + \big(a_2(s,t)\big)e_2 = A_1 e_1 + A_2 e_2 \end{aligned} \quad \textbf{(3.3)}$$

The product of two infinite matrices $A_{\mathbb{BC}}$ and $B_{\mathbb{BC}}$ with bicomplex terms is expressed as

$$\begin{aligned} A_{\mathbb{BC}} B_{\mathbb{BC}} &= (A_1 e_1 + A_2 e_2)(B_1 e_1 + B_2 e_2) \\ &= (A_1 B_1) e_1 + (A_2 B_2) e_2. \end{aligned} \quad \textbf{(3.4)}$$

This definition essentially states that a matrix with bicomplex terms can be decomposed into two matrices with complex terms, each multiplied by idempotent units e_1 or e_2, and their multiplication is performed component-wise.

Definition 3.3. Let $\mathcal{X} \subset w_{\mathbb{BC}}$ and $\mathcal{Y} \subset w_{\mathbb{BC}}$. Also, let $x = \big(x(s)\big) \in \mathcal{X} = \mathcal{X}_1 e_1 + \mathcal{X}_2 e_2$ with $x(s) = x_1(s)e_1 + x_2(s)e_2$ for every $s \in \mathbb{N}$. If $A_{\mathbb{BC}}^k(x) = \sum_{s=0}^{\infty} a(k,s) \otimes x(s)$ for every $k \in \mathbb{N}$ and $x \in \mathcal{X}$ is $\mathbb{D}$-convergent and $\left(A_{\mathbb{BC}}^k(x)\right)_{k \in \mathbb{N}} \in \mathcal{Y}$, then the class of all infinite matrices $A_{\mathbb{BC}}$ with bicomplex terms is denoted by $(\mathcal{X}, \mathcal{Y})$. Thus, for all $k \in \mathbb{N}$ and all $x \in \mathcal{X}$, we write

$$(\mathcal{X}, \mathcal{Y}) = \left\{A_{\mathbb{BC}} : A_{\mathbb{BC}}^k \in \mathcal{X}^{\beta_{\mathbb{D}}} \wedge \left(A_{\mathbb{BC}}^k(x)\right)_k \in \mathcal{Y}\right\}. \quad \textbf{(3.5)}$$

Definition 3.4. The set defined as

$$\mathcal{X}_{A_{\mathbb{BC}}} = \left\{ w \in w_{\mathbb{BC}} \colon \left(A_{\mathbb{BC}}^{k}(w) \right)_{k \in \mathbb{N}} \in \mathcal{X} \right\} \qquad \textbf{(3.6)}$$

is called the domain of the matrix $A_{\mathbb{BC}}$ over the space $\mathcal{X}$.

Definition 3.5. Let $\mu = (\mu(s)) \in w_{\mathbb{BC}}$ with $\mu(s) = \mu_1(s)e_1 + \mu_2(s)e_2$ for every $s \in \mathbb{N}$. The set defined as

$$\mu^{-1} * \mathcal{X} = \{ w \in w_{\mathbb{BC}} \colon \mu \otimes w \in \mathcal{X} \} \qquad \textbf{(3.7)}$$

is called the $\mathbb{D}$-reflection of the bicomplex sequence μ over the space $\mathcal{X}$.

Furthermore, the set of all bicomplex sequences $\mu = (\mu(s))$ with $\mu_1(s) \neq 0$ and $\mu_2(s) \neq 0$ for every $s \in \mathbb{N}$ is denoted by $\mathcal{U}_{\mathbb{BC}}$. The multiplicative inverse of each $u \in \mathcal{U}_{\mathbb{BC}}$ is written as

$$\frac{1}{u(s)} = \frac{1}{u_1(s)} e_1 + \frac{1}{u_2(s)} e_2 \qquad \textbf{(3.8)}$$

(Işık & Duyar, 2023).

For any negative index s, we use the rule $w(s) = \theta$, namely $w_1(s) = 0$ and $w_2(s) = 0$. The symbols $\Delta_{\mathbb{BC}}$ and $\Delta_{\mathbb{BC}}^{+}$ denote the following operators:

$$\begin{aligned} \Delta_{\mathbb{BC}} w &= \left(\Delta_{\mathbb{BC}} w(s) \right) = \left(w(s) - w(s-1) \right) \\ &= \left(\Delta w_1(s) e_1 + \Delta w_2(s) e_2 \right) \\ &= \Delta w_1 e_1 + \Delta w_2 e_2 \end{aligned} \qquad \textbf{(3.9)}$$

and

$$\begin{aligned} \Delta_{\mathbb{BC}}^{+} w &= \left(\Delta_{\mathbb{BC}}^{+} w(s) \right) = \left(w(s) - w(s+1) \right) \\ &= \left(\Delta^{+} w_1(s) \right) e_1 + \left(\Delta^{+} w_2(s) \right) e_2 \\ &= \Delta^{+} w_1 e_1 + \Delta^{+} w_2 e_2 \,. \end{aligned} \qquad \textbf{(3.10)}$$

In essence, these definitions introduce backward and forward difference operators for bicomplex sequences. They work by taking the difference between consecutive terms of the sequence, and this operation is applied independently to each idempotent component (e_1 and e_2) of the bicomplex numbers in the sequence.

Definition 3.6. Let $t(k,s) = t_1(k,s)e_1 + t_2(k,s)e_2$ for all $k,s \in \mathbb{N}$. A infinite matrix $\mathcal{T}_{\mathbb{BC}} = \big(t(k,s)\big)_{k,s\in\mathbb{N}}$ with bicomplex terms is called a $\mathbb{D}$-triangular matrix, if its entries satisfy the following conditions:

- If $s > k$, then $t(k,s) = \theta$ (equivalently $t_1(k,s) = 0$ and $t_2(k,s) = 0$).
- If $s = k$, then $t(k,s) \neq \theta$.

In simpler terms, a $\mathbb{D}$-triangular matrix is an infinite matrix with bicomplex terms, where all entries above the main diagonal are zero and the entries on the main diagonal are non-zero. This is analogous to the definition of a lower triangular matrix in standard matrix theory, extended to bicomplex numbers and their idempotent representation.

Definition 3.7. For every $k,s \in \mathbb{N}$, let $\hbar(k,s) = \hbar_1(k,s)e_1 + \hbar_2(k,s)e_2 \in \mathbb{BC}$ with

$$\hbar_i(k,s) = \begin{cases} 1, & 0 \le s \le k \\ 0, & \text{otherwise} \end{cases}, i = 1,2$$

and let $\mathcal{H}_{\mathbb{BC}} = \big(\hbar(k,s)\big)_{k,s\in\mathbb{N}}$. Given any $u, v \in \mathcal{U}_{\mathbb{BC}}$ and the subset $\mathcal{R} = \mathcal{R}_1 e_1 + \mathcal{R}_2 e_2 \subset w_{\mathbb{BC}}$, the space defined by

$$\mathcal{Z}_{\mathbb{BC}} = \mathcal{Z}_{\mathbb{BC}}(u, v; \mathcal{R}) = v^{-1} * (u^{-1} * \mathcal{R})_{\mathcal{H}_{\mathbb{BC}}} \qquad \textbf{(3.11)}$$

is called a bicomplex $\mathcal{Z}$-space.

The equality (3.11) can be expanded as follows:

$$\mathcal{Z}_{\mathbb{BC}} = v^{-1} * (u^{-1} * \mathcal{R})_{\mathcal{H}_{\mathbb{BC}}}$$

$$= \{w \in w_{\mathbb{BC}} : v \otimes w \in (u^{-1} * \mathcal{R})_{\mathcal{H}_{\mathbb{BC}}}\}$$

$$= \{w \in w_{\mathbb{BC}}: \mathcal{H}_{\mathbb{BC}}(v \otimes w) \in u^{-1} * \mathcal{R}\}$$
$$= \{w \in w_{\mathbb{BC}}: u \otimes \mathcal{H}_{\mathbb{BC}}(v \otimes w) \in \mathcal{R}\}.$$

For every $s \in \mathbb{N}$, since

$$u(s) \otimes \left(\mathcal{H}^s_{\mathbb{BC}}(v \otimes w)\right)$$
$$= u(s) \otimes \left(\sum_{k=0}^{\infty} h(s,k) \otimes \left(v(k) \otimes \ w(k)\right)\right)$$
$$= u(s) \otimes \left(\sum_{k=0}^{s}\left(v(k) \otimes w(k)\right)\right)$$
$$= \sum_{i=1}^{2}(u_i(s) \sum_{k=0}^{s} v_i(k) w_i(k))e_i,$$

the $\mathcal{Z}_{\mathbb{BC}}$-space can be expressed as

$$\{w: (\sum_{i=1}^{2}(u_i(s) \sum_{k=0}^{s} v_i(k) w_i(k))e_i)_{s=0}^{\infty} \in \mathcal{R}\}$$
$$= \{w_1 \in \mathcal{W}: (u_1(s) \sum_{k=0}^{s} v_1(k) w_1(k))_{s=0}^{\infty} \in \mathcal{R}_1\}e_1$$
$$+\{w_2 \in \mathcal{W}: (u_2(s) \sum_{k=0}^{s} v_2(k) w_2(k))_{S=0}^{\infty} \in \mathcal{R}_2\}e_2.$$

Consequently, this proves that the $\mathcal{Z}_{\mathbb{BC}}$-space with bicomplex terms can be written as the linear combination of two complex $\mathcal{Z}$-spaces as follows:

$$\mathcal{Z}_{\mathbb{BC}}(u, v; \mathcal{R}) = \mathcal{Z}(u_1, v_1; \mathcal{R}_1)e_1 + \mathcal{Z}(u_2, v_2; \mathcal{R}_2)e_2 \quad \textbf{(3.12)}$$

This derivation shows that operations on bicomplex sequence spaces often split into independent operations on the corresponding idempotent components, which simplifies the analysis considerably.

Example 3.1. Let $c = \left(c(s)\right)$ be a sequence with $c(s) = c_1(s)e_1 + c_2(s)e_2 = 1e_1 + 1e_2 = 1_{\mathbb{D}}$ for all $s \in \mathbb{N}$. If $u = v = c$, then, we have $\mathcal{Z}_{\mathbb{BC}}(c, c; C_{\mathbb{BC}}) = \mathcal{CS}_{\mathbb{BC}}$ and $\mathcal{Z}_{\mathbb{BC}}(c, c; \Phi_{\mathbb{BC}}) = \mathcal{BS}_{\mathbb{BC}}$.

Solution. It is a known result that $\mathcal{Z}(c, c; C) = \mathcal{CS}$ and $\mathcal{Z}(c, c; \ell_\infty) = \mathcal{BS}$ (Malkowsky & Savaş, 2004). Thus, we have

$$\mathcal{Z}_{\mathbb{BC}}(c, c; C_{\mathbb{BC}}) = \mathcal{Z}_{\mathbb{BC}}(c, c; Ce_1 + Ce_2)$$
$$= \mathcal{Z}(c_1, c_1; C)e_1 + \mathcal{Z}(c_2, c_2; C)e_2$$

$$= \mathcal{CS}e_1 + \mathcal{CS}e_2 = \mathcal{CS}_{\mathbb{BC}}$$

and

$$\mathcal{Z}_{\mathbb{BC}}(c, c; \Phi_{\mathbb{BC}}) = \mathcal{Z}_{\mathbb{BC}}(c, c; \ell_\infty e_1 + \ell_\infty e_2)$$
$$= \mathcal{Z}(c_1, c_1; \ell_\infty)e_1 + \mathcal{Z}(c_2, c_2; \ell_\infty)e_2$$
$$= \mathcal{BS}e_1 + \mathcal{BS}e_2 = \mathcal{BS}_{\mathbb{BC}}.$$

For every $s \in \mathbb{N}$, let $q(s) = q_1(s)e_1 + q_2(s)e_2 \succ \theta$. Consider the sequences $\mathscr{v} = q = (q(s))$, $Q_1(m) = \sum_{s=0}^{m} q_1(s)$ and $Q_2(m) = \sum_{s=0}^{m} q_2(s)$. Let the sequence $Q = (Q(m))$ with $Q(m) = Q_1(m)e_1 + Q_2(m)e_2$ for every m$\in \mathbb{N}$ and $\mathscr{u} = 1/Q$ be given. In this scenario, if we use equation (3.12), then, for any $\tau = \tau_1 e_1 + \tau_2 e_2 \in \mathcal{Z}_{\mathbb{BC}}$, we get

$$\tau \in \mathcal{Z}(\mathscr{u}_1, \mathscr{v}_1; \mathcal{R}_1)e_1 + \mathcal{Z}(\mathscr{u}_2, \mathscr{v}_2; \mathcal{R}_2)e_2$$
$$\Leftrightarrow \tau_1 \in \mathcal{Z}\left(\frac{1}{Q_1}, q_1; \mathcal{R}_1\right) \wedge \tau_2 \in \mathcal{Z}\left(\frac{1}{Q_2}, q_2; \mathcal{R}_2\right).$$

Additionally, the following equalities have been established for $i = 1,2$ (Jarrah & Malkowsky, 1998):

- $\mathcal{Z}\left(\frac{1}{Q_i}, q_i; C_0\right) = (\overline{\mathcal{N}}, q_i)_0$
- $\mathcal{Z}\left(\frac{1}{Q_i}, q_i; C\right) = (\overline{\mathcal{N}}, q_i)$
- $\mathcal{Z}\left(\frac{1}{Q_i}, q_i; \ell_\infty\right) = (\overline{\mathcal{N}}, q_i)_\infty.$

Using these equalities and (3.12), we can write the bicomplex $\mathcal{Z}$-spaces as

$$\mathcal{Z}_{\mathbb{BC}}\left(\frac{1}{Q}, q; C_{\mathbb{BC}}^{0}\right) = (\overline{\mathcal{N}}, q)_{\mathbb{BC}}^{0}$$
$$= \mathcal{Z}\left(\frac{1}{Q_1}, q_1; C_0\right)e_1 + \mathcal{Z}\left(\frac{1}{Q_2}, q_2; C_0\right)e_2$$
$$= (\overline{\mathcal{N}}, q_1)_0 e_1 + (\overline{\mathcal{N}}, q_2)_0 e_2, \qquad \textbf{(3.13)}$$

$$\mathcal{Z}_{\mathbb{BC}}\left(\frac{1}{Q}, q; C_{\mathbb{BC}}\right) = \left(\overline{\mathcal{N}}, q\right)_{\mathbb{BC}}$$

$$= \mathcal{Z}\left(\frac{1}{Q_1}, q_1; C\right) e_1 + \mathcal{Z}\left(\frac{1}{Q_2}, q_2; C\right) e_2$$

$$= \left(\overline{\mathcal{N}}, q_1\right) e_1 + \left(\overline{\mathcal{N}}, q_2\right) e_2, \quad \textbf{(3.14)}$$

$$\mathcal{Z}_{\mathbb{BC}}\left(\frac{1}{Q}, q; \Phi_{\mathbb{BC}}\right) = \left(\overline{\mathcal{N}}, q\right)_{\mathbb{BC}}^{\infty}$$

$$= \mathcal{Z}\left(\frac{1}{Q_1}, q_1; \ell_\infty\right) e_1 + \mathcal{Z}\left(\frac{1}{Q_2}, q_2; \ell_\infty\right) e_2$$

$$= \left(\overline{\mathcal{N}}, q_1\right)_\infty e_1 + \left(\overline{\mathcal{N}}, q_2\right)_\infty e_2. \quad \textbf{(3.15)}$$

These sets are the sets of $\mathbb{D}$**-**null**,** $\mathbb{D}$-convergent**,** and $\mathbb{D}$-bounded sequences of $\mathbb{D}$-weighted means with bicomplex terms, respectively.

Lemma 3.1. Let $\mathcal{X} = \mathcal{X}_1 e_1 + \mathcal{X}_2 e_2 \subset w_{\mathbb{BC}}$, $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}_1 e_1 + \widehat{\mathcal{X}}_2 e_2 \subset w_{\mathbb{BC}}$, $\mathcal{Y} = \mathcal{Y}_1 e_1 + \mathcal{Y}_2 e_2 \subset w_{\mathbb{BC}}$ and $\widehat{\mathcal{Y}} = \widehat{\mathcal{Y}}_1 e_1 + \widehat{\mathcal{Y}}_2 e_2 \subset w_{\mathbb{BC}}$, and also let $u \in \mathcal{U}_{\mathbb{BC}}$. The following propositions hold:

a) If $\widehat{\mathcal{X}} \subset \mathcal{X}$, then $\mathcal{M}_{\mathbb{BC}}(\mathcal{X}, \mathcal{Y}) \subset \mathcal{M}_{\mathbb{BC}}\left(\widehat{\mathcal{X}}, \mathcal{Y}\right)$.

b) If $\mathcal{Y} \subset \widehat{\mathcal{Y}}$, then $\mathcal{M}_{\mathbb{BC}}(\mathcal{X}, \mathcal{Y}) \subset \mathcal{M}_{\mathbb{BC}}\left(\mathcal{X}, \widehat{\mathcal{Y}}\right)$.

c) $\mathcal{M}_{\mathbb{BC}}(u^{-1} * \mathcal{X}, \mathcal{Y}) = (1/u)^{-1} * \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, \mathcal{Y})$.

Proof. a) Assume $\widehat{\mathcal{X}} \subset \mathcal{X}$. This implies that $\widehat{\mathcal{X}}_1 \subset \mathcal{X}_1$ and $\widehat{\mathcal{X}}_2 \subset \mathcal{X}_2$. It is a known result in (Malkowsky & Savaş, 2004) that if $\widehat{\mathcal{X}}_i \subset \mathcal{X}_i$ for $i = 1{,}2$ and $\mathcal{T} \subset \mathcal{W}$, then $\mathcal{M}(\mathcal{X}_i, \mathcal{T}) \subset \mathcal{M}\left(\widetilde{\mathcal{X}}_i, \mathcal{T}\right)$. Using (3.1), we have

$$\begin{aligned} \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, \mathcal{Y}) &= \mathcal{M}(\mathcal{X}_1, \mathcal{Y}_1) e_1 + \mathcal{M}(\mathcal{X}_2, \mathcal{Y}_2) e_2 \\ &\subset \mathcal{M}\left(\widehat{\mathcal{X}}_1, \mathcal{Y}_1\right) e_1 + \mathcal{M}\left(\widehat{\mathcal{X}}_2, \mathcal{Y}_2\right) e_2 \\ &= \mathcal{M}_{\mathbb{BC}}\left(\widehat{\mathcal{X}}, \mathcal{Y}\right). \end{aligned}$$

b) Assume $\mathcal{Y} \subset \hat{\mathcal{Y}}$. This implies that $\mathcal{Y}_1 \subset \hat{\mathcal{Y}}_1$ and $\mathcal{Y}_2 \subset \hat{\mathcal{Y}}_2$. It is a known result in (Malkowsky & Savaş, 2004) that if $\mathcal{Y}_i \subset \hat{\mathcal{Y}}_i$ and $\mathcal{P} \subset \mathcal{W}$, (a general complex sequence space), then $\mathcal{M}(\mathcal{P}, \mathcal{Y}_i) \subset \mathcal{M}\left(\mathcal{P}, \hat{\mathcal{Y}}_i\right)$. Thus,

$$\begin{aligned}\mathcal{M}_{\mathbb{BC}}(\mathcal{X}, \mathcal{Y}) &= \mathcal{M}(\mathcal{X}_1, \mathcal{Y}_1)\mathrm{e}_1 + \mathcal{M}(\mathcal{X}_2, \mathcal{Y}_2)\mathrm{e}_2 \\ &\subset \mathcal{M}\left(\mathcal{X}_1, \hat{\mathcal{Y}}_1\right)e_1 + \mathcal{M}\left(\mathcal{X}_2, \hat{\mathcal{Y}}_2\right)e_2 \\ &= \mathcal{M}_{\mathbb{BC}}\left(\mathcal{X}, \hat{\mathcal{Y}}\right)\end{aligned}$$

is obtained.

c) First, let us show the equality $u^{-1} * \mathcal{X} = ({u_1}^{-1} * \mathcal{X}_1)e_1 + ({u_2}^{-1} * \mathcal{X}_2)e_2$. Let $v \in u^{-1} * \mathcal{X}$. By (3.7), we get

$$\begin{aligned}v \in u^{-1} * \mathcal{X} &\Leftrightarrow u \otimes v \in \mathcal{X} = \mathcal{X}_1 e_1 + \mathcal{X}_2 e_2 \\ &\Leftrightarrow u_1 v_1 e_1 + u_2 v_2 e_2 \in \mathcal{X}_1 e_1 + \mathcal{X}_2 e_2 \\ &\Leftrightarrow u_1 v_1 \in \mathcal{X}_1 \wedge u_2 v_2 \in \mathcal{X}_2 \\ &\Leftrightarrow v_1 \in {u_1}^{-1} * \mathcal{X}_1 \wedge v_2 \in {u_2}^{-1} * \mathcal{X}_2 \\ &\Leftrightarrow v \in ({u_1}^{-1} * \mathcal{X}_1)e_1 + ({u_2}^{-1} * \mathcal{X}_2)e_2.\end{aligned}$$

Thus, the desired is achieved. Also, using (3.1), we have

$$\begin{aligned}&\mathcal{M}_{\mathbb{BC}}(u^{-1} * \mathcal{X}, \mathcal{Y}) \\ &\quad = \mathcal{M}({u_1}^{-1} * \mathcal{X}_1, \mathcal{Y}_1)e_1 + \mathcal{M}({u_2}^{-1} * \mathcal{X}_2, \mathcal{Y}_2)e_2.\end{aligned}$$

It is known that $\mathcal{M}({u_i}^{-1} * \mathcal{X}_i, \mathcal{Y}_i) = (1/u_i)^{-1} * \mathcal{M}(\mathcal{X}_i, \mathcal{Y}_i)$ for $i = 1,2$, (Malkowsky & Savaş, 2004). Substituting this into the bicomplex expression, we have

$$\mathcal{M}_{\mathbb{BC}}(u^{-1} * \mathcal{X}, \mathcal{Y}) = \sum_{i=1}^{2}\left((1/u_i)^{-1} * \mathcal{M}(\mathcal{X}_i, \mathcal{Y}_i)\right)e_i.$$

Let $\lambda = \lambda_1 e_1 + \lambda_2 e_2 \in \mathcal{M}_{\mathbb{BC}}(u^{-1} * \mathcal{X}, \mathcal{Y}))$. Then, we get, for $i = 1,2$,

$$\lambda_i \in (1/u_i)^{-1} * \mathcal{M}(\mathcal{X}_i, \mathcal{Y}_i) \Leftrightarrow (1/u_i)\lambda_i \in \mathcal{M}(\mathcal{X}_i, \mathcal{Y}_i)$$

$$\Leftrightarrow (1/u) \otimes \lambda \in \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, \mathcal{Y})$$
$$\Leftrightarrow \lambda \in (1/u)^{-1} * \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, \mathcal{Y}).$$

This completes the proof of part c).

Lemma 3.2. The following equalities are true:

(a) $\mathcal{M}_{\mathbb{BC}}(C^0_{\mathbb{BC}}, C^0_{\mathbb{BC}}) = \Phi_{\mathbb{BC}}$,

(b) $\mathcal{M}_{\mathbb{BC}}(C_{\mathbb{BC}}, C_{\mathbb{BC}}) = C_{\mathbb{BC}}$,

(c) $\mathcal{M}_{\mathbb{BC}}(\Phi_{\mathbb{BC}}, C^0_{\mathbb{BC}}) = C^0_{\mathbb{BC}}$,

(d) If $1 \le q < \infty$, $\mathcal{M}_{\mathbb{BC}}\left(\mathcal{L}^q_{\mathbb{BC}}, C^0_{\mathbb{BC}}\right) = \Phi_{\mathbb{BC}}$.

Proof. It is well known that $\mathcal{M}(C_0, C_0) = \ell_\infty$, $\mathcal{M}(C, C) = C$, $\mathcal{M}(\ell_\infty, C_0) = C_0$ and if $1 \le q < \infty$, then $\mathcal{M}(\ell^q, C_0) = \ell_\infty$ (Malkowsky & Savaş, 2004). Let us use Lemma 2.1.

(a) $$\mathcal{M}_{\mathbb{BC}}(C^0_{\mathbb{BC}}, C^0_{\mathbb{BC}}) = \mathcal{M}_{\mathbb{BC}}(C_0e_1 + C_0e_2, C_0e_1 + C_0e_2)$$
$$= \mathcal{M}(C_0, C_0)e_1 + \mathcal{M}(C_0, C_0)e_2$$
$$= \ell_\infty e_1 + \ell_\infty e_2 = \Phi_{\mathbb{BC}},$$

(b) $$\mathcal{M}_{\mathbb{BC}}(C_{\mathbb{BC}}, C_{\mathbb{BC}}) = \mathcal{M}_{\mathbb{BC}}(Ce_1 + Ce_2, Ce_1 + Ce_2)$$
$$= \mathcal{M}(C, C)e_1 + \mathcal{M}(C, C)e_2$$
$$= Ce_1 + Ce_2 = C_{\mathbb{BC}},$$

(c) $$\mathcal{M}_{\mathbb{BC}}(\Phi_{\mathbb{BC}}, C^0_{\mathbb{BC}}) = \mathcal{M}_{\mathbb{BC}}(\ell_\infty e_1 + \ell_\infty e_2, C_0e_1 + C_0e_2)$$
$$= \mathcal{M}(\ell_\infty, C_0)e_1 + \mathcal{M}(\ell_\infty, C_0)e_2$$
$$= C_0e_1 + C_0e_2 = C^0_{\mathbb{BC}},$$

(d) $$\mathcal{M}_{\mathbb{BC}}\left(\mathcal{L}^q_{\mathbb{BC}}, C^0_{\mathbb{BC}}\right) = \mathcal{M}_{\mathbb{BC}}(\ell^q e_1 + \ell^q e_2, C_0e_1 + C_0e_2)$$
$$= \mathcal{M}(\ell^q, C_0)e_1 + \mathcal{M}(\ell^q, C_0)e_2$$
$$= \ell_\infty e_1 + \ell_\infty e_2 = \Phi_{\mathbb{BC}}.$$

Lemma 3.3. Let $\mathcal{X} = \mathcal{X}_1 e_1 + \mathcal{X}_2 e_2 \subset w_{\mathbb{BC}}$ and $\mathcal{Y} = \mathcal{X}_{\mathcal{H}_{\mathbb{BC}}}$. Also, let $\mathcal{K}_1 = \left(\mathcal{X}^{\beta_{\mathbb{D}}}\right)_{\Delta^+_{\mathbb{BC}}}$, $\mathcal{K}_2 = \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, C_{\mathbb{BC}})$ and $\mathcal{K}_3 = \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, C^0_{\mathbb{BC}})$. Then

$$\mathcal{K}_1 \cap \mathcal{K}_2 \subset \mathcal{Y}^{\beta_{\mathbb{D}}} \qquad \textbf{(3.16)}$$

and if $\mathcal{X}$ is a $\mathbb{D}$-normal set, then

$$\mathcal{Y}^{\beta_{\mathbb{D}}} = \mathcal{K}_1 \cap \mathcal{K}_3. \qquad \textbf{(3.17)}$$

Moreover, for every $a = \big(a(s)\big) \in \mathcal{Y}^{\beta_{\mathbb{D}}}$ and $y = \big(y(s)\big) \in \mathcal{Y}$,

$$\sum_{s=0}^{\infty} a(s) \otimes y(s) = \sum_{s=0}^{\infty} \Delta^+_{\mathbb{BC}}\big(a(s)\big) \otimes \mathcal{H}^s_{\mathbb{BC}}(y). \qquad \textbf{(3.18)}$$

Proof. Let any $y = \big(y(s)\big) = (y_1(s)e_1 + y_2(s)e_2) \in \mathcal{Y}$ be given. Then $\mathcal{H}_{\mathbb{BC}}(y) \in \mathcal{X}$ and

$$\mathcal{H}^s_{\mathbb{BC}}(y) = \sum_{k=0}^{\infty} h(s,k) \otimes y(k) = \sum_{k=0}^{s} y(k) = w(s),$$
$w = \big(w(s)\big) \in \mathcal{X}$.

Also, for all $s \in \mathbb{N}$, we have

$$\begin{aligned} w(s) &= \left(\textstyle\sum_{k=0}^{s} w(k)\right) - \left(\textstyle\sum_{k=0}^{s-1} w(k)\right) \\ &= \textstyle\sum_{k=0}^{s}\big(w(k) - w(k-1)\big) \\ &= \textstyle\sum_{k=0}^{s} \Delta_{\mathbb{BC}} w(k) = \sum_{k=0}^{s} y(k) = \mathcal{H}^s_{\mathbb{BC}}(y). \end{aligned}$$

Thus, using $\mathcal{Y} = \mathcal{X}_{\mathcal{H}_{\mathbb{BC}}}$, we can write

$$w \in \mathcal{X} \Leftrightarrow w = \mathcal{H}_{\mathbb{BC}}(y) \in \mathcal{X} \Leftrightarrow y \in \mathcal{X}_{\mathcal{H}_{\mathbb{BC}}}. \qquad \textbf{(3.19)}$$

Additionally, for each $a = \big(a(s)\big) = (a_1(s)e_1 + a_2(s)e_2) \in w_{\mathbb{BC}}$, we get

$$\sum_{k=0}^{s} a(k) \otimes y(k) = \sum_{k=0}^{s} a(k) \otimes \Delta_{\mathbb{BC}} w(k)$$
$$= \left(\sum_{k=0}^{s} a_1(k)\big(\Delta w_1(k)\big)\right)e_1 + \left(\sum_{k=0}^{s} a_2(k)\big(\Delta w_2(k)\big)\right)e_2. \qquad \textbf{(3.20)}$$

Also, using $\Delta w_i(-1) = 0$, we have

$$\sum_{k=0}^{s} a_i(k)\big(\Delta w_i(k)\big) = \sum_{k=0}^{s} a_i(k)\big(w_i(k) - w_i(k-1)\big)$$
$$= \sum_{k=0}^{s} a_i(k)w_i(k) - \sum_{k=0}^{s} a_i(k)w_i(k-1)$$
$$= \sum_{k=0}^{s-1} a_i(k)w_i(k) + a_i(s)w_i(s) - \sum_{k=0}^{s-1} a_i(k+1)w_i(k)$$
$$= \sum_{k=0}^{s-1}\{a_i(k) - a_i(k+1)\}w_i(k) + a_i(s)w_i(s)$$
$$= \sum_{k=0}^{s-1}\big(\Delta^{+} a_i(k)\big)w_i(k) + a_i(s)w_i(s).$$

If we substitute this in (3.20) and use (3.10), then we obtain

$$\sum_{k=0}^{s} a(k) \otimes y(k)$$
$$= \sum_{k=0}^{s-1} \Delta^{+}_{\mathbb{BC}} a(k) \otimes w(k) \oplus a(s) \otimes w(s). \quad \textbf{(3.21)}$$

Now, let an arbitrary $a = \big(a(s)\big) \in \mathcal{K}_1 \cap \mathcal{K}_2$ be given. Then, since $a \in \mathcal{K}_1 = \big(\mathcal{X}^{\beta_{\mathbb{D}}}\big)_{\Delta^{+}_{\mathbb{BC}}}$, we can write $\Delta^{+}_{\mathbb{BC}} a \in \mathcal{X}^{\beta_{\mathbb{D}}}$. Accordingly, for every $w \in \mathcal{X}$, we have $\Delta^{+}_{\mathbb{BC}}(a) \otimes w \in \mathcal{CS}_{\mathbb{BC}}$, and so

$$\sum_{s=0}^{\infty}\big(\Delta^{+}_{\mathbb{BC}} a(s)\big) \otimes w(s) \prec \infty_{\mathbb{D}}. \quad \textbf{(3.22)}$$

Also, we can write

$$\big(\Delta^{+}_{\mathbb{BC}} a(s) \otimes w(s)\big) = \Delta^{+}_{\mathbb{BC}}(a) \otimes w$$
$$= \Delta^{+}_{\mathbb{BC}}(a) \otimes \mathcal{H}_{\mathbb{BC}}(y) \in \mathcal{CS}_{\mathbb{BC}}. \quad \textbf{(3.23)}$$

Furthermore, since $a \in \mathcal{K}_2 = \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, C_{\mathbb{BC}})$, by (3.19), we have

$$a \otimes w = a \otimes \mathcal{H}_{\mathbb{BC}}(y) \in C_{\mathbb{BC}} \quad \textbf{(3.24)}$$

for every $w \in \mathcal{X}$. If (3.21), (3.22), and (3.24) are used, then we get $\sum_{k=0}^{\infty} a(k) \otimes y(k) \prec \infty_{\mathbb{D}}$, which means $a \otimes y \in \mathcal{CS}_{\mathbb{BC}}$ for every $y \in \mathcal{Y}$. This shows that $a \in \mathcal{Y}^{\beta_{\mathbb{D}}}$, and (3.16) is proved.

If $C^{0}_{\mathbb{BC}} \subset C_{\mathbb{BC}}$ and Lemma 3.1. (b) are used, we find

$$\mathcal{K}_3 = \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, C^{0}_{\mathbb{BC}}) \subset \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, C_{\mathbb{BC}}) = \mathcal{K}_2$$

and thus

$$\mathcal{K}_1 \cap \mathcal{K}_3 \subset \mathcal{K}_1 \cap \mathcal{K}_2 \subset \mathcal{Y}^{\beta_{\mathbb{D}}} \quad \textbf{(3.25)}$$

If $\mathcal{X}$ is a $\mathbb{D}$-normal set and $a \in \mathcal{Y}^{\beta_{\mathbb{D}}}$, then $a \otimes y = (a(s) \otimes y(s)) \in \mathcal{CS}_{\mathbb{BC}}$ for every $y \in \mathcal{Y}$, and thus $(a_1(s)y_1(s)) \in \mathcal{CS}$ and $(a_2(s)y_2(s)) \in \mathcal{CS}$. From this, we obtain $a_1(s)y_1(s) \to 0$ and $a_2(s)y_2(s) \to 0$ as $s \to \infty$. Thus, it is observed that $a(s) \otimes y(s) \to \theta$. In this case, we get $a \otimes \Delta_{\mathbb{BC}}w \in C^0_{\mathbb{BC}}$ for every $w \in \mathcal{X}$. Now, let $\tilde{x} = (\tilde{x}(s))$ with $\tilde{x}(s) = (-1)^s|w(s)|_\xi$ for all $w \in \mathcal{X}$ and $s \in \mathbb{N}$. Then, we have

$$\left|(-1)^s \odot |w(s)|_\xi\right|_\xi \preccurlyeq |w(s)|_\xi$$

for every $s \in \mathbb{N}$ and using the fact that $\mathcal{X}$ is a $\mathbb{D}$-normal set, we obtain $\tilde{x} \in \mathcal{X}$, and so $a \otimes \Delta_{\mathbb{BC}}\tilde{x} \in C^0_{\mathbb{BC}}$. Also, since

$$\begin{aligned} a \otimes \Delta_{\mathbb{BC}}\left((-1)^s \odot |w(s)|_\xi\right) &= a \otimes \Delta_{\mathbb{BC}}((-1)^s|w_1(s)|e_1 + (-1)^s|w_2(s)|e_2) \\ &= (-1)^s\left(\textstyle\sum_{i=1}^2 a_i(\mathrm{s})(|w_i(\mathrm{s})| + |w_i(s-1)|)e_i\right), \end{aligned}$$

we obtain $a_1w_1 = (a_1(s)w_1(s)) \to 0$ and $a_2w_2 = (a_2(s)w_2(s)) \to 0$. Thus, $a \otimes w \to \theta$ for every $w \in \mathcal{X}$. This shows that $a \in \mathcal{K}_3 = \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, C^0_{\mathbb{BC}})$.

Furthermore, if we take the $\mathbb{D}$-limit of both sides of expression (3.21) and use the fact that $a \otimes w \in C^0_{\mathbb{BC}}$ for every $w \in \mathcal{X}$, then we get

$$\begin{aligned} &\lim_{s\to\infty}{}_{\mathbb{D}} \textstyle\sum_{k=0}^s a(k)\, y(k) \\ &\qquad = \lim_{s\to\infty}{}_{\mathbb{D}} \left(\textstyle\sum_{k=0}^{s-1}\left(\Delta^+_{\mathbb{BC}}a(k)\right) \otimes w(k) \oplus a(s) \otimes w(s)\right), \end{aligned}$$

and so

$$\textstyle\sum_{k=0}^\infty a(k) \otimes y(k) = \sum_{k=0}^\infty\left(\Delta^+_{\mathbb{BC}}a(k)\right) \otimes w(k). \quad \textbf{(3.26)}$$

On the other hand, since $a \in \mathcal{Y}^{\beta_{\mathbb{D}}}$, both sums are finite. Since this holds for every $w \in \mathcal{X}$, then $\Delta^+_{\mathbb{BC}}a = (\Delta^+_{\mathbb{BC}}a(s)) \in \mathcal{X}^{\beta_{\mathbb{D}}}$, and therefore $a \in \left(\mathcal{X}^{\beta_{\mathbb{D}}}\right)_{\Delta_{\mathbb{BC}}^{+}} = \mathcal{K}_1$. Thus, we have $a \in \mathcal{K}_1 \cap \mathcal{K}_3$ for every $a \in \mathcal{Y}^{\beta_{\mathbb{D}}}$ and therefore

$$\mathcal{Y}^{\beta_{\mathbb{D}}} \subset \mathcal{K}_1 \cap \mathcal{K}_3 \tag{3.27}$$

If (3.25) and (3.27) are used, we get $\mathcal{Y}^{\beta_{\mathbb{D}}} = \mathcal{K}_1 \cap \mathcal{K}_3$.

Additionally, if the equality $w = (w(s)) = (\mathcal{H}_{\mathbb{BC}}^{s}(y)) = \mathcal{H}_{\mathbb{BC}}(y)$ is used in expression (3.26), then (3.18) is true.

Lemma 3.4. For arbitrary sequences $w = (w(s)) \in \mathcal{X} = \mathcal{X}_1 e_1 + \mathcal{X}_2 e_2 \subset w_{\mathbb{BC}}$ and $u = (u(s)), v = (v(s)) \in \mathcal{U}_{\mathbb{BC}}$ with $w(s) = w_1(s)e_1 + w_2(s)e_2$, $u(s) = u_1(s)e_1 + u_2(s)\ \ e_2$ and $v(s) = v_1(s)e_1 + v_2(s)e_2$ for all $s \in \mathbb{N}$, the following equalities hold:

a) $w \otimes \frac{1}{u} = \frac{1}{u} \otimes w = \frac{w}{u}$,

b) $w \otimes \frac{1}{u} \otimes \frac{1}{v} = \frac{w}{u \otimes v}$,

c) $w \otimes \frac{u}{v} = \frac{w \otimes u}{v}$.

Proof. For $b_1 \neq 0 \neq b_2$, the equality holds

$$\frac{a_1 e_1 + a_2 e_2}{b_1 e_1 + b_2 e_2} = \frac{a_1}{b_1} e_1 + \frac{a_2}{b_2} e_2$$

exists (Luna–Elizarraras & et al., 2015). If this equation and the vector multiplication operation defined on $w_{\mathbb{BC}}$ are used, (a), (b) and (c) are easily obtained.

Lemma 3.5. Let $\mathcal{X} = \mathcal{X}_1 e_1 + \mathcal{X}_2 e_2 \subset w_{\mathbb{BC}}$, $u = (u(s)) \in \mathcal{U}_{\mathbb{BC}}$ and $v = (v(s)) \in \mathcal{U}_{\mathbb{BC}}$. If the set $\mathcal{X}$ is $\mathbb{D}$-normal, then the set $u^{-1} * \mathcal{X}$ is also $\mathbb{D}$-normal. Furthermore, the following equality holds:

$$\mathcal{M}_{\mathbb{BC}}\left(v^{-1} * (u^{-1} * \mathcal{X})_{\mathcal{H}_{\mathbb{BC}}}, \mathcal{CS}_{\mathbb{BC}}\right)$$

$$= (1/v)^{-1} * \left((u^{-1} * \mathcal{X})_{\mathcal{H}_{\mathbb{BC}}}\right)^{\beta_{\mathbb{D}}} \tag{3.28}$$

Proof. Let $\mu = (\mu(s)) \in \mathcal{u}^{-1} * \mathcal{X}$ with $\mu(s) = \mu_1(s)e_1 + \mu_2(s)e_2$ for every $s \in \mathbb{N}$, and $|\delta(s)|_\xi \preccurlyeq |\mu(s)|_\xi$. In this case, $\mathcal{u} \otimes \mu \in \mathcal{X}$, $|\delta_1(s)| \le |\mu_1(s)|$ and $|\delta_2(s)| \le |\mu_2(s)|$ are obtained. Then, $|(\mathcal{u} \otimes \delta)(s)|_\xi \preccurlyeq |(\mathcal{u} \otimes \mu)(s)|_\xi$ is written and if is used that the set $\mathcal{X}$ is $\mathbb{D}$-normal, then we obtain $\mathcal{u} \otimes \delta \in \mathcal{X}$, hence

$\delta = (\delta(s)) \in \mathcal{u}^{-1} * \mathcal{X}$. Thus, it is seen that the set $\mathcal{u}^{-1} * \mathcal{X}$ is also $\mathbb{D}$-normal. By Lemma 3.1 (c), we get

$$\begin{aligned} &\mathcal{M}_{\mathbb{BC}}\left(\mathcal{v}^{-1} * (\mathcal{u}^{-1} * \mathcal{X})_{\mathcal{H}_{\mathbb{BC}}}, \mathcal{CS}_{\mathbb{BC}}\right) \\ &\qquad = (1/\mathcal{v})^{-1} * \mathcal{M}_{\mathbb{BC}}\left((\mathcal{u}^{-1} * \mathcal{X})_{\mathcal{H}_{\mathbb{BC}}}, \mathcal{CS}_{\mathbb{BC}}\right) \\ &\qquad = (1/\mathcal{v})^{-1} * \left((\mathcal{u}^{-1} * \mathcal{X})_{\mathcal{H}_{\mathbb{BC}}}\right)^{\beta_{\mathbb{D}}}. \end{aligned}$$

Theorem 3.1. Let $\mathcal{u} = (\mathcal{u}(s)) \in \mathcal{U}_{\mathbb{BC}}, \mathcal{v} = (\mathcal{v}(s)) \in \mathcal{U}_{\mathbb{BC}}$, $\mathcal{X} = \mathcal{X}_1 e_1 + \mathcal{X}_2 e_2 \subset w_{\mathbb{BC}}$ and $\mathcal{Z}_{\mathbb{BC}} = \mathcal{Z}_{\mathbb{BC}}(\mathcal{u}, \mathcal{v}; \mathcal{X})$. Then

$$\mathcal{Z}_{\mathbb{BC}}^{\beta_{\mathbb{D}}} \supset \left\{a: \frac{1}{\mathcal{u}} \otimes \Delta_{\mathbb{BC}}^{+}\left(\frac{a}{\mathcal{v}}\right) \in \mathcal{X}^{\beta_{\mathbb{D}}} \wedge \frac{a}{\mathcal{u} \otimes \mathcal{v}} \in \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, C_{\mathbb{BC}})\right\} \quad \textbf{(3.29)}$$

and if $\mathcal{X}$ is a $\mathbb{D}$-normal set, then

$$\mathcal{Z}_{\mathbb{BC}}^{\beta_{\mathbb{D}}} = \left\{a: \frac{1}{\mathcal{u}} \otimes \Delta_{\mathbb{BC}}^{+}\left(\frac{a}{\mathcal{v}}\right) \in \mathcal{X}^{\beta_{\mathbb{D}}} \wedge \frac{a}{\mathcal{u} \otimes \mathcal{v}} \in \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, C_{\mathbb{BC}}^{0})\right\} \quad \textbf{(3.30)}$$

holds. Moreover, for $a = (a(s)) \in \mathcal{Z}_{\mathbb{BC}}^{\beta_{\mathbb{D}}}$ and $\tau = (\tau(s)) \in \mathcal{Z}_{\mathbb{BC}}$, we obtain

$$\sum_{s=0}^{\infty} a(s) \otimes \tau(s) = \sum_{s=0}^{\infty} \Delta_{\mathbb{BC}}^{+}\left(\frac{a(s)}{\mathcal{v}(s)}\right) \otimes \mathcal{H}_{\mathbb{BC}}^{s}(\mathcal{v} \otimes \tau) \quad \textbf{(3.31)}$$

Proof. Firstly, by (3.28), we can write $\mathcal{Z}_{\mathbb{BC}}^{\beta_{\mathbb{D}}} = (1/\mathcal{v})^{-1} * \mathcal{Y}^{\beta_{\mathbb{D}}}$ with $\mathcal{Y} = (\mathcal{u}^{-1} * \mathcal{X})_{\mathcal{H}_{\mathbb{BC}}}$. In this case, we write $(1/\mathcal{v}) \otimes a \in \mathcal{Y}^{\beta_{\mathbb{D}}}$ for each $a \in \mathcal{Z}_{\mathbb{BC}}^{\beta_{\mathbb{D}}}$. Hence, there exists a $y = (y(s)) \in \mathcal{Y}^{\beta_{\mathbb{D}}}$ such that

$$y(s) = y_1(s)e_1 + y_2(s)e_2$$

with

$$\mathcal{y}_1(s) = \frac{a_1(s)}{v_1(s)}, \mathcal{y}_2(s) = \frac{a_2(s)}{v_2(s)}.$$

If we take $u^{-1} * \mathcal{X}$ instead of $\mathcal{X}$ in Lemma 3.3, then $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}_3$ are written as

$$\mathcal{K}_1 = \left((u^{-1} * \mathcal{X})^{\beta_{\mathbb{D}}}\right)_{\Delta_{\mathbb{BC}}^{+}} = \left((1/u)^{-1} * \mathcal{X}^{\beta_{\mathbb{D}}}\right)_{\Delta_{\mathbb{BC}}^{+}}$$

$$\mathcal{K}_2 = \mathcal{M}_{\mathbb{BC}}\big((u^{-1} * \mathcal{X}), C_{\mathbb{BC}}\big) = (1/u)^{-1} * \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, C_{\mathbb{BC}})$$

and

$$\mathcal{K}_3 = \mathcal{M}_{\mathbb{BC}}\left((u^{-1} * \mathcal{X}), C_{\mathbb{BC}}^{0}\right) = (1/u)^{-1} * \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, C_{\mathbb{BC}}^{0}).$$

In this case, if (3.16) is used, then

$$\mathcal{K}_1 \cap \mathcal{K}_2 = \left((1/u)^{-1} * \mathcal{X}^{\beta_{\mathbb{D}}}\right)_{\Delta_{\mathbb{BC}}^{+}} \cap \{(1/u)^{-1} * \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, C_{\mathbb{BC}})\}$$
$$\subset \mathcal{Y}^{\beta_{\mathbb{D}}} = (1/v) \otimes \mathcal{Z}_{\mathbb{BC}}^{\beta_{\mathbb{D}}}$$

holds. Accordingly,

$$\left\{v \otimes w \in w_{\mathbb{BC}}: w \in \left(\left(\frac{1}{u}\right)^{-1} * \mathcal{X}^{\beta_{\mathbb{D}}}\right)_{\Delta_{\mathbb{BC}}^{+}} \wedge w \in \left\{\left(\frac{1}{u}\right)^{-1} * \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, C_{\mathbb{BC}})\right\}\right\} \subset \mathcal{Z}_{\mathbb{BC}}^{\beta_{\mathbb{D}}}.$$

If we receive $v \otimes w = a$ and use Lemma 3.4, then we write

$$\left\{a: \frac{1}{u} \otimes \Delta_{\mathbb{BC}}^{+}\left(\frac{a}{v}\right) \in \mathcal{X}^{\beta_{\mathbb{D}}} \wedge \frac{a}{u \otimes v} \in \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, C_{\mathbb{BC}})\right\} \subset \mathcal{Z}_{\mathbb{BC}}^{\beta_{\mathbb{D}}},$$

and thus (3.29) is obtained.

Now, let $\mathcal{X}$ be a $\mathbb{D}$-normal set. If Lemma 3.5 and (3.17) are used, we write

$$\mathcal{K}_1 \cap \mathcal{K}_3 = \mathcal{Y}^{\beta_{\mathbb{D}}} = (1/v) \otimes \mathcal{Z}_{\mathbb{BC}}^{\beta_{\mathbb{D}}}$$

and thus

$$\left\{a: \frac{1}{u} \otimes \Delta^{+}_{\mathbb{BC}}\left(\frac{a}{v}\right) \in \mathcal{X}^{\beta_{\mathbb{D}}} \wedge \frac{a}{u \otimes v} \in \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, C^{0}_{\mathbb{BC}})\right\} = \mathcal{Z}^{\beta_{\mathbb{D}}}_{\mathbb{BC}}.$$

This proves (3.30).

If $\tau = (\tau(k)) \in \mathcal{Z}_{\mathbb{BC}}$, then, for $s \in \mathbb{N}$, as in Definition 3.7, we write

$$x_i(s) = u_i(s) \sum_{k=0}^{s} v_i(k)\tau_i(k),\ i = 1,2$$

and so

$$(x_i/u_i)(s) = \sum_{k=0}^{s} v_i(k)\tau_i(k),\ i = 1,2$$

with $x = (x(s)) = (x_1(s)e_1 + x_2(s)e_2) \in \mathcal{X}$. Also, for every $s \in \mathbb{N}$,

$$\begin{aligned} v_i(s)\tau_i(s) &= \sum_{k=0}^{s} v_i(k)\tau_i(k) - \sum_{k=0}^{s-1} v_i(k)\tau_i(k) \\ &= (x_i/u_i)(s) - (x_i/u_i)(s-1) \\ &= \Delta\big((x_i/u_i)(s)\big), i = 1,2 \end{aligned}$$

and hence

$$\tau_i(s) = (1/v_i(s))\Delta\big((x_i/u_i)(s)\big)$$

is written. From this, for every $s \in \mathbb{N}$,

$$\begin{aligned} \tau(s) &= \sum_{i=1}^{2} \tau_i(s)e_i = \sum_{i=1}^{2} (1/v_i(s))\Delta\big((x_i/u_i)(s)\big)e_i \\ &= \left(\sum_{i=1}^{2}(1/v_i(s))e_i\right) \otimes \Delta_{\mathbb{BC}}\left(\sum_{i=1}^{2}(x_i(s)/u_i(s))e_i\right) \\ &= \big((1/v) \otimes \Delta_{\mathbb{BC}}(x/u)\big)(s). \end{aligned}$$

is obtained. Now, let $a \in \mathcal{Z}^{\beta_{\mathbb{D}}}_{\mathbb{BC}}$. If we take $\tau \in \mathcal{Z}_{\mathbb{BC}}(u, v; \mathcal{X})$ with $\tau = (1/v) \otimes \Delta_{\mathbb{BC}}(x/u)$, then we write

$$\left(\sum_{k=0}^{s} a(k) \otimes \tau(k)\right)_{s=0}^{\infty}$$

$$= \left(\sum_{k=0}^{s}\left(a(k) \otimes \left(\frac{1}{v(k)}\right) \otimes \Delta_{\mathbb{BC}}\left(\frac{x(k)}{u(k)}\right)\right)\right)_{s=0}^{\infty} \in C_{\mathbb{BC}}$$

and, by (3.21), we have

$$\sum_{k=0}^{s} a(k) \otimes \tau(k)$$

$$= \sum_{k=0}^{s-1} \left(\frac{1}{u(k)} \otimes \Delta_{\mathbb{BC}}^{+} \left(\frac{a(k)}{v(k)} \right) \right) \otimes x(k) \oplus \frac{a(s)}{v(s) \otimes u(s)} x(s). \quad \textbf{(3.32)}$$

Now, since $a \in \mathcal{Z}_{\mathbb{BC}}^{\beta_{\mathbb{D}}}$, by (3.30), $\frac{a}{u \otimes v} \in \mathcal{M}_{\mathbb{BC}}(\mathcal{X}, C_{\mathbb{BC}}^{0})$, and, by (3.32), we have

$$\sum_{k=0}^{s} \frac{a(k)}{v(k)} \otimes \Delta_{\mathbb{BC}} \left(\frac{x(k)}{u(k)} \right) = \sum_{k=0}^{s-1} \Delta_{\mathbb{BC}}^{+} \left(\frac{a(k)}{v(k)} \right) \otimes \frac{x(k)}{u(k)} \quad \textbf{(3.33)}$$

If $x = u \otimes \mathcal{H}_{\mathbb{BC}}(v \otimes \tau)$ in (3.33) is used, then, for every $k \in \mathbb{N}$, $\frac{x(k)}{u(k)} = \mathcal{H}_{\mathbb{BC}}^{k}(v \otimes \tau)$, thus

$$\sum_{k=0}^{\infty} a(k) \otimes \tau(k) = \sum_{k=0}^{\infty} \Delta_{\mathbb{BC}}^{+} \left(\frac{a(k)}{v(k)} \right) \otimes \mathcal{H}_{\mathbb{BC}}^{k}(v \otimes \tau)$$

is obtained.

Lemma 3.6. For β-dual spaces of some sequences with complex terms,

$$C_0^{\beta} = C^{\beta} = \ell_{\infty}^{\beta} = \ell_1, \quad \textbf{(3.34)}$$

and

$$\ell_{q}^{\beta} = \ell_{p} \quad \textbf{(3.35)}$$

with $1 \leq q < \infty$ and $(1/p) + (1/q) = 1$ (Başar & Çolak, 2011).

Theorem 3.2. Some spaces of sequences with bicomplex terms satisfy the following equalities:

$$(C_{\mathbb{BC}}^{0})^{\beta_{\mathbb{D}}} = C_{\mathbb{BC}}^{\beta_{\mathbb{D}}} = \Phi_{\mathbb{BC}}^{\beta_{\mathbb{D}}} = \mathcal{L}_{\mathbb{BC}}^{1}, \quad \textbf{(3.36)}$$

and

$$\left(\mathcal{L}_{\mathbb{BC}}^{q}\right)^{\beta_{\mathbb{D}}} = \mathcal{L}_{\mathbb{BC}}^{p} \quad \textbf{(3.37)}$$

with $1 < q < \infty$ and $(1/p) + (1/q) = 1$, also

$$(\mathcal{L}^1_{\mathbb{BC}})^{\beta_{\mathbb{D}}} = \mathcal{L}^\infty_{\mathbb{BC}} = \Phi^{\beta_{\mathbb{D}}}_{\mathbb{BC}}.$$

Proof. If the equalities $\mathcal{M}_{\mathbb{BC}}(\mathcal{X}, \mathcal{CS}_{\mathbb{BC}}) = \mathcal{M}(\mathcal{X}_1, \mathcal{CS})e_1 + \mathcal{M}(\mathcal{X}_2, \mathcal{CS})e_2$, and (3.34), (3.35) in Lemma 3.6 are used, then

$$\begin{aligned}(C^0_{\mathbb{BC}})^{\beta_{\mathbb{D}}} &= \mathcal{M}_{\mathbb{BC}}(C^0_{\mathbb{BC}}, \mathcal{CS}_{\mathbb{BC}}) = \mathcal{M}_{\mathbb{BC}}(C_0e_1 + C_0e_2, \mathcal{CS}_{\mathbb{BC}})\\ &= \mathcal{M}(C_0, \mathcal{CS})e_1 + \mathcal{M}(C_0, \mathcal{CS})e_2\\ &= \ell_1e_1 + \ell_1e_2 = \mathcal{L}^1_{\mathbb{BC}},\end{aligned}$$

$$\begin{aligned}C^{\beta_{\mathbb{D}}}_{\mathbb{BC}} &= \mathcal{M}_{\mathbb{BC}}(C_{\mathbb{BC}}, \mathcal{CS}_{\mathbb{BC}}) = \mathcal{M}_{\mathbb{BC}}(Ce_1 + Ce_2, \mathcal{CS}_{\mathbb{BC}})\\ &= \mathcal{M}(C, \mathcal{CS})e_1 + \mathcal{M}(C, \mathcal{CS})e_2\\ &= \ell_1e_1 + \ell_1e_2 = \mathcal{L}^1_{\mathbb{BC}},\end{aligned}$$

$$\begin{aligned}\Phi^{\beta_{\mathbb{D}}}_{\mathbb{BC}} &= \mathcal{M}_{\mathbb{BC}}(\Phi_{\mathbb{BC}}, \mathcal{CS}_{\mathbb{BC}}) = \mathcal{M}_{\mathbb{BC}}(\ell_\infty e_1 + \ell_\infty e_2, \mathcal{CS}_{\mathbb{BC}})\\ &= \mathcal{M}(\ell_\infty, \mathcal{CS})e_1 + \mathcal{M}(\ell_\infty, \mathcal{CS})e_2\\ &= \ell_1e_1 + \ell_1e_2 = \mathcal{L}^1_{\mathbb{BC}}\end{aligned}$$

and so (3.36) holds.

Now, let $1 \le q < \infty$ and $(1/p) + (1/q) = 1$. In this case, we get

$$\begin{aligned}\left(\mathcal{L}^q_{\mathbb{BC}}\right)^{\beta_{\mathbb{D}}} &= \mathcal{M}_{\mathbb{BC}}\left(\mathcal{L}^q_{\mathbb{BC}}, \mathcal{CS}_{\mathbb{BC}}\right) = \mathcal{M}_{\mathbb{BC}}(\ell^q e_1 + \ell^q e_2, \mathcal{CS}_{\mathbb{BC}})\\ &= \mathcal{M}(\ell^q, \mathcal{CS})e_1 + \mathcal{M}(\ell^q, \mathcal{CS})e_2\\ &= \ell^p e_1 + \ell^p e_2 = \mathcal{L}^p_{\mathbb{BC}},\end{aligned}$$

thereby yielding (3.37). Here, if $q = 1$ is taken, then $(\mathcal{L}^1_{\mathbb{BC}})^{\beta_{\mathbb{D}}} = \mathcal{L}^\infty_{\mathbb{BC}} = \Phi^{\beta_{\mathbb{D}}}_{\mathbb{BC}}$ is written.

Lemma 3.7. The spaces $\mathcal{L}^q_{\mathbb{BC}}$ with $1 \le q < \infty$ and $(1/p) + (1/q) = 1$, $C^0_{\mathbb{BC}}$ and $\Phi_{\mathbb{BC}}$ are $\mathbb{D}$-normal sets. Also, the space $C_{\mathbb{BC}}$ is not a $\mathbb{D}$-normal set.

Proof. First, let $c = \big(c(s)\big) \in C^0_{\mathbb{BC}}$ be given with $c(s) = c_1(s)e_1 + c_2(s)e_2$ and $|\tilde{c}(s)|_\xi \preccurlyeq |c(s)|_\xi$ for every $s \in \mathbb{N}$. In this case, we write $c_1(s) \to 0$ and $c_2(s) \to 0$ in $\mathbb{C}$. Furthermore, if the $\mathbb{D}$-ordering is used, then $|\tilde{c}_1(s)| \leq |c_1(s)|$ and $|\tilde{c}_2(s)| \leq |c_2(s)|$ hold. Then, $\tilde{c}_1(s) \to 0$, $\tilde{c}_2(s) \to 0$ and thus $\tilde{c}(s) = \tilde{c}_1(s)e_1 + \tilde{c}_2(s)e_2 \to \theta$. From this, $\tilde{c} = \big(\tilde{c}(s)\big) \in C^0_{\mathbb{BC}}$, and thus the space $C^0_{\mathbb{BC}}$ is a $\mathbb{D}$-normal set.

If $c = \big(c(s)\big) \in \Phi_{\mathbb{BC}}$ is taken, then $\sup_{\mathbb{D}\, s\in\mathbb{N}} |c(s)|_\xi \prec \infty_{\mathbb{D}}$, and thus $\sup_{s\in\mathbb{N}} |c_i(s)| < \infty$ for $i = 1,2$. Now, if $|\tilde{c}(s)|_\xi \preccurlyeq |c(s)|_\xi$ for all $s \in \mathbb{N}$, then we get

$$\begin{aligned} \sup_{\mathbb{D}\, s\in\mathbb{N}} |\tilde{c}(s)|_\xi &= \sup_{s\in\mathbb{N}} |\tilde{c}_1(s)| e_1 + \sup_{s\in\mathbb{N}} |\tilde{c}_2(s)| e_2 \\ &\preccurlyeq \sup_{s\in\mathbb{N}} |c_1(s)| e_1 + \sup_{s\in\mathbb{N}} |c_2(s)| e_2 \\ &= \sup_{\mathbb{D}\, s\in\mathbb{N}} |c(s)|_\xi \prec \infty_{\mathbb{D}} \end{aligned}$$

and hence $\Phi_{\mathbb{BC}}$ is a $\mathbb{D}$-normal set.

Again, let $c = \big(c(s)\big) \in \mathcal{L}^{q}_{\mathbb{BC}}$ and $|\tilde{c}(s)|_\xi \preccurlyeq |c(s)|_\xi$ for all $s \in \mathbb{N}$. In this case, $\sum_{s=0}^{\infty} \big(|c(s)|_\xi\big)^{q} \prec \infty_{\mathbb{D}}$, and thus $\sum_{s=0}^{\infty} |\tilde{c}_i(s)|^{q} \leq \sum_{s=0}^{\infty} |c_i(s)|^{q} < \infty$ for $i = 1,2$. Then

$$\begin{aligned} \sum_{s=0}^{\infty} \big(|\tilde{c}(s)|_\xi\big)^{q} &= \sum_{i=1}^{2} \Big(\sum_{s=0}^{\infty} |\tilde{c}_i(s)|^{q}\, e_i\Big) \\ &\preccurlyeq \sum_{s=0}^{\infty} |c_1(s)|^{q}\, e_1 + \sum_{s=0}^{\infty} |c_2(s)|^{q}\, e_2 \\ &= \sum_{s=0}^{\infty} \big(|c(s)|_\xi\big)^{q} \prec \infty_{\mathbb{D}}. \end{aligned}$$

This shows that $\tilde{c} = \big(\tilde{c}(s)\big) \in \mathcal{L}^{q}_{\mathbb{BC}}$ and therefore $\mathcal{L}^{q}_{\mathbb{BC}}$ is a $\mathbb{D}$-normal set.

Now, we show that $C_{\mathbb{BC}}$ is not a $\mathbb{D}$-normal set. For this purpose, let $c = \big(c(s)\big)$ and $\tilde{c} = \big(\tilde{c}(s)\big)$ be sequences with $c(s) =$

$\left(1+\frac{i}{s}\right)e_1+\left(1+\frac{i}{s}\right)e_2$ and $\tilde{c}(s)=i^s e_1+i^s e_2$ for every $s\in\mathbb{N}$. Hence $c(s)\to 1e_1+1e_2=1$ as s and $|\tilde{c}(s)|_\xi \preccurlyeq |c(s)|_\xi$ for every $s\in\mathbb{N}$. Due to $c=\big(c(s)\big)\in C_{\mathbb{BC}}$ and $\tilde{c}\notin C_{\mathbb{BC}}$, the space $C_{\mathbb{BC}}$ is not a $\mathbb{D}$-normal set.

Corollary 3.1. Let $u,v\in\mathcal{U}_{\mathbb{BC}}$, $1\le q<\infty$ and $(1/p)+(1/q)=1$. Then, the following equalities hold:

a) $\left(\mathcal{Z}_{\mathbb{BC}}(u,v;C^0_{\mathbb{BC}})\right)^{\beta_{\mathbb{D}}}=\left\{a:\frac{1}{u}\otimes\Delta^+_{\mathbb{BC}}\left(\frac{a}{v}\right)\in\mathcal{L}^1_{\mathbb{BC}}\wedge\frac{a}{u\otimes v}\in\Phi_{\mathbb{BC}}\right\}$,

b) $\left(\mathcal{Z}_{\mathbb{BC}}(u,v;\Phi_{\mathbb{BC}})\right)^{\beta_{\mathbb{D}}}=\left\{a:\frac{1}{u}\otimes\Delta^+_{\mathbb{BC}}\left(\frac{a}{v}\right)\in\mathcal{L}^1_{\mathbb{BC}}\wedge\frac{a}{u\otimes v}\in C^0_{\mathbb{BC}}\right\}$,

c) $\left(\mathcal{Z}_{\mathbb{BC}}(u,v;\mathcal{L}^q_{\mathbb{BC}})\right)^{\beta_{\mathbb{D}}}=\left\{a:\frac{1}{u}\otimes\Delta^+_{\mathbb{BC}}\left(\frac{a}{v}\right)\in\mathcal{L}^p_{\mathbb{BC}}\wedge\frac{a}{u\otimes v}\in\Phi_{\mathbb{BC}}\right\}$,

d) $\left(\mathcal{Z}_{\mathbb{BC}}(u,v;C_{\mathbb{BC}})\right)^{\beta_{\mathbb{D}}}=\left\{a:\frac{1}{u}\otimes\Delta^+_{\mathbb{BC}}\left(\frac{a}{v}\right)\in\mathcal{L}^1_{\mathbb{BC}}\wedge\frac{a}{u\otimes v}\in C_{\mathbb{BC}}\right\}$.

Proof. In Lemma 3.7, it was shown that the spaces $C^0_{\mathbb{BC}}$, $\Phi_{\mathbb{BC}}$ and $\mathcal{L}^q_{\mathbb{BC}}$ are $\mathbb{D}$-normal sets. If Theorem 3.1, and (3.30), (3.36), (3.37) are used, then (a), (b) and (c) are easily obtained.

For the sake of brevity in our calculations, we denote $\left(\mathcal{Z}_{\mathbb{BC}}(u,v;C_{\mathbb{BC}})\right)^{\beta_{\mathbb{D}}}=\mathrm{F}$.

Since $C_{\mathbb{BC}}$ is not a $\mathbb{D}$-normal set, by (3.29) and Lemma 3.2, we get

$$\mathrm{F}\supset\left\{a:\frac{1}{u}\otimes\Delta^+_{\mathbb{BC}}\left(\frac{a}{v}\right)\in C^{\beta_{\mathbb{D}}}_{\mathbb{BC}}\wedge\frac{a}{u\otimes v}\in\mathcal{M}_{\mathbb{BC}}(C_{\mathbb{BC}},C_{\mathbb{BC}})\right\}$$

, and so

$$\mathrm{F}=\left\{a\in w_{\mathbb{BC}}:\frac{1}{u}\otimes\Delta^+_{\mathbb{BC}}\left(\frac{a}{v}\right)\in\mathcal{L}^1_{\mathbb{BC}}\wedge\frac{a}{u\otimes v}\in C_{\mathbb{BC}}\right\}.\quad\textbf{(3.38)}$$

Also, $\mathcal{Z}_{\mathbb{BC}}(u,v;C^0_{\mathbb{BC}})\subset\mathcal{Z}_{\mathbb{BC}}(u,v;C_{\mathbb{BC}})$ and by Lemma 3.1 (a), we have

$$\mathcal{M}_{\mathbb{BC}}(\mathcal{Z}_{\mathbb{BC}}(u,v;C_{\mathbb{BC}}),\mathcal{CS}_{\mathbb{BC}})\subset\mathcal{M}_{\mathbb{BC}}(\mathcal{Z}_{\mathbb{BC}}(u,v;C^0_{\mathbb{BC}}),\mathcal{CS}_{\mathbb{BC}}),$$

and so

$$\mathrm{F} \subset \left(\mathcal{Z}_{\mathbb{BC}}(u, v; C^0_{\mathbb{BC}})\right)^{\beta_{\mathbb{D}}}.$$

Hence, we write

$$\mathrm{F} \subset \left\{a \in w_{\mathbb{BC}} : \frac{1}{u} \otimes \Delta^+_{\mathbb{BC}}\left(\frac{a}{v}\right) \in \mathcal{L}^1_{\mathbb{BC}}\right\}. \qquad \textbf{(3.39)}$$

If (3.36) is used, then

$$\mathrm{F} \subset \left\{a : x \in C_{\mathbb{BC}}, \sum_{k=0}^{\infty}\left(\frac{1}{u(k)} \otimes \Delta^+_{\mathbb{BC}}\left(\frac{a(k)}{v(k)}\right)\right) \otimes x(k) \prec \infty_{\mathbb{D}}\right\}$$

holds. Now, let $a \in \mathrm{F}$. Then, given any $\tau \in \mathcal{Z}_{\mathbb{BC}}(u, v; C_{\mathbb{BC}})$ with $\tau = (1/v) \otimes \Delta_{\mathbb{BC}}(x/u)$, if (3.32) is used, then

$$\sum_{k=0}^{s} a(k) \otimes \tau(k)$$

$$= \left(\sum_{k=0}^{s-1}\left(\frac{1}{u(k)} \otimes \Delta^+_{\mathbb{BC}} \frac{a(k)}{v(k)}\right) \otimes x(k)\right) \oplus \frac{a(s)}{v(s) \otimes u(s)} \otimes x(s). \qquad \textbf{(3.40)}$$

Here, if

$$G_s = \sum_{k=0}^{s} a(k) \otimes \tau(k),\ \ I_s = \sum_{k=0}^{s-1} \frac{1}{u(k)} \otimes \Delta^+_{\mathbb{BC}} \frac{a(k)}{v(k)} \otimes x(k)$$

are taken and (3.39) is used, we write

$$\frac{a(s)}{v(s) \otimes u(s)} \otimes x(s) = G_s - I_s$$

thus, for every $x \in C_{\mathbb{BC}}$,

$$\frac{a}{u \otimes v} \otimes x = \left(\frac{a(s)}{v(s) \otimes u(s)} \otimes x(s)\right) = (G_s - I_s) \in C_{\mathbb{BC}}$$

and hence

$$\frac{a}{u \otimes v} \in \mathcal{M}_{\mathbb{BC}}(C_{\mathbb{BC}}, C_{\mathbb{BC}}) = C_{\mathbb{BC}} \qquad \textbf{(3.41)}$$

is obtained. Thus, if (3.39) and (3.40) are used

$$\mathrm{F} \subset \left\{a \in w_{\mathbb{BC}} : \frac{1}{u} \otimes \Delta^+_{\mathbb{BC}}\left(\frac{a}{v}\right) \in \mathcal{L}^1_{\mathbb{BC}} \wedge \frac{a}{u \otimes v} \in C_{\mathbb{BC}}\right\} \qquad \textbf{(3.42)}$$

is written. Now, if the inclusions (3.38) and (3.42) are considered jointly:

$$\mathrm{F} = \left\{a \in w_{\mathbb{BC}}: \frac{1}{u} \otimes \Delta_{\mathbb{BC}}^{+}\left(\frac{a}{v}\right) \in \mathcal{L}_{\mathbb{BC}}^{1} \wedge \frac{a}{u \otimes v} \in C_{\mathbb{BC}}\right\}.$$

Lemma 3.8. Let $\mathcal{CS}$ be the space of convergent series and $\mathcal{BS}$ be the space of bounded series with complex terms. Then $\mathcal{CS}^{\beta} = \mathcal{BV}$ and $\mathcal{BS}^{\beta} = \mathcal{BV}_0 = \mathcal{BV} \cap C_0$ hold (Malkowsky & Savaş, 2004).

Theorem 3.3. The equalities $\mathcal{CS}_{\mathbb{BC}}^{\beta_{\mathbb{D}}} = \mathcal{BV}_{\mathbb{BC}}$ and $\mathcal{BS}_{\mathbb{BC}}^{\beta_{\mathbb{D}}} = \mathcal{BV}_{\mathbb{BC}}^{0} = \mathcal{BV}_{\mathbb{BC}} \cap C_{\mathbb{BC}}^{0}$ hold.

Proof. If Lemma 3.8 are used, then we have

$$\begin{aligned}\mathcal{CS}_{\mathbb{BC}}^{\beta_{\mathbb{D}}} &= \mathcal{M}_{\mathbb{BC}}(\mathcal{CS}_{\mathbb{BC}}, \mathcal{CS}_{\mathbb{BC}}) = \mathcal{M}_{\mathbb{BC}}(\mathcal{CS}e_1 + \mathcal{CS}e_2, \mathcal{CS}e_1 + \mathcal{CS}e_2)\\ &= \mathcal{M}(\mathcal{CS}, \mathcal{CS})e_1 + \mathcal{M}(\mathcal{CS}, \mathcal{CS})e_2\\ &= \mathcal{BV}e_1 + \mathcal{BV}e_2 = \mathcal{BV}_{\mathbb{BC}},\end{aligned}$$

$$\begin{aligned}\mathcal{BS}_{\mathbb{BC}}^{\beta_{\mathbb{D}}} &= \mathcal{M}_{\mathbb{BC}}(\mathcal{BS}_{\mathbb{BC}}, \mathcal{CS}_{\mathbb{BC}}) = \mathcal{M}_{\mathbb{BC}}(\mathcal{BS}e_1 + \mathcal{BS}e_2, \mathcal{CS}e_1 + \mathcal{CS}e_2)\\ &= \mathcal{M}(\mathcal{BS}, \mathcal{CS})e_1 + \mathcal{M}(\mathcal{BS}, \mathcal{CS})e_2\\ &= \mathcal{BV}_0 e_1 + \mathcal{BV}_0 e_2 = \mathcal{BV}_{\mathbb{BC}} \cap C_{\mathbb{BC}}^{0}\end{aligned}$$

is obtained.

Example 3.2. For any $a = \big(a(s)\big) \in w_{\mathbb{BC}}$, let $\varphi_i(s) = Q_i(s)\left|\frac{a_i(s)}{q_i(s)} - \frac{a_i(s+1)}{q_i(s+1)}\right| e_i$, $i = 1,2$. Then the following equalities hold:

a) $\left(\left(\overline{\mathcal{N}}, q\right)_{\mathbb{BC}}^{0}\right)^{\beta_{\mathbb{D}}} = \left\{a: \sum_{s=0}^{\infty}\left(\sum_{i=1}^{2} \varphi_i(s)\right) \prec \infty_{\mathbb{D}} \wedge \frac{Q \otimes a}{q} \in \Phi_{\mathbb{BC}}\right\}$,

b) $\left(\overline{\mathcal{N}}, q\right)_{\mathbb{BC}}^{\beta_{\mathbb{D}}} = \left\{a: \sum_{s=0}^{\infty}\left(\sum_{i=1}^{2} \varphi_i(s)\right) \prec \infty_{\mathbb{D}} \wedge \frac{Q \otimes a}{q} \in C_{\mathbb{BC}}\right\}$,

c) $\left(\left(\overline{\mathcal{N}}, q\right)_{\mathbb{BC}}^{\infty}\right)^{\beta_{\mathbb{D}}} = \left\{a: \sum_{s=0}^{\infty}\left(\sum_{i=1}^{2} \varphi_i(s)\right) \prec \infty_{\mathbb{D}} \wedge \frac{Q \otimes a}{q} \in C_{\mathbb{BC}}^{0}\right\}$.

Solution. First, let us recall the sequences $v = q = \big(q(s)\big)$ such that $q(s) = q_1(s)e_1 + q_2(s)e_2 \succ \theta$ and $Q = \big(Q(s)\big) = (Q_1(s)e_1 + Q_2(s)e_2\,)$ with $Q_1(s) = \sum_{k=0}^{s} q_1(k)$ and $Q_2(s) = \sum_{k=0}^{s} q_2(k)$ for every $s \in \mathbb{N}$. Also, let $u = \big(u(s)\big)$ be a sequence with $u(s) = 1/Q(s)$, i.e., $u(s) = 1/Q = 1/Q_1(s)\,e_1 + 1/Q_2(s)\,e_2$. Using the equation (a) in Corollary 3.1, we have

$$\left(\mathcal{Z}_{\mathbb{BC}}(u, v; C_{\mathbb{BC}}^{0})\right)^{\beta_{\mathbb{D}}} = \left\{a \colon \frac{1}{(1/Q)} \otimes \Delta_{\mathbb{BC}}^{+}\left(\frac{a}{q}\right) \in \mathcal{L}_{\mathbb{BC}}^{1} \wedge \frac{a}{\frac{1}{Q}\otimes q} \in \Phi_{\mathbb{BC}}\right\}.$$

Now, according to the properties of bicomplex numbers written in idempotent representation, since

$$\frac{1}{\frac{1}{Q(s)}} = \frac{1e_1+1e_2}{\left(\frac{1e_1+1e_2}{Q_1(s)e_1+Q_2(s)e_2}\right)} = \frac{1e_1+1e_2}{\frac{1}{Q_1(s)}e_1+\frac{1}{Q_2(s)}e_2}$$
$$= Q_1(s)e_1 + Q_2(s)e_2 = Q(s),$$

we have

$$\frac{a(s)}{\frac{v(s)}{Q(s)}} = \frac{a_1(s)e_1+a_2(s)e_2}{\left(\frac{v_1(s)e_1+v_2(s)e_2}{Q_1(s)e_1+Q_2(s)e_2}\right)} = \frac{a_1(s)Q_1(s)}{v_1(s)}e_1 + \frac{a_2(s)Q_2(s)}{v_2(s)}e_2$$
$$= \frac{a_1(s)Q_1(s)e_1+a_2(s)Q_2(s)e_2}{v_1(s)e_1+v_2(s)e_2} = \frac{a(s)\otimes Q(s)}{v(s)}$$

for every $s \in \mathbb{N}$ and thus

$$\left(\left(\overline{\mathcal{N}}, q\right)_{\mathbb{BC}}^{0}\right)^{\beta_{\mathbb{D}}} = \left\{a \colon Q \otimes \Delta_{\mathbb{BC}}^{+}\left(\frac{a}{q}\right) \in \mathcal{L}_{\mathbb{BC}}^{1} \wedge \frac{a\otimes Q}{q} \in \Phi_{\mathbb{BC}}\right\}$$
$$= \left\{a \colon \sum_{s=0}^{\infty} \left|Q(s) \otimes \Delta_{\mathbb{BC}}^{+}\left(\frac{a(s)}{q(s)}\right)\right|_{\xi} \prec \infty_{\mathbb{D}} \wedge \frac{a\otimes Q}{q} \in \Phi_{\mathbb{BC}}\right\}$$
$$= \left\{a \colon \sum_{s=0}^{\infty}\left(\sum_{i=1}^{2} Q_i(s)\left|\left(\frac{a_i(s)}{q_i(s)} - \frac{a_i(s+1)}{q_i(s+1)}\right)\right| e_i\right) \prec \infty_{\mathbb{D}} \wedge \frac{a\otimes Q}{q} \in \Phi_{\mathbb{BC}}\right\}$$
$$= \left\{a \colon \sum_{s=0}^{\infty}\left(\sum_{i=1}^{2} \varphi_i(s)\right) \prec \infty_{\mathbb{D}} \wedge \frac{a\otimes Q}{q} \in \Phi_{\mathbb{BC}}\right\}.$$

Again, if the equality $\mathcal{M}_{\mathbb{BC}}(C_{\mathbb{BC}}, C_{\mathbb{BC}}) = C_{\mathbb{BC}}$ is used for the set $(\overline{\mathcal{N}}, q)_{\mathbb{BC}}^{\beta_{\mathbb{D}}}$ and the equality $\mathcal{M}_{\mathbb{BC}}(\Phi_{\mathbb{BC}}, C_{\mathbb{BC}}^0) = C_{\mathbb{BC}}^0$ is used for the set $\left((\overline{\mathcal{N}}, q)_{\mathbb{BC}}^{\infty}\right)^{\beta_{\mathbb{D}}}$, then (c) and (b) are easily seen.

Example 3.3. Let $\varpi = (\varpi(s))$ be given with $\varpi(s) = \varpi_1(s)e_1 + \varpi_2(s)e_2 = 1e_1 + 1e_2$ for every $s \in \mathbb{N}$. If the sequences $\mathcal{v} = \varpi$ and $u = (u(s)) = (u_1(s)e_1 + u_2(s)e_2) = \left(\frac{1}{s+1}e_1 + \frac{1}{s+1}e_2\right)$ are taken. Then for any $a = (a(s)) \in w_{\mathbb{BC}}$ the following equalities hold:

a) $\left(\mathcal{Z}_{\mathbb{BC}}(u, \mathcal{v}; \mathcal{L}_{\mathbb{BC}}^1)\right)^{\beta_{\mathbb{D}}} = \left\{a : \sup_{\mathbb{D}\, s\in\mathbb{N}} \pi(s) \prec \infty_{\mathbb{D}} \wedge (\mu(s)) \in \Phi_{\mathbb{BC}}\right\}$,

b) For $1 < q < \infty$ and $(q-1)(p-1) = 1$,

$$\left(\mathcal{Z}_{\mathbb{BC}}(u, \mathcal{v}; \mathcal{L}_{\mathbb{BC}}^q)\right)^{\beta_{\mathbb{D}}} = \left\{a : \sum_{s=0}^{\infty}(\pi(s))^{p} \prec \infty_{\mathbb{D}} \wedge (\mu(s)) \in \Phi_{\mathbb{BC}}\right\},$$

c) $\left(\mathcal{Z}_{\mathbb{BC}}(u, \mathcal{v}; \Phi_{\mathbb{BC}})\right)^{\beta_{\mathbb{D}}} = \left\{a : \sum_{s=0}^{\infty} \pi(s) \prec \infty_{\mathbb{D}} \wedge (\mu(s)) \in C_{\mathbb{BC}}^0\right\}$,

where $\pi(s) = (s+1) \odot \left|\Delta_{\mathbb{BC}}^+(a(s))\right|_{\xi}$, $\mu(s) = (s+1) \odot a(s)$.

Solution. If Corollary 3.1 and Theorem 3.2 are used:

$$\left(\mathcal{Z}_{\mathbb{BC}}(u, \mathcal{v}; \mathcal{L}_{\mathbb{BC}}^1)\right)^{\beta_{\mathbb{D}}}$$

$$= \left\{a : \frac{1}{u} \otimes \Delta_{\mathbb{BC}}^+\left(\frac{a}{\mathcal{v}}\right) \in (\mathcal{L}_{\mathbb{BC}}^1)^{\beta_{\mathbb{D}}} \wedge \frac{a}{u \otimes \mathcal{v}} \in \Phi_{\mathbb{BC}}\right\}$$

$$= \left\{a : \left(\frac{1}{u} \otimes \Delta_{\mathbb{BC}}^+\left(\frac{a_1(s)e_1 + a_2(s)e}{\varpi_1(s)e_1 + \varpi_2(s)e_2}\right)\right) \in \Phi_{\mathbb{BC}} \wedge \frac{a}{u \otimes \varpi} \in \Phi_{\mathbb{BC}}\right\}.$$

$$= \left\{a : \left(\frac{1}{u} \otimes \Delta_{\mathbb{BC}}^+\left(\frac{a_1(s)}{1}e_1 + \frac{a_2(s)}{1}e_2\right)\right) \in \Phi_{\mathbb{BC}} \wedge \frac{a}{u} \in \Phi_{\mathbb{BC}}\right\}.$$

$$= \left\{a : \sup_{s\in\mathbb{N}}{}_{\mathbb{D}}\pi(s) \prec \infty_{\mathbb{D}} \wedge \left(\mu(s)\right) \in \Phi_{\mathbb{BC}}\right\},$$

$$\left(\mathcal{Z}_{\mathbb{BC}}\left(u, v; \mathcal{L}_{\mathbb{BC}}^{q}\right)\right)^{\beta_{\mathbb{D}}}$$

$$= \left\{a: \frac{1}{u} \otimes \Delta_{\mathbb{BC}}^{+}\left(\frac{a}{v}\right) \in \left(\mathcal{L}_{\mathbb{BC}}^{q}\right)^{\beta_{\mathbb{D}}} \wedge \frac{a}{u\otimes v} \in \Phi_{\mathbb{BC}}\right\}$$

$$= \left\{a: \left(\pi(s)\right) \in \mathcal{L}_{\mathbb{BC}}^{p} \wedge \left(\mu(s)\right) \in \Phi_{\mathbb{BC}}\right\}$$

$$= \left\{a: \sum_{s=0}^{\infty}\left(\pi(s)\right)^{p} \prec \infty_{\mathbb{D}} \wedge \left(\mu(s)\right) \in \Phi_{\mathbb{BC}}\right\},$$

$$\left(\mathcal{Z}_{\mathbb{BC}}(u, v; \Phi_{\mathbb{BC}})\right)^{\beta_{\mathbb{D}}} = \left\{a: \frac{1}{u} \otimes \Delta_{\mathbb{BC}}^{+}\left(\frac{a}{v}\right) \in \Phi_{\mathbb{BC}}^{\beta_{\mathbb{D}}} \wedge \frac{a}{u\otimes v} \in C_{\mathbb{BC}}^{0}\right\}$$

$$= \left\{a: \left(\pi(s)\right) \in \mathcal{L}_{\mathbb{BC}}^{1} \wedge \left(\mu(s)\right) \in C_{\mathbb{BC}}^{0}\right\}$$

$$= \left\{a: \sum_{s=0}^{\infty} \pi(s) \prec \infty_{\mathbb{D}} \wedge \left(\mu(s)\right) \in C_{\mathbb{BC}}^{0}\right\}$$

are obtained.

4. Results and Discussion

In this study, bicomplex $\mathcal{Z}_{\mathbb{BC}}$-spaces, which constitute bicomplex generalizations of the classical $\mathcal{Z}$ spaces, have been introduced and systematically investigated. In this framework, the spaces $\Phi_{\mathbb{BC}}$, $C_{\mathbb{BC}}$, $C_{\mathbb{BC}}^{0}$, $\mathcal{L}_{\mathbb{BC}}^{p}$ $(1 \leq p < \infty)$, $\mathcal{CS}_{\mathbb{BC}}$, $\mathcal{BS}_{\mathbb{BC}}$ and $\mathcal{BV}_{\mathbb{BC}}$ were defined and analyzed. These constructions extend the topological structure of sequence spaces from the complex setting to the bicomplex number framework, thereby enriching the theory of sequence spaces with additional algebraic and topological features arising from bicomplex analysis.

Subsequently, the $\beta_{\mathbb{D}}$-duals of the newly defined bicomplex $\mathcal{Z}_{\mathbb{BC}}$-spaces were determined. In particular, it was shown that the $\beta_{\mathbb{D}}$-duals of $C_{\mathbb{BC}}^{0}$, $C_{\mathbb{BC}}$, and $\Phi_{\mathbb{BC}}$ all coincide with the space $\mathcal{L}_{\mathbb{BC}}^{1}$. This result provides a precise characterization of bicomplex sequences that generate convergent series when multiplied by elements of these $\mathcal{Z}_{\mathbb{BC}}$-spaces. From a functional-analytic perspective, this characterization plays a role analogous to that of classical duality results in complex sequence space theory, while also reflecting the intrinsic structure induced by the presence of zero divisors in the bicomplex setting.

The findings of this work contribute to the development of functional analysis over bicomplex numbers by supplying new examples of bicomplex sequence spaces and by clarifying their dual relationships. These results offer useful tools and insights for researchers working on sequence spaces, duality theory, and related aspects of bicomplex functional analysis. Moreover, the techniques employed here may be adapted to the study of other classes of bicomplex sequence spaces.

Future research directions include the investigation of weighted bicomplex sequence spaces and the analysis of their duals and matrix transformations. In addition, exploring potential applications of bicomplex sequence spaces in areas where bicomplex numbers naturally arise-such as signal processing, quantum mechanics, and electromagnetism-appears to be a promising avenue for further study.

Overall, this study establishes a foundational framework for the analysis of $\mathcal{Z}_{\mathbb{BC}}$-spaces in the bicomplex setting and opens new perspectives for both theoretical advancements and applied research in bicomplex functional analysis.

References

[1] Altay, B. & Başar, F. (2001). *Matrix mappings on the space bs(p) and its α−, β− and γ−duals*, Aligarh Bull. Math. 21(1), 79-91.

[2] Altay, B. & Başar, F. (2006). *Some paranormed sequence spaces of non-absolute type derived by weighted mean*, J. Math. Anal. Appl. 319(2), 494-508.

[3] Altay, B.,& Başar, F. (2007). *Certain topological properties and duals of the matrix domain of a triangle matrix in a sequence space*, ibid. 336(1), 632-645.

[4] Başar, F. & Çolak, R. (2011). *Summability theory and ıts applications*, url: www.ifrro.org, eISBN: 978-1-60805-452-3.

[5] Segre, C. (1892). *Rappresentazioni reali delle forme complesse e gli enti peralgebrici*, Math. Ann., 40. No. 3, 413-467.

[6] Işık, G. & Duyar, C. (2023). *On Bicomplex Lorentz Sequence, Explorations in Mathematical Analysis*, İlker Eryılmaz (Ed.) Explorations in Mathematical Analysis: A Collection of Diverse Papers (1 st Edition, 6-32). Bidge Publications.

[7] Jarrah, A. M. & Malkowsky, E. (1998). *BK spaces, bases and linear operators*, Rendiconti Circ. Mat. Palermo II.

[8] Luna–Elizarraras, M. E.; Shapiro, M.; Struppa, D. C. & Vajiac, A. (2015). *Bicomplex Holomorphic Functions: The Algebra, Geometry and Analysis of Bicomplex Numbers*, International Publishing Switzerland.

[9] Maddox, I. J. (1968). *Paranormed sequence spaces generated by infinite matrices*. Proc. Cambridge Philos. Soc. 64, 335–340.

[10] Maddox, I. J. (1969). *Continuous and Kothe Toeplitz duals of certain sequence spaces*, Proc. Cambridge Philos. Soc. (65), 431–435.

[11] Malkowsky, E. (1989). *Absolute and ordinary Köthe-Toeplitz duals of some sets of sequences and matrix transformations*, Publ. Inst. Math. (Beograd) (N.S.) 46(60), 97-103.

[12] Malkowsky, E. & Rakocevic, V. (2001). *Measure of noncompactness of linear operators between spaces of sequences that are* $(\overline{\mathcal{N}}, q)$ *summable or bounded*, Czechoslovak Mathematical Journal, Vol. 51, No. 3, 505–522.

[13] Malkowsky, E. & Savaş, E. (2004). *Matrix transformations between sequence spaces of generalized weighted means*, Appl. Math. Comput. 147, 333-345.

[14] Ng, P.-N. And Lee, P.-Y. (1978). *Cesaro sequence spaces of non-absolute type*, Commentationes Mathematicae XX 429–433.

[15] Sager, N. & Sağır, B. (2020). *On completeness of some bicomplex sequence spaces*, Palest. J. Math., Vol:9, No.2, 891-902.

[16] Sağır, B. & Değirmen, N. (2022). $\mathbb{D}-$*Topological Duals of Bicomplex* $\mathbb{BC}-$*Modules* $\ell_p^{\mathbb{BC}}$, Journal of Mathematical Extension, Vol:16, No.10, (6) 1-25.

[17] Toksoy, E. & Sağır, B. (2024) *On geometrical characteristics and inequalities of new bicomplex Lebesgue Spaces with hyperbolic-valued norm*, Georgian Math. J., 31 (3), 483-495.

[18] Wilansky, A. (1984). *Summability through Functional Analysis,* North-Holland Mathematics Studies 85, Amsterdam-New York-Oxford.

CHAPTER 3

ON THE CONTINUITY OF THE WIGNER-VILLE DISTRIBUTION IN H^1 AND BMO SPACES

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Introduction

As a member of the Cohen class, the Wigner-Ville distribution is a quadratic time-frequency representation used to extract key signal characteristics, including marginal properties, mean instantaneous frequency, and group delay. Unlike Short-Time Fourier Transform -based spectrograms, Wigner-Ville distribution does not require any windowing function, this eliminates biases arising from window type selection in the analysis. Thanks to its time and frequency shift invariance property, signal components shifted on the time axis retain their morphological integrity in the time-frequency plane. Wigner-Ville distribution offers superior performance, particularly in situations requiring high time-frequency resolution, where components are far apart, or where feature extraction is required from single-component signals.

This study investigates the mapping properties of the Wigner-Ville distribution, a fundamental tool in time-frequency

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analysis, within the context of Hardy and BMO spaces, which are standard frameworks for harmonic analysis.

For the purpose of conceptual clarity, the key terms are defined below.

Let h denote a function on $\mathbb{R}$. The modulation operator of h is specified as $M_\mu h(y) = h(y)e^{2\pi i\mu y}$ for $\mu, y \in \mathbb{R}$, while the translation operator is on $T_u h(y) = h(y-u)$ for $y \in \mathbb{R}$. T and M are sometimes known as the time and frequency shift operators, respectively. Operators $T_u M_\mu$ or $M_\mu T_u$ are known as time frequency shifts. T and M do not commute. However, we observe instantly the canonical commutation relations

$$M_\mu T_u = e^{2\pi i u\mu} T_u M_\mu.$$

It is evident that L and M commute iff $u\mu \in \mathbb{Z}$.

The dilation operator, denoted by D_λ, is defined as $D_\lambda h(y) = \frac{1}{\lambda} h\left(\frac{y}{\lambda}\right)$, where $\lambda > 0$.

If $p \in [1, \infty[$, the Lebesgue spaces which is denoted by $L^p(\mathbb{R})$, is defined as the set of complex-valued measurable functions on $\mathbb{R}$ that satisfy

$$\int_{\mathbb{R}} |h(y)|^p dy < \infty.$$

If $h \in L^p(\mathbb{R})$, the L^p norm of h is defined by

$$\|h\|_{L^p} = \|h\|_p = \left(\int_{\mathbb{R}} |h(y)|^p dy\right)^{1/p} < \infty.$$

Under the norm $\|\cdot\|_p$, the set of functions $L^p(\mathbb{R})$ forms a complete normed vector space.

We define a complex-valued function h on $\mathbb{R}$ as locally integrable provided that the condition $\int_K |h(x)| dx < \infty$ holds for all compact $K \subset \mathbb{R}$. $L^1_{loc}(\mathbb{R})$ denotes the spaces of locally integrable functions.

Let $h \in L^1(\mathbb{R})$, let us define $\hat{h}$ (or $\mathcal{F}h$) by

$$\mathcal{F}h(z) = \hat{h}(z) = \int_{\mathbb{R}} h(x) e^{-2\pi i x z} dx, \quad z \in \mathbb{R}.$$

The expression $\hat{h}$ denotes the Fourier transform of h, (Gröchenig, 2001), (Debnath & Shah, 2015).

The following definition constitutes the cross-Wigner Ville distribution of functionals h and g, which are elements of the $L^2(\mathbb{R})$ space:

$$W(h,g)(u,\mu) = \int_{\mathbb{R}} e^{-2\pi i \mu t} h(u + t/2) \overline{g(u - t/2)} dt.$$

If we write h instead of g, then $W(h,h) = Wh$ is known as the Wigner distribution of h. In the context of analysing non-stationary signals, it is imperative to employ both time and frequency representations, as the Fourier analysis, a valuable instrument for the study of stationary signals, is inadequate for the comprehensive analysis of non-stationary signals. The Wigner distribution is the most often used time-frequency representation because it offers a high-resolution representation in both time and frequency for non-stationary signals, (Wiener, 1932), (Gröchenig, 2001), (Debnath & Shah, 2015).

There are significant connections between the theory of Hardy Spaces and many areas of mathematical study, such as Fourier analysis, harmonic analysis, operator theory and singular integrals, signal and image processing, and control theory. Research shows that for specific problems in harmonic analysis, Hardy spaces offer a more suitable framework than Lebesgue spaces. The maximal functions can be defined as follows, and this will allow us to give an equivalent definition of $H^1(\mathbb{R})$: We are going to take a function that is both integrable and smooth. This function will be denoted by ' φ ' and its domain will be the Euclid space, with its support lying in the unit ball. In addition, it should be $\int_{\mathbb{R}} \varphi = 1$. Let us set $\varphi_t(y) = 1/t\, \varphi(y/t),\ t > 0$. For an integrable function h, the maximal operator, represented by the symbol $\mathcal{M}_\varphi$, is defined as follows:

$$\mathcal{M}_\varphi h(y) = \sup_{t>0} |h * \varphi_t(y)|.$$

The $H^1(\mathbb{R})$ represents the linear space of all $h \in L^1(\mathbb{R})$ if, for some $\varphi \in \mathcal{S}(\mathbb{R})$ with $\int_{\mathbb{R}} \varphi = 1$, $\mathcal{M}_\varphi h$ is in $L^1(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz space. If h belongs to Hardy Space, then both the dilation operator $D_\lambda h$ and the translation operator $T_u h$ are in Hardy Space and fulfill

$$\|D_\lambda h\|_{H^1} = \|h\|_{H^1} \quad \text{and} \quad \|T_u h\|_{H^1} = \|h\|_{H^1}.$$

The space of Bounded Mean Oscillation, or BMO, consists of functions whose average deviation from their mean over cubes remains bounded, (also called the John-Nirenberg space). In (John & Nirenberg, 1961), John and Nirenberg developed the space $BMO(\mathbb{R})$ of functions with bounded mean oscillation. $BMO(\mathbb{R})$ represents the space of all functions $h \in L^1_{loc}(\mathbb{R})$ such that

$$\|h\|_{BMO} = \sup_{Q\subset\mathbb{R}} |Q|^{-1} \int_Q |h(x) - Q(h)| dx < \infty,$$

where the integral is over Q and the supremum is taken over the balls Q in $\mathbb{R}$ of measure $|Q|$, and $Q(h)$ stands for the mean of h on Q, namely,

$$Q(h) = |Q|^{-1} \int_Q h(x)\, dx \le |Q|^{-1} \int_Q |h(x)|\, dx \le M < \infty.$$

One of the most significant results in harmonic analysis is that the dual of the Hardy space is precisely the space $BMO(\mathbb{R})$. These spaces are thoroughly explored in the literature, particularly in (John & Nirenberg, 1961), (Bennet & Sharpley, 1979), (Stein & Murphy, 1993), (Edmunds & Evans, 2004), (Chuong & Duong, 2013), (Verma & Gupta, 2021).

Continuity of Wigner Wille Distribution on Hardy Space

We now examine the continuity of the Wigner-Ville distribution on Hardy space.

Lemma 1.1. If $h \in L^1(\mathbb{R})$, $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $W(h,g)(\cdot,\mu) \in L^1(\mathbb{R})$.

Proof. For a fixed μ in $\mathbb{R}$, the function $W(h,g)(u,\mu)$ depends on u. By changing variable $u - t/2 = z$ and applying the Fubini's Theorem, we have

$$
\begin{aligned}
\|W(h,g)(\cdot,\mu)\|_1 &= \int_{\mathbb{R}} |W(h,g)(u,\mu)| \, du \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{-2\pi i\mu} \; h\left(u + \frac{t}{2}\right) \overline{g\left(u - \frac{t}{2}\right)} \, dt \right| du \\
&= 2 \int_{\mathbb{R}} \left| \int_{\mathbb{R}} e^{-4\pi i\mu} \; h(2u - z) \overline{g(z)} \; e^{4\pi i\mu z} dz \right| du \\
&\leq 2 \int_{\mathbb{R}} |\overline{g(z)}| \left(\int_{\mathbb{R}} |h(2u - z)| \, du \right) dz \\
&= \int_{\mathbb{R}} |\overline{g(z)}| \left(\int_{\mathbb{R}} |D_{1/2} T_z h(u)| \, du \right) dz \\
&= \int_{\mathbb{R}} |\overline{g(z)}| \, \|D_{1/2} T_z h\|_1 dz
\end{aligned}
$$

Again, using the dilation invariant and translation invariant properties of Lebesgue space, we obtain

$$\|W(h,g)(\cdot,\mu)\|_1 = \|h\|_1 \int_{\mathbb{R}} \left|\overline{g(z)}\right| \, dz = \|h\|_1 \|g\|_1.$$

Hence, $W(h,g)(\cdot,\mu) \in L^1(\mathbb{R})$.

Theorem 1.2. Let g be in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. The function $h \to W(h,g)(\cdot,\mu)$ defines a continuous operator from $H^1(\mathbb{R})$ to itself. Moreover,

$$\|W(h,g)(\cdot,\mu)\|_{H^1} \leq \|h\|_{H^1} \, \|g\|_1.$$

Proof. Every function h in the Hardy space $H^1(\mathbb{R})$ is also an element of $L^1(\mathbb{R})$. Then By Lemma 1.1, we obtain $W(h,g) \in L^1(\mathbb{R})$. If Wigner Ville distribution rewrite in the following form

$$\begin{aligned} W(h,g)(u,\mu) &= 2e^{4\pi iu} \int_{\mathbb{R}} e^{-4\pi ia\mu} \, h(a)\overline{g(a-2u)} \, da \\ &= 2e^{4\pi iu\mu} \int_{\mathbb{R}} e^{-4\pi i(z+2u)\mu} \, h(z+2u)\overline{\tilde{g}(z)} \, dz \\ &= 2e^{4\pi iu\mu} \int_{\mathbb{R}} M_{-2\mu} \, h(z+2u)\overline{\tilde{g}(z)} \, dz, \end{aligned}$$

where $\tilde{g}(z)$ is defined as $g(-z)$, we have

$$\begin{aligned} &(W(h,g)(\cdot,\mu) * \varphi_t)(x) \\ &= \int_{\mathbb{R}} W(h,g)(x-y,\mu) \, \varphi_t(y) \, dy \\ &= \int_{\mathbb{R}} \left(2e^{4\pi i(x-y)\mu} \int_{\mathbb{R}} M_{-2\mu} \, h\big(z+2(x-y)\big)\overline{\tilde{g}(z)} \, dz \right) \varphi_t(y) \, dy \\ &= 2\int_{\mathbb{R}} e^{4\pi i(x-y)\mu}\overline{\tilde{g}(z)} \left(\int_{\mathbb{R}} \frac{1}{2} D_{\frac{1}{2}} T_{-z} M_{-2\mu} h(x-y)\varphi_t(y) \, dy \right) dz \\ &= \int_{\mathbb{R}} e^{4\pi i(x-y)\mu}\overline{\tilde{g}(z)} \left(\left(D_{\frac{1}{2}} T_{-z} M_{-2\mu} h \right) * \varphi_t \right)(x) dz. \end{aligned}$$

Leveraging the translation and dilation invariance of Hardy space, we obtain

$$\|W(h,g)(\cdot,\mu)\|_{H^1} = \int_{\mathbb{R}} \sup_{t>0} |(W(h,g)(\cdot,\mu) * \varphi_t)(x)|\, dx$$

$$\leq \int_{\mathbb{R}} \left|\overline{\tilde{g}(z)}\right| \left(\int_{\mathbb{R}} \sup_{t>0} \left| \left(\left(D_{\frac{1}{2}} T_{-z} M_{-2\mu} h \right) * \varphi_t \right)(x) \right| dx \right) dz$$

$$= \int_{\mathbb{R}} \left|\overline{\tilde{g}(z)}\right| \left\| D_{\frac{1}{2}} T_{-z} M_{-2\mu} h \right\|_{H^1} dz$$

$$= \|h\|_{H^1} \int_{\mathbb{R}} |\tilde{g}(z)| dz = \|h\|_{H^1} \|g\|_1.$$

Thus, the assertion is proved.

Theorem 1.3. Let ϕ and ψ belong to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. If $h_1,\ h_2 \in H^1(\mathbb{R})$, then

$$\|W(h_1,\phi)(\cdot,\mu) - W(h_2,\psi)(\cdot,\mu)\|_{H^1} \leq \|\phi - \psi\|_1 \|h_1\|_{H^1} + \|\psi\|_1 \|h_1 - h_2\|_{H^1}.$$

Proof. Since $W(h_1,\psi)(u,\mu) - W(h_2,\psi)(u,\mu) = W(h_1 - h_2,\psi)(u,\mu)$, we obtain

$$\Big(\big(W(h_1,\psi)(u,\mu) - W(h_2,\psi)(u,\mu) \big) * \varphi_t(\cdot) \Big)(x)$$

$$= \int_{\mathbb{R}} e^{4\pi i (x-y)\mu} \overline{\tilde{g}(z)} \left(\left(D_{\frac{1}{2}} T_{-z} M_{-2\mu} (h_1 - h_2) \right) * \varphi_t \right)(x) dz.$$

Then by Theorem 1.2, we get

$$\|W(h_1,\psi)(\cdot,w) - W(h_2,\psi)(\cdot,w)\|_{H^1} \leq \|\psi\|_1 \|h_1 - h_2\|_{H^1}. \qquad (1)$$

Moreover, it is evident that

$$W(h_1,\phi)(u,\mu) - W(h_1,\psi)(u,\mu) = W(h_1,\phi - \psi)(u,\mu)$$

and

$$\left(\left(W(h_1,\phi) - W(h_1,\psi)\right)(\cdot,\mu) * \varphi_t(\cdot)\right)(x)$$

$$= \int_{\mathbb{R}} e^{4\pi\ (x-y)\mu}\, \overline{(\widetilde{g_1 - g_2})(z)}\, \left(\left(D_{\frac{1}{2}} T_{-z} M_{-2\mu} h_1\right) * \varphi_t\right)(x) dz.$$

Again by Theorem 1.2, we write

$\|W(h_1,\phi)(\cdot,\mu) - W(h_1,\psi)(\cdot,\mu)(\cdot,w)\|_{H^1} \leq \|\phi - \psi\|_1 \|h_1\|_{H^1}.$ (2)

Then from the equations (1) and (2), we obtain

$$\|W(h_1,\phi)(\cdot,\mu) - W(h_2,\psi)(\cdot,\mu)\|_{H^1}$$

$$\leq \|W(h_1,\phi)(\cdot,\mu) - W(h_1,\psi)(\cdot,\mu)\|_{H^1} + \|W(h_1,\psi)(\cdot,\mu) - W(h_2,\psi)(\cdot,\mu)\|_{H^1}$$

$$\leq \|\phi - \psi\|_1 \|h_1\|_{H^1} + \|\psi\|_1 \|h_1 - h_2\|_{H^1}.$$

Continuity of Wigner Wille Distribution on BMO Space

Now, we will investigate the BMO- countinuity of Wigner Ville distribution. In order to proceed, the following lemma must be established.

Lemma 1.4. Let us assume that g is a function belonging to the $L^1(\mathbb{R})$ and is compactly supported (cs). If $h \in L^1_{loc}(\mathbb{R})$, then $W(h,g)(\cdot,\mu) \in L^1_{loc}(\mathbb{R})$.

Proof. Since $W(h,g)(u,\mu)$ is a function of the first variable and

$$|W(h,g)(u,\mu)| = \left| 2e^{4\pi i u\mu} \int_{\mathbb{R}} e^{-4\pi i(a+2u)\mu}\, h(a+2u)\overline{\tilde{g}(a)}\, da \right|$$

$$\leq 2\int_{\mathbb{R}} |h(a+2u)| \left|\overline{\tilde{g}(a)}\right| da,$$

we can get for any compact ball $B \subset \mathbb{R}$

$$\int_B |W(h,g)(u,\mu)|\, du \le \int_{\mathbb{R}} |\tilde{g}(a)| \left(\int_B |h(a+2u)| du \right) da.$$

Let $K \subset a + B$. As $K \subset suppg + B$, where $suppg$ is the closure of the set $\{x \in \mathbb{R}^d | g(x) \neq 0\}$, is a closed and bounded set in $\mathbb{R}$ and $h \in L^1_{loc}(\mathbb{R})$, hence we get

$$\int_B |W(h,g)(u,\mu)|\, du \le \int_{\mathbb{R}} |\tilde{g}(a)| \left(\int_K |h(b)| db \right) da = N\, \|g\|_1.$$

So, $W(h,g)(\cdot,\mu)$ is a locally integrable function.

Theorem 1.5. Assume that $g \in L^1(\mathbb{R})$ is a function whose closed support is a compact set. The function $h \to W(h,g)(\cdot,\mu)$ defines a continuous operatör from $BMO(\mathbb{R})$ to itself. Moreover,

$$\|W(h,g)(\cdot,\mu)\|_{BMO} \le (\|h\|_{BMO} + 2M)\, \|g\|_1.$$

Proof. Let $Q \subset \mathbb{R}$ be an arbitrary ball and $h \in BMO(\mathbb{R})$. Then $h \in L^1_{loc}(\mathbb{R})$ and so $W(h,g) \in L^1_{loc}(\mathbb{R})$ by Lemma 1.4. From Fubini's Theorem, it follows that:

$$Q\big(W(h,g)\big) = |Q|^{-1} \int_Q W(h,g)(z,\mu)\, dz$$

$$= |Q|^{-1} \int_Q \left(2 \int_{\mathbb{R}} h(a+2z) \tilde{g}(a) e^{-4\pi i \mu (a+2z)} e^{4\pi i \mu z} da \right) dz$$

$$= 2 \int_{\mathbb{R}} \tilde{g}(a) \left(|Q|^{-1} \int_Q M_{-2\mu} h(a+2z) e^{4\pi i \mu z} dz \right) da$$

$$= 2 \int_{\mathbb{R}} \tilde{g}(a) \left(|Q|^{-1} \int_Q \frac{1}{2} D_{1/2} T_{-a} M_{-2\mu} h(z) e^{4\pi i \mu z} dz \right) da$$

$$= \int_{\mathbb{R}} \tilde{g}(a) \left(|Q|^{-1} \int_Q M_{2\mu} D_{1/2} T_{-a} M_{-2\mu} h(z) dz \right) da,$$

and from here, we write

$$\|W(h,g)(\cdot,\mu)\|_{BMO}$$
$$= \sup_{Q\subset\mathbb{R}} |Q|^{-1} \int_Q \left| W(h,g)(u,\mu) - Q\big(W(h,g)\big) \right| du$$
$$\leq \int_{\mathbb{R}} |\tilde{g}(a)| \left(\sup_{Q\subset\mathbb{R}} |Q|^{-1} \int_Q \left| \big(M_{2\mu} D_{1/2} T_{-a} M_{-2\mu} h\big)(u) - Q\big(M_{2\mu} D_{1/2} T_{-a} M_{-2\mu} h\big) \right| du \right) da$$
$$= \int_{\mathbb{R}} |\tilde{g}(a)| \left\| M_{2\mu} D_{1/2} T_{-a} M_{-2\mu} h \right\|_{BMO} da,$$

also by using Lemma 2.2 in (Sandıkçı, 2023) and the dilation invariance of BMO, we obtain

$$\|W(h,g)(\cdot,\mu)\|_{BMO} \leq \int_{\mathbb{R}} |\tilde{g}(a)| (\|h\|_{BMO} + 4M) da$$
$$= (\|h\|_{BMO} + 2M) \int_{\mathbb{R}} |\tilde{g}(a)| da$$
$$= (\|h\|_{BMO} + 2M)\ \|g\|_1.$$

Therefore, the premise has been established.

Theorem 1.6. Let $\phi,\ \psi \in L^1(\mathbb{R})$ be two compactly supported functions. If $h_1,\ h_2 \in BMO(\mathbb{R})$, then we have

$$\|W(h_1,\psi)(\cdot,\mu) - W(h_2,\phi)(\cdot,\mu)\|_{BMO}$$
$$\leq \|\psi - \phi\|_1 (\|h_1\|_{BMO} + 4M) + \|\phi\|_1 (\|h_1 - h_2\|_{BMO} + 4M).$$

Proof. Let $\phi,\ \psi \in L^1(\mathbb{R})$ be two cs functions and $h_1,\ h_2 \in BMO(\mathbb{R})$. So, $h_1,\ h_2 \in L^1_{loc}(\mathbb{R})$ and so $W(h_1,\psi)(\cdot,\mu)$, $W(h_2,\phi)(\cdot,\mu) \in L^1_{loc}(\mathbb{R})$ by Lemma 1.4. Morever, since $W(h_1,\psi)(u,\mu) - W(h_1,\phi)(u,\mu) = W(h_1,\psi - \phi)(u,\mu)$ and $W(h_1,\phi)(u,\mu) - W(h_2,\phi)(u,\mu) = W(h_1 - h_2,\phi)(u,\mu)$, we obtain by Theorem 1.5,

$$\|W(h_1,\psi)(\cdot,\mu) - W(h_2,\phi)(\cdot,\mu)\|_{BMO}$$
$$\leq \|W(h_1,\psi)(\cdot,\mu) - W(h_1,\phi)(\cdot,\mu)\|_{BMO} + \|W(h_1,\phi)(\cdot,\mu) - W(h_2,\phi)(\cdot,\mu)\|_{BMO}$$
$$= \|W(h_1,\psi - \phi)(\cdot,\mu)\|_{BMO} + \|W(h_1 - h_2,\phi)(\cdot,\mu)\|_{BMO}$$
$$\leq \|\psi - \phi\|_1(\|h_1\|_{BMO} + 4M) + \|\phi\|_1(\|h_1 - h_2\|_{BMO} + 4M).$$

Consequently, the hypothesis is validated.

Results and Discussion

In the present work, we examined how the Wigner Ville distribution, a cornerstone of time-frequency analysis, maps onto Hardy and BMO spaces. These spaces serve as the classical environments for conducting harmonic analysis. This theoretical foundation provides a framework for translating abstract concepts into practical engineering solutions.

References

Bennet, C. & Sharpley, R. (1979). Weak-type inequalities for H^p and BMO, Proc. Sympos. Pure Math., 35 (1): 201-229.

Chuong, N. M. & Duong, D. V. (2013). Boundedness of the wavelet integral operator on weighted function spaces, Russ. J. Math. Phys., 20(3): 268-275.

Debnath, L. & Shah, F. A. (2015). Wavelet Transforms and Their Applications, Birkhauser, New York.

Edmunds, D. E. & Evans, W. D. (2004). Hardy Operators Function Spaces and Embeddings, Springer Berlin Heidelberg New York.

Gröchenig, K. (2001). Foundations of Time-Freuquency Analysis, Birkhauser, Boston.

John, F. & Nirenberg, L. (1961). On functions of bounded mean oscillation, Comm. Pure Appl. Math. 14 (3): 415–426.

Sandıkçı, A. (2023). Characterization of the boundedness of τ-Wigner transform on Hardy and BMO spaces, U.P.B. Sci. Bull., Series A., 85 (1): 43-52.

Stein, E. M. & Murphy, T. S. (1993). Harmonic analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals, Princeton, NJ: Princeton University Press.

Verma, A. K. & Gupta, B. (2021). Certain properties of continuous fractional wavelet transform on Hardy space and Morrey space, Opuscula Math., 41(5): 701-723.

Wiener E. (1932). On the quantum correction for thermodynamic equilibrium, Phys. Rev., 40(5), 749-759.

CHAPTER 4

THE WEBER–FECHNER LAW AS AN ANAGEOMETRIC MODEL

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Introduction

Non-Newtonian calculus was first introduced by Michael Grossman and Robert Katz between 1967 and 1970 as an alternative framework to classical calculus. The researchers initially described an infinite family of calculi consisting of classical, geometric, harmonic, and quadratic analysis, and later expanded this family by defining the bigeometric, biharmonic, and biquadratic calculi. More recently, this family has been further enriched by logarithmic–geometric approaches such as anageometric calculus, a system in

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which variations in function values are still evaluated through linear differences, while changes in the independent variable are interpreted through ratios rather than increments, providing a multiplicative perspective not captured by classical methods. The structure of anageometric calculus also resonates with the broader efforts to unify discrete and continuous frameworks, such as Hilger's time scales theory introduced in 1988, which similarly reformulates analytical concepts by altering the underlying measurement of change. Since all these systems deviate fundamentally from classical calculus, Grossman and Katz referred to them collectively as "non-Newtonian calculi" (Grossman & Katz, 1972; Grossman, 1979, 1983).

Non-Newtonian calculi offer tools that are often more suitable than classical calculus for capturing proportional change, multiplicative growth, and scale-dependent behavior in mathematical models. For this reason, they have found wide application in differential equations, functional analysis, numerical methods, biology, economics, image processing, artificial intelligence, blood viscosity modeling, elasticity theory, and many other fields (Bashirov et al., 2008, 2011; Çakmak & Başar, 2012; Boruah & Hazarika, 2018).

Within this broad family, anageometric calculus has emerged as another important member based on a geometric perspective of variation. Anageometric calculus measures change not by additive differences but by logarithmic differences, i.e.,

$$\ln(b) - \ln(a),$$

which represent multiplicative (geometric) change on the positive real axis. This replaces the classical notion of distance with a geometric distance, allowing the behavior of functions to be studied in terms of proportional variation. In this framework, the anageometric derivative quantifies the response of a function to infinitesimal multiplicative perturbations of its argument, whereas the anageometric integral is formulated as a Stieltjes integral weighted along the logarithmic axis.

Anageometric calculus provides a natural analytical setting for problems involving the Weber–Fechner Law, stellar magnitude, scale invariance in the argument, growth induced by proportional variation in the argument, linearity in the logarithm of the argument, and processes defined on the positive real axis. Because of these properties, anageometric analysis offers a more suitable mathematical model in many contexts where classical or geometric calculus is insufficient.

In summary, anageometric calculus occupies a distinctive position within the family of non-Newtonian calculi. It brings together

- the multiplicative structure of geometric calculus,
- the limit, derivative, and integral concepts of classical calculus,

to form a logarithmically grounded differential and integral framework.

The Weber–Fechner law constitutes a foundational principle of psychophysics by formalizing the relationship between the physical intensity of a stimulus and its subjective perceptual magnitude. Early experimental work by Ernst Heinrich Weber demonstrated that the just-noticeable difference (JND) between two stimuli is proportional to the baseline stimulus intensity rather than being an absolute quantity, a relationship now known as Weber's law (Weber, 1834). Building on this empirical insight, Gustav Theodor Fechner proposed a logarithmic formulation linking physical stimulus intensity I to subjective sensation S, expressed as $S = k\log I + C$, thereby establishing the Weber–Fechner law (Fechner, 1860). This logarithmic scaling implies that sensation grows arithmetically as stimulus intensity increases geometrically, a mechanism often interpreted as an efficient sensory coding strategy that compresses a wide dynamic range of environmental inputs into a manageable internal representation (Portugal & Svaiter, 2011). Although later empirical research, most notably Stevens' power law, has shown that a power function may better capture perception in certain sensory modalities, the Weber–Fechner framework remains a central theoretical model in perceptual psychology and cognitive neuroscience, with influential applications in areas such as numerical cognition and the logarithmic mental number line described by Stanislas Dehaene (Dehaene, 2003).

In the following sections, we introduce the fundamental concepts of anageometric analysis, including the anageometric derivative, anageometric mean, and anageometric integral, and discuss their relationship to classical calculus as well as their potential applications. Finally we give the Weber-Fechner law as an anageometric model.

1. Basic Concepts of Non-Newtonian Analysis

In this section, some fundamental definitions and concepts used in anageometric analysis, which is one of the non-Newtonian analytical approaches, will be introduced. These concepts have been discussed in various studies in the literature and form the theoretical foundation of alternative analytical systems based on multiplicative changes in functions (Türkmen & Başar, 2012; Çakmak & Başar, 2012).

Definition 1: A function $\alpha: \mathbb{R}_{\text{exp}} \to \mathbb{R}$ which is bijective (one-to-one and onto) and continuous is called a generator function.

Definition 2 : The non-Newtonian (N.N.) number sets are defined as follows:

	N.N Number Sets	Exponential Number Sets
N.N real numbers	$\mathbb{R}_\alpha = \{\alpha(t) \mid t \in \mathbb{R}\}$	$\mathbb{R}_{\exp} = (0, \infty)$,
N.N positive real numbers	$\mathbb{R}_\alpha^+ = \{\alpha(t) \mid t > 0\}$	$\mathbb{R}_{\exp}^+ = (1, \infty)$,
N.N negative real numbers	$\mathbb{R}_\alpha^- = \{\alpha(t) \mid t < 0\}$	$\mathbb{R}_{\exp}^- = (0,1)$,
N.N non-negative real numbers:	$\mathbb{R}_\alpha^{+,0} = \{\alpha(t) \mid t \geq 0\}$	$\mathbb{R}_{\exp}^{+,0} = [1, \infty)$,
N.N non-positive real numbers	$\mathbb{R}_\alpha^{-,0} = \{\alpha(t) \mid t \geq 0\}$	$\mathbb{R}_{\exp}^{-,0} = (0,1]$

Definition 3: The following operations define the α-arithmetic in non-Newtonian analysis:

- α-addition: $x \dot{+} y = \alpha\left(\alpha^{-1}(x) + \alpha^{-1}(y)\right)$
- α-subtraction: $x \dot{-} y = \alpha\left(\alpha^{-1}(x) - \alpha^{-1}(y)\right)$
- α-multiplication: $x \dot{\times} y = \alpha\left(\alpha^{-1}(x) \times \alpha^{-1}(y)\right)$
- α-division: $x \dot{/} y = \alpha\left(\alpha^{-1}(x) \div \alpha^{-1}(y)\right)$, for $y \neq \alpha(y)$
- α-ordering: $x \dot{<} y \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y)$

Definition 4: If the generator function is chosen as $\alpha = \exp$ then the geometric arithmetic operations in non-Newtonian analysis for all $a, b \in \mathbb{R}_{\exp}$ are given as follows (Grossman & Katz, 1972; Grossman, 1979, 1983; Boruah & Hazarika, 2018a,b):

- **Geometric addition:**

$$a \oplus b = \exp\{\ln(a) + \ln(b)\} = e^{\ln(a)+\ln(b)} = a \cdot b$$

- **Geometric subtraction:**

$$a \ominus b = \exp\{\ln(a) - \ln(b)\} = e^{\ln(a)-\ln(b)} = \frac{a}{b}$$

- **Geometric multiplication:**

$$a \odot b = \exp\{\ln(a) \cdot \ln(b)\} = e^{\ln(a)\cdot\ln(b)} = a^{\ln(b)}$$

- **Geometric division:**

$$a \oslash b \ = \exp\left\{\frac{\ln(a)}{\ln(b)}\right\} = e^{\frac{\ln(a)}{\ln(b)}} = a^{\frac{1}{\ln(b)}}, \qquad (b \neq 1)$$

Additional properties of geometric (multiplicative) exponentiation, roots, inverse elements, and absolute value are as follows $\left(\text{For all } a, b \in \mathbb{R}_{\exp}, r \in \mathbb{R}\right)$

- $a^{2\odot} = a \odot a = a^{\ln a}$,
- $a^{b\odot} = \exp\{(\ln a)^b\}$,
- $\sqrt{a}_* = e^{(\ln a)^{\frac{1}{2}}}$,
- $\sqrt{a^{2\odot}}_* = |a|_* = e^{|\ln(a)|}$
- $a^{\ominus 1} \ = \ e^{\frac{1}{\ln a}}, \ (a \neq 1)$
- $a \odot e = a \oplus 1 = a$,
- $e^r \odot a = a^r$,
- $|a \oplus b|_* \leq_* |a|_* \oplus |b|_*$
- $|a \oslash b|_* = |a|_* \oslash |b|_*$
- $|a \ominus b|_* \geq_* |a|_* \ominus |b|_*$
- $|a \odot b|_* = |a|_* \odot |b|_*$
- $\ominus e \odot (a \ominus b) = b \ominus a$,
- $|e^r|_* = e^{|r|}$,
- $|a|_* = \begin{cases} a, & a > 1 \\ 1, & a = 1 \\ \frac{1}{a}, & a < 1. \end{cases}$

2. Fundamental Concepts and Theorems of Anageometric Analysis

Classical calculus investigates change through linear differences and limiting processes. In contrast, anageometric analysis characterizes variation not by additive increments but by geometric (multiplicative) changes in the argument. That is, a change in the argument of a function is not understood as

$$x \mapsto x + \Delta x,$$

but rather as a proportional transformation of the form

$$x \mapsto x \cdot \Delta_G x$$

Consequently, a change in the argument of a function is quantified not by the classical difference $x - a$, but by $\ln(x) - \ln(a)$, which is sensitive to multiplicative increments. This leads to a derivative and integral structure adapted to multiplicative variation.

Functions on the Positive Real Axis and Geometric Intervals

Anageometric analysis is developed exclusively on the domain of positive real arguments, that is on the set $\mathbb{R}^+ = (0, \infty)$. This restriction is not merely technical; it emerges naturally from the fact that anageometric change is measured through logarithmic differences, which are only well defined for positive inputs. Consequently, the fundamental objects replacing classical linear intervals are geometric intervals.

For two points $a < b$ in $\mathbb{R}^+$, the interval $[a, b]$ is assigned a geometric extent defined by $G(a, b) = \frac{b}{a}$. Unlike the classical notion of interval length $b - a$, the geometric extent captures the geometric displacement between the endpoints. Hence, the “size’’ of an interval is measured not by linear separation but by the ratio of its endpoints. This shift from additive to multiplicative structure is central to the conceptual framework of anageometric calculus: changes in arguments are evaluated through the induced logarithmic variation

$$\ln(b) - \ln(a) = \ln\left(\frac{b}{a}\right),$$

which encodes the same multiplicative information.

Anageometrically Uniform Functions

A function $f: (0, \infty) \to \mathbb{R}$ is said to be anageometrically uniform if its classical increment is determined solely by the geometric extent of the interval on which it is evaluated. Formally, the function satisfies

$$\frac{b}{a} = \frac{d}{c} \Longrightarrow f(b) - f(a) = f(d) - f(c),$$

for all positive quadruples $a < b$ and $c < d$. In other words, equal multiplicative changes in the argument yield equal additive changes in the value of the function.

An equivalent and structurally illuminating characterization shows that every anageometrically uniform function necessarily has the form

$$f(x) = \ln(Cx^m) = \ln C + m \cdot \ln x,$$

where $C > 0$ and $m \in \mathbb{R}$ are constants. Thus, anageometrically uniform functions are precisely those that appear as affine functions on the logarithmic scale; when plotted on a semi-logarithmic axis (logarithmic in x, linear in $f(x)$), such functions trace a straight line. Their anageometric slope is exactly the parameter m.

A further consequence of this structure is that if the argument values form a geometric progression, then the corresponding function values form an arithmetic progression. This multiplicative-to-additive correspondence is a defining hallmark of anageometric uniformity and explains why such functions serve as the “local linear models’’ in anageometric calculus the analogue of linear (affine) functions in classical differential calculus.

Stellar Magnitude as an Anageometric Model

The stellar (apparent) magnitude scale used to quantify stellar brightness originates from the ancient classification traditionally attributed to Hipparchus (ca. 130 BCE), in which brighter stars were assigned smaller numerical values, a convention that persists in modern astronomy through a logarithmic formulation (Hearnshaw, 1996). In contemporary terms, the apparent magnitude m of a star is defined in terms of the measured light flux F relative to a reference (zero-point) flux F_0 by

$$m = -2.5\log_{10}\left(\frac{F}{F_0}\right).$$

The negative sign preserves the historical convention that brighter objects correspond to smaller (and possibly negative) magnitude values, while the coefficient 2.5 encodes the Pogson relation, according to which a difference of five magnitudes corresponds to a factor of exactly 100 in flux (Norman Pogson, 1856; Carroll & Ostlie, 2017). The reference flux F_0 is not universal but depends on the adopted photometric system: in Vega-based systems, it is calibrated so that the star Vega has approximately zero magnitude in a given band (Johnson & Morgan, 1953), whereas in modern systems such as the AB magnitude system, the zero point is defined by a constant spectral flux density rather than by a specific star (Oke & Gunn, 1983). This logarithmic formulation and its physical interpretation constitute a foundational framework for stellar photometry and are central to modern astrophysics and observational cosmology (Ryden & Peterson, 2010).

The Anageometric Gradient on $[\boldsymbol{a}, \boldsymbol{b}]$

Given a function $f:(0,\infty)\to\mathbb{R}$ and two positive real numbers $a<b$, the anageometric gradient of f over the geometric interval $[a,b]$ is defined as the slope of the *unique* anageometrically uniform function that interpolates the points $(a,f(a))$ and $(b,f(b))$.

Since anageometrically uniform functions are precisely those of the form

$$f(x)=\ln(Cx^m)=\ln C+m\cdot\ln x,$$

their classical slope on a logarithmic x-axis is the constant parameter m. The anageometric gradient of f over $[a,b]$ is therefore determined by selecting the value of m that makes f pass through the two prescribed points. Solving

$$f(b)-f(a)=m(\ln(b)-\ln(a))$$

yields the explicit formula:

$$\underset{\sim}{G}_a^b f=\frac{f(b)-f(a)}{\ln(b)-\ln(a)}$$

This expression generalizes the classical *secant slope*, but replaces the linear increment $b-a$ with the logarithmic increment $\ln(b)-\ln(a)$, reflecting the multiplicative geometry of the underlying domain.

The definition exhibits two notable structural properties:

Invariance under unit transformations

Because it depends only on differences of logarithms, the anageometric gradient is insensitive to rescalings of the input space.

A uniform scaling by a factor k leaves the logarithmic difference invariant, since

$$\ln(kb)-\ln(ka)=\ln(b)-\ln(a)$$

by cancellation of the additive $\ln k$ terms. So, the anageometric gradient is unchanged.

This invariance is one rationale for adopting logarithmic measures of change in anageometric calculus.

Limiting behavior and the emergence of the anageometric derivative

When $b \to a$, the expression

$$\frac{f(b) - f(a)}{\ln(b) - \ln(a)}$$

develops the indeterminate form $0/0$. The limit, when it exists, is precisely the anageometric derivative of f at a, which will be developed in detail in the next section. Thus the gradient serves as the natural finite-interval analogue of the differential notion.

Interpretation on semi-logarithmic coordinates

If one plots f on a semi-logarithmic graph linear scale in the vertical direction, logarithmic scale in the horizontal direction then

- the points $(a, f(a))$ and $(b, f(b))$ appear at horizontal positions $\ln(a)$ and $\ln(b)$,
- and the anageometric gradient $G_a^b f$ is exactly the classical slope of the straight line joining these transformed points.

This graphical representation mirrors the role of secant lines in classical calculus, and further highlights why logarithmic differences play the role of “geometric increments’’ in the anageometric framework.

We can give the correspondence between anageometric gradient and gradient in the classical sense as follows:

Let $a < b$, and set

$$\alpha = \ln(a)\,, \;\; \beta = \ln(b)$$

Then the anageometric gradient

$$\underset{\sim}{G}_a^b f = \frac{f(b) - f(a)}{\ln(b) - \ln(a)}$$

corresponds directly to the classical gradient of the transformed function $F(u) = f(e^u)$ over the interval $[\alpha, \beta]$:

$$G_\alpha^\beta F = \frac{F(\beta) - F(\alpha)}{\beta - \alpha}.$$

Thus:

$$\underset{\sim}{G}_a^b f = G_\alpha^\beta F$$

This identity highlights that anageometric gradients are nothing more than classical gradients evaluated after mapping the domain through the logarithmic transformation.

Logarithmic Foundations of Anageometric Differentiation

Classical differentiation measures infinitesimal change relative to *additive* perturbations of the argument, that is, variations of the form $x \mapsto x + \Delta x$. In contrast, anageometric analysis is built upon the principle that meaningful change in the argument should be assessed multiplicatively. Thus, the fundamental perturbation is

$$x \mapsto x \cdot \Delta_G x, \qquad \Delta_G x \in \mathbb{R}_{\text{exp}},$$

and the appropriate quantitative measure of this perturbation is the induced logarithmic increment

$$\ln(x) - \ln(a) = \ln\left(\frac{x}{a}\right)$$

This multiplicative viewpoint leads naturally to a differential operator that behaves as the classical derivative *with respect to* the logarithmic coordinate $\ln(x)$. In this sense, the anageometric derivative captures the rate of change of a function under geometric (i.e., scale-based) displacements of its input.

Definition of the Anageometric Derivative

Let $f: (0, \infty) \to \mathbb{R}$ be defined on a positive interval containing the point $a > 0$. The anageometric derivative of f at a is defined by the limit

$$\underset{\sim}{D} f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{\ln(x) - \ln(a)}$$

provided the limit exists.

This definition mirrors the classical secant ratio

$$\frac{f(x) - f(a)}{x - a}$$

but replaces the additive increment with its logarithmic counterpart. When the above limit exists, we say that f *is anageometrically differentiable at* a.

Geometric Increment and the Anageometric Derivative

Let a very small geometric increment of x be denoted by $\Delta_G x \in \mathbb{R}_{\text{exp}}$ and $\Delta_G x \to 1$. In this case, $x \cdot \Delta_G x \to x$, and the anageometric derivative of a function $f: (0, \infty) \to \mathbb{R}$ at the point x can be written as

$$\underset{\sim}{D} f(x) = \lim_{\Delta_G x \to 1} \frac{f(x \cdot \Delta_G x) - f(x)}{\ln\,(x \cdot \Delta_G x) - \ln\,(x)}.$$

Using the logarithmic identity $\ln\,(x \cdot \Delta_G x) - \ln\,(x) = \ln\,(\Delta_G x)$, this expression reduces to

$$\underset{\sim}{D} f(x) = \lim_{\Delta_G x \to 1} \frac{f(x \cdot \Delta_G x) - f(x)}{\ln\,(\Delta_G x)}.$$

If the geometric increment at x is denoted by $K := \Delta_G x$, then the anageometric derivative takes the equivalent form

$$\underset{\sim}{D} f(x) = \lim_{K \to 1} \frac{f(Kx) - f(x)}{\ln(K)}$$

which measures the rate of change under geometric (scale-based) displacements.

The relation between geometric increment Δ_G and additive increment Δ is as follows

$$\Delta_G x = e^{\Delta(\ln x)}$$

Definition 5: The geometric differential is defined as follows

$$d_G x = e^{d(\ln x)} = e^{dx/x}.$$

By the definition of the geometric differential we have

$$\ln(d_G x) = d(\ln x).$$

Relation to the Classical Derivative

Using the definition of $\underset{\sim}{D} f(x)$, one obtains the fundamental relation between classical and anageometric integral as

$$\underset{\sim}{D} f(x) = \frac{df}{d(\ln x)} = x f'(x)$$

or

$$\underset{\sim}{D} f(x) = x D f(x)$$

since $\frac{d}{d(\ln x)} = x\frac{d}{dx}$. This shows that the anageometric derivative is equivalent to the classical derivative scaled by the argument.

Moreover we have

$$\underset{\sim}{D} f(x) = \frac{df}{\ln(d_G x)}$$

since $\ln(d_G x) = d(\ln x)$.

Linearity Properties of the Operator $\underset{\sim}{D}$

The anageometric derivative operator satisfies the two foundational linearity properties:

1. **Additivity:**

$$\underset{\sim}{D}(f+g)=\underset{\sim}{D}f+\underset{\sim}{D}g.$$

2. **Homogeneity:**

$$\underset{\sim}{D}(c\cdot f)=c\cdot\underset{\sim}{D}f,\quad c\in\mathbb{R}.$$

These properties mirror those of the classical derivative and ensure that the anageometric differential calculus forms a consistent linear theory.

Constantness of the Derivative and Uniformity

In direct analogy with classical calculus, the source establishes that:

- If f is anageometrically uniform, then $\underset{\sim}{D}f$ is constant.
- Conversely, if $\underset{\sim}{D}f$ is constant throughout $\mathbb{R}_{\text{exp}}$, then f must be anageometrically uniform.

Given the explicit form $f(x)=\ln(Cx^m)$, we find

$$\underset{\sim}{D}f(x)=m \text{ for all } x>0.$$

This result reinforces the classification of log affine functions as the "linear objects'' in the anageometric setting.

Example: The Function $h(x)=mx$

Let $h(x)=mx,\ \ x>0$. Its anageometric derivative satisfies the identity

$$\underset{\sim}{D}h=h.$$

Indeed,

$$\underset{\sim}{D}h(a)=a\,h'(a)=a\cdot m=ma=h(a).$$

This equality highlights the fundamentally different behavior of the anageometric derivative compared to the classical one: The multiplicative structure of the domain ensures that linear functions preserve their form under anageometric differentiation; this behavior is analogous to that of exponential functions in classical calculus.

Transition to Anageometric Integration

The anageometric derivative serves as the infinitesimal counterpart to the anageometric gradient introduced earlier. As the geometric increment $\ln(b) - \ln(a)$ shrinks to zero, the finite gradient converges to $\underset{\sim}{D}f(a)$. This convergence sets the stage for a parallel development of integration, where averaging over geometric partitions gives rise to the anageometric integral.

The next section will formalize this connection by introducing the anageometric average and establishing the structural foundations for the anageometric integral.

The Anageometric Average

Since anageometric calculus evaluates changes in arguments through *ratios* rather than additive differences, the natural discretization of an interval $[a, b] \subset (0, \infty)$ must respect this multiplicative structure. For this reason, the appropriate analogue of a classical partition is a geometric partition.

Definition 6 (Geometric partition):

A *geometric partition* of $[a, b]$ is any finite sequence

$$a = a_1 < a_2 < \cdots < a_n = b$$

such that the ratio

$$\frac{a_{k+1}}{a_k}$$

is constant for all $k = 1, \cdots, n-1$.
Equivalently, the points form a geometric progression. If the sequence contains n points, it is called an n-fold geometric partition. This structure is mandated by the multiplicative nature of anageometric change: equal "steps" in a geometric partition correspond to equal increments of $\ln(x)$. Thus geometric partitions are precisely those partitions that become *uniform* in classical sense when transferred to the logarithmic axis.

Definition of the Anageometric Average

Let $f: [a, b] \to \mathbb{R}$ be a continuous function. For a geometric partition

$$a_1, a_2, \dots, a_n,$$

consider the corresponding arithmetic mean of sampled values:

$$A_n(f) = \frac{1}{n}\big(f(a_1) + f(a_2) + \cdots + f(a_n)\big).$$

Because the partition points form a geometric progression, these sample points reflect equal spacing in logarithmic coordinates, making $A_n(f)$ the natural analogue of Riemann sums in classical calculus.

Definition 7 (Anageometric average):

The anageometric average of f over $[a, b]$ is defined by

$$\underset{\sim}{M}_a^b f = \lim_{n\to\infty} A_n(f),$$

provided the limit exists (which it does for all continuous f). Thus, $\underset{\sim}{M}_a^b f$ represents the limiting mean value of f sampled along increasingly refined geometric partitions of $[a, b]$.

Comparison with the Classical Arithmetic Average

It is important to emphasize that the anageometric average is not identical to the classical arithmetic average over an interval. The difference arises from the fact that geometric partitions weight the domain multiplicatively, introducing a distortion relative to the uniform linear discretization used in standard calculus.

Below is an example that illustrates this distinction:

Let $f(x) = x$. Then

$$\underset{\sim}{M}_a^b f = \frac{b - a}{\ln(b) - \ln(a)}$$

whereas the classical arithmetic mean over $[a, b]$ is

$$M_a^b f = \frac{a + b}{2}$$

The anageometric average privileges the behavior of f under geometric scaling, not under translation.

For a continuous function f, the anageometric average over $[a, b]$ satisfies:

$$\underset{\sim}{M}_a^b f = M_\alpha^\beta F$$

where $F(u) = f(e^u)$, and the right-hand side is the classical average of F over $[\alpha = \ln(a), \beta = \ln(b)]$.

That is:

$$M_\alpha^\beta F = \frac{1}{\beta - \alpha} \int_\alpha^\beta F(u)\, du$$

This identity reveals that anageometric averaging corresponds precisely to classical averaging under the logarithmic reparameterization.

It also reflects the fact that geometric partitions in $[a, b]$ correspond to uniform partitions in the logarithmic interval $[\alpha, \beta]$.

Linearity Properties of the Anageometric Average

The operator $\underset{\sim}{M}_a^b$ satisfies two essential structural properties analogous to those in classical analysis:

1. **Additivity:**

$$\underset{\sim}{M}_a^b(f+g) = \underset{\sim}{M}_a^b f + \underset{\sim}{M}_a^b g.$$

2. **Homogeneity:**

$$\underset{\sim}{M}_a^b(c \cdot f) = c \cdot \underset{\sim}{M}_a^b f \qquad (c \in R).$$

These properties follow directly from the additivity and homogeneity of arithmetic means at each partition level.

Anageometrically Uniform Functions and Their Averages

For an anageometrically uniform function

$$h(x) = \ln(Cx^m) = \ln C + m \cdot \ln x$$

the anageometric average admits a remarkable simplification:

- It equals the arithmetic mean of the endpoint values:

$$\underset{\sim}{M}_a^b h = \frac{h(a) + h(b)}{2}$$

- It also equals the value of h evaluated at the geometric mean of the endpoints:

$$\underset{\sim}{M}_a^b h = h\left(\sqrt{ab}\right)$$

This property reflects the log-affine structure of uniform functions, which behave linearly on the logarithmic scale. Thus, the anageometric average generalizes the midpoint rule of classical calculus, but in a multiplicative rather than additive framework.

Characterization Theorem for the Anageometric Average

The anageometric average is the *unique* operator that satisfies three fundamental properties:

1. **Normalization on constants:**
   For every constant $f(x) = c \in \mathbb{R}$,
   $$\underset{\sim}{M}_a^b f = c.$$
2. **Monotonicity:**
   If $f(x) < g(x)$ for all $x \in [a, b]$, then
   $$\underset{\sim}{M}_a^b f < \underset{\sim}{M}_a^b g.$$
3. **Logarithmic additivity across subintervals:**
   For any $a < s < b$,
   $$[\ln(s) - \ln(a)]\, \underset{\sim}{M}_a^s f + [\ln(b) - \ln(s)]\, \underset{\sim}{M}_s^b f = [\ln(b) - \ln(a)]\, \underset{\sim}{M}_a^b f$$

This final property mirrors the mean-value structure of classical integrals, but with logarithmic weights replacing linear lengths. It also foreshadows the construction of the anageometric integral, which appears naturally from this characterization.

Role of the Anageometric Average in the Calculus Framework

The anageometric average fits *naturally* into the structure of anageometric calculus because:

- It reflects the multiplicative geometry of the domain.
- It provides the correct limiting behavior needed for anageometric integration.
- It preserves the essential properties expected of a mean operator in a geometrically structured space.

In particular, the basic theorem of anageometric calculus (developed in the next section) hinges directly on the interplay between the anageometric average and the anageometric derivative.

The Basic Theorem of Anageometric Calculus

The anageometric average introduced in the previous section acquires its full significance through its relationship with the anageometric derivative. This relationship, parallels the classical connection between averages of derivatives and secant slopes.

Let $h: [a, b] \to \mathbb{R}$ be a function whose anageometric derivative $\underset{\sim}{D}h$ exists and is continuous on $[a, b]$. Then the theorem states:

$$\underset{\sim}{M}_a^b(\underset{\sim}{D}h) = \frac{h(b) - h(a)}{\ln(b) - \ln(a)}$$

That is the anageometric average of the anageometric derivative over $[a, b]$ equals the anageometric gradient of h over the same interval.

This result provides a multiplicative analogue of the classical fact that the average of h' over $[a, b]$ equals the slope of the secant line between $(a, h(a))$ and $(b, h(b))$.

Here, however, the denominator is the logarithmic increment $\ln(b) - \ln(a)$, which encodes geometric displacement rather than additive separation.

The theorem reveals several deep structural features:

(1) Consistency of the anageometric framework: It confirms that the integral-like quantity

$$[\ln(b) - \ln(a)]\, \underset{\sim}{M}_a^b(\underset{\sim}{D}h)$$

recovers the finite change $h(b) - h(a)$, which means that the anageometric derivative and average are perfectly compatible dual notions.

(2) Preparation for the Fundamental Theorems: This identity functions as a precursor to the fundamental theorems of anageometric calculus, just as the classical mean value relationships support the development of standard integral calculus.

(3) Justification for the integral definition: Most importantly, it *motivates the definition* of the anageometric integral, since one wishes for an operator that inverts differentiation in the sense captured by this theorem.

Thus, the basic theorem stands as the central conceptual bridge between differentiation and integration in the multiplicative regime.

The Anageometric Integral

Definition 8 (Anageometric Integral):

Let $f: [a, b] \to \mathbb{R}$ be continuous. The anageometric integral of f over $[a, b]$ is defined by:

$$\int_{\underset{\sim}{a}}^{b} f = \underset{\sim}{M}_a^b\{[\ln(b) - \ln(a)] \cdot f\}$$

The anageometric integral from a to a is set to be 0:

$$\int_{\underset{\sim}{a}}^{a} f = 0.$$

Thus, the anageometric integral is a *logarithmically weighted anageometric average*. This matches the structure derived from the basic theorem and ensures that integrating a derivative reproduces the correct secant change of the original function.

Interpretation via Geometric Riemann Sums

The anageometric integral can be equivalently formulated as the limit of a sequence of geometric Riemann sums.

Consider an n-fold geometric partition of $[a, b]$,

$$a_1 = a, \qquad a_2, \dots, a_n = b$$

with common ratio

$$k_n = \frac{a_{j+1}}{a_j} \quad \text{(independent of } j\text{)}$$

Then the corresponding sum is:

$$S_n(f) = (\ln k_n) f(a_1) + (\ln k_n) f(a_2) + \cdots + (\ln k_n)\, f(a_{n-1}).$$

The anageometric integral is

$$\int_{\underset{\sim}{a}}^{b} f = \lim_{n\to\infty} S_n(f)$$

This formulation parallels the classical Riemann sum construction, except that:

- the increments are *logarithmic* $(\ln k_n)$,
- the partition is geometric,
- and the integral sums over multiplicative displacements.

Such a structure naturally expresses integration with respect to $\ln(x)$, which is the Stieltjes measure underlying the anageometric theory.

Relationship to Stieltjes Integration

$\int_{\underset{\sim}{a}}^{b} f$ is the Stieltjes integral of f with respect to $\ln(x)$:

$$\int_{\underset{\sim}{a}}^{b} f = \int_{a}^{b} f(x) d(\ln x)$$

This clarifies that the underlying measure is

$$d(\ln x) = \frac{1}{x}\, dx$$

and hence the anageometric integral is essentially a *weighted* classical integral, but one whose weighting precisely encodes multiplicative scaling.

Using change of variables $(u = \ln x)$, the relation between anageometric and classical integral can be written as

$$\int_{\underset{\sim}{a}}^{b} f = \int_{\ln(a)}^{\ln(b)} f(e^u)du.$$

If we define $F(u) = f(e^u)$ and $\alpha = \ln(a), \beta = \ln(b)$ then we get

$$\int_{\underset{\sim}{a}}^{b} f = \int_{\alpha}^{\beta} F(u)du$$

Linearity and Additivity Properties

The anageometric integral satisfies the same algebraic laws as classical integration:

1. **Linearity:**

$$\int_{\underset{\sim}{a}}^{b} (c \cdot f) = c \cdot \int_{\underset{\sim}{a}}^{b} f, \qquad c \in \mathbb{R}$$

2. **Monotonicity:**
   If $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_{\underset{\sim}{a}}^{b} f \leq \int_{\underset{\sim}{a}}^{b} g$$

3. **Additivity over subintervals:**
   For any $a < s < b$,

$$\int_{\underset{\sim}{a}}^{b} f = \int_{\underset{\sim}{a}}^{s} f + \int_{\underset{\sim}{s}}^{b} f$$

These properties are identical to those in the classical theory but are derived from the geometric nature of the underlying partitions and the logarithmic weighting.

The Fundamental Theorems of Anageometric Calculus

The anageometric derivative and the anageometric integral, each developed as multiplicative analogues of their classical counterparts, are connected through two foundational results that mirror the fundamental theorems of classical calculus. The key difference is that these results operate on the logarithmic geometry of the positive real line.

The First Fundamental Theorem of Anageometric Calculus

Let $f: [a, b] \to \mathbb{R}$ be continuous, and define a function g on $[a, b]$ by

$$g(x) = \int_{\underset{\sim}{a}}^{x} f$$

Then:

$$\underset{\sim}{D} g(x) = f(x) \text{ for all } x \in [a, b].$$

This result asserts that anageometric integration is the inverse operation of anageometric differentiation.

This theorem closely parallels the classical statement that

$$\underset{\sim}{D}\left(\int_{\underset{\sim}{a}}^{x} f(t)dt\right) = f(x),$$

but with integration and differentiation performed with respect to the geometric measure $d(\ln x)$.

The Second Fundamental Theorem of Anageometric Calculus

Let $h: [a, b] \to \mathbb{R}$ be such that its anageometric derivative $\underset{\sim}{D}h$ exists and is continuous. Then:

$$\int_{\underset{\sim}{a}}^{b} \underset{\sim}{D}h = h(b) - h(a).$$

This identity is the multiplicative analogue of the classical fundamental theorem:

$$\int_{a}^{b} h'(x)\, dx = h(b) - h(a).$$

Here the integral is taken with respect to $d(\ln x)$, and the identity follows directly from the basic theorem of anageometric calculus:

$$\underset{\sim}{M}_a^b(\underset{\sim}{D}h) = \frac{h(b) - h(a)}{\ln(b) - \ln(a)}.$$

Multiplying both sides by $\ln(b) - \ln(a)$ yields exactly the desired expression.

3. An Anageometric Model

In this section, firstly we give Weber-Fechner law in classical analysis. Then we will present the Weber-Fechner law in anageometric analysis.

Weber–Fechner Law in Classical Analysis

Definition 9 (Physical stimulus intensity): A physical stimulus perceived by a sensory system is modeled as a positive intensity variable

$$I > 0.$$

Definition 10 (Just-Noticeable Difference – JND): The *just-noticeable difference* (JND) is defined as the smallest increment in stimulus intensity that an observer can reliably discriminate between two stimuli. The JND is an experimentally measured, discrete quantity (Ernst Heinrich Weber, 1834).

Definition 11 (Sensation magnitude): The sensation magnitude is defined as an abstract measurement variable depending on the physical stimulus intensity,

$$S = S(I).$$

This variable is not directly measurable; it is defined only through discrimination thresholds (Gustav Theodor Fechner, 1860).

Axiom 1 (Weber's Law- Empirical Axiom): For a given sensory modality, the ratio of the smallest discriminable stimulus increment to the stimulus intensity is constant:

$$\frac{\Delta I}{I} = k_W,$$

where $k_W > 0$ is the *Weber fraction*, an experimentally determined, modality-specific constant (Weber, 1834; John C. Baird & Eiichi Noma, 1978).
This law pertains solely to the physical stimulus space; the sensation variable S has not yet been introduced.

Axiom 2 below is about Fechner's Assumption (Measurement Axiom).

Axiom 2 (Equal JND = Equal Sensation Increment): Each JND corresponds to an equal increment on the sensation scale:

$$\Delta S = k_S,$$

where $k_S > 0$ is a scale constant depending on the chosen unit of sensation (Fechner, 1860).

By an appropriate choice of units, one may set $k_S = 1$; this is a normalization convention.

Assuming that JNDs are sufficiently small, the discrete structure is idealized as continuous:

$$\Delta I \to dI, \Delta S \to dS.$$

This transition is a standard idealization in continuous psychophysical modeling (Stanley Smith Stevens, 1957).

Theorem 1 (Weber–Fechner Differential Law): Under Axiom 1 and Axiom 2, the following differential relationship holds between sensation increment and stimulus increment:

$$dS = k_F \frac{dI}{I},$$

where

$$k_F := \frac{k_S}{k_W}.$$

Proof.
By Weber's law, we have

$$\Delta I = k_W I.$$

for one JND. And By Fechner's assumption, for the same JND, we have

$$\Delta S = k_S.$$

For sufficiently small JND, they can be related as

$$\frac{dS}{k_S} = \frac{dI/I}{k_W}.$$

Hence,

$$dS = \frac{k_S}{k_W}\frac{dI}{I} = k_F\frac{dI}{I}.$$

The proof is complete. □

Integrating the equation in Weber–Fechner differential law we get Fechner's formula.

Theorem 2 (Fechner's Law): The following relationship holds between sensation and stimulus:

$$S(I) = k_F \ln\ I + C.$$

Proof. Integrating the relation in Theorem 1 yields the relation in Theorem 2.

This result shows that sensation depends logarithmically on physical stimulus intensity and represents the classical form of the Weber–Fechner law (Fechner, 1860).

Weber-Fechner Law in Anageometric Analysis

Definition 12 (Geometric Change of Stimulus Intensity):

Let $I > 0$ denote a physical stimulus intensity. A geometric (scale-based) change of I is defined by $\Delta_G I$ such as

$$I \longmapsto I \cdot (\Delta_G I), \qquad \Delta_G I > 0, \qquad \Delta_G I \to 1$$

where $\Delta_G I$ is a dimensionless scaling factor. As the limit $\Delta_G I \to 1$ we get an infinitesimal geometric (multiplicative) increment which is denoted by $d_G I$.

For a geometric increment $I \mapsto I \cdot \Delta_G I$, the natural step length is $\ln(\Delta_G I)$.

Axiom 3 (Anageometric Weber's Law): For a given sensory modality, the smallest discriminable geometric increment of stimulus is an exponential positive constant:

$$\Delta_G I = k_W^{(G)}, \qquad k_W^{(G)} > 1 \text{ is constant.}$$

And $k_W^{(G)}$ is called geometric *Weber constant*.

As $\Delta_G I \to 1$ the discrete structure is idealized as continuous:

$$\Delta_G I \to d_G I, \qquad \Delta S \to dS.$$

Theorem 3 (Anageometric Weber–Fechner Differential Law): Under Axiom 3 and Axiom 2, the anagaeometric derivative of sensation is a constant:

$$\underset{\sim}{D} S(I) = k_F$$

where

$$k_F := \frac{k_S}{\ln\left(k_W^{(G)}\right)}$$

Proof.
By Fechner's assumption, we have

$$\Delta S = k_S.$$

And By Weber's law, we have

$$\ln(\Delta_G I) = \ln\left(k_W^{(G)}\right).$$

For $\Delta_G I \to 1$, they can be related as

$$\frac{dS}{\ln(d_G I)} = \frac{k_S}{\ln\left(k_W^{(G)}\right)}.$$

Hence,

$$\underset{\sim}{D} S(I) = k_F.$$

The proof is complete. □

Theorem 4 (Fechner's Law in Anageometric Analysis): The following relationships hold for sensation and stimulus:

$$S(I) = \int_{\sim} k_F$$

$$S(I) = k_F \ln(I) + C.$$

Proof. Taking the anageometric integral of both sides of the equation in anageometric Weber–Fechner differential law we write

$$\int_{\sim} \underset{\sim}{D} S(I) = \int_{\sim} k_F$$

which yields

$$S(I) = \int_{\sim} k_F$$

$$S(I) = k_F \ln(I) + C.$$

Conclusion

In this study, the fundamental structure of anageometric calculus is given. This structure forms a framework in which variation on the positive real axis is measured not through linear increments but through multiplicative (geometric) changes encoded by logarithmic differences. The essential idea behind anageometric analysis is that meaningful change on $(0, \infty)$ is naturally expressed through the quantity $\ln(b) - \ln(a)$ which replaces the classical linear displacement $b - a$. This shift establishes a calculus grounded in the geometry of the logarithmic axis.

We first formalized the notion of positive intervals and their geometric extent, defined by the ratio b/a whose logarithm provides the measure of distance in anageometric calculus. This reinterpretation shows that anageometric calculus is effectively classical calculus re-expressed under the transformation $x \to \ln(x)$, yet retaining a distinct geometric interpretation based on proportional change.

The anageometric derivative was then introduced as the limit

$$\underset{\sim}{D} f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{\ln x - \ln a}$$

which coincides with the classical derivative of f with respect to $\ln x$. Thus, the derivative captures the sensitivity of a function to infinitesimal multiplicative perturbations of its argument. This formulation also reveals that functions with constant anageometric derivatives are precisely the logarithmic affine functions, reinforcing their role as the "linear models'' of the multiplicative framework.

The development continued with the anageometric average, defined via geometric partitions of an interval. Unlike classical arithmetic means, this average samples the function with uniform spacing in the logarithmic coordinate. As a consequence, the anageometric integral

$$\int_{\underset{\sim}{a}}^{b} f = [\ln(b) - \ln(a)] \underset{\sim}{M}_a^b f$$

emerges as a logarithmically weighted mean and is exactly the Stieltjes integral with respect to $\ln x$. This establishes a rigorous connection between geometric scaling and integration.

Anageometric calculus is isomorphic to classical calculus under the logarithmic change of variables:

- Every theorem of classical calculus yields a corresponding theorem in the anageometric context via the substitution $x = e^u$.
- Conversely, every statement in anageometric calculus can be translated into the classical setting by expressing the function in logarithmic coordinates.

The fundamental theorems of anageometric calculus establish that differentiation and integration are mutually inverse operations within a multiplicative geometric setting, operating entirely through logarithmic increments.

The last part of the chapter is devoted to an application of anageometric analysis which is Weber-Fechner law in anageometric analysis. It is seen that Weber-Fechner differential law can be formulated simply such that the anageometric derivative of sensation is a constant in anageometric analysis. And Fechner law can be stated as the sensation is equal to anageometric integral of some constant.

References

Baird, J. C., & Noma, E. (1978). *Fundamentals of scaling and psychophysics*. Wiley.

Bashirov, A. E., Kurpınar, E. M., & Özyapıcı, A. (2008). Multiplicative calculus and its applications. *Journal of Mathematical Analysis and Applications*, 337(1), 36–48. https://doi.org/10.1016/j.jmaa.2007.03.081

Bashirov, A. E., Mısırlı, E., Tandoğdu, Y., & Özyapıcı, A. (2011). On modeling with multiplicative differential equations. *Applied Mathematics: A Journal of Chinese Universities*, 26(4), 425–438.

Boruah, K., & Hazarika, B. (2018a). G-calculus. *TWMS Journal of Applied and Engineering Mathematics*, 8(1), 94–105.

Boruah, K., & Hazarika, B. (2018b). Bigeometric integral calculus. *TWMS Journal of Applied and Engineering Mathematics*, 8(2), 374–385.

Boruah, K., Hazarika, B., & Bashirov, A. E. (2018). Solvability of bigeometric differential equations by numerical methods. *Boletim da Sociedade Paranaense de Matemática*, 39(2), 203–222.

Carroll, B. W., & Ostlie, D. A. (2017). *An introduction to modern astrophysics* (2nd ed.). Cambridge University Press.

Çakmak, A. F., & Başar, F. (2012). Some new results on sequence spaces with respect to non-Newtonian calculus. *Journal of Inequalities and Applications*, 2012, Article 123. https://doi.org/10.1186/1029-242X-2012-123

Çakmak, A. F., & Başar, F. (2014a). Certain spaces of functions over the field of non-Newtonian complex numbers. *Abstract and Applied Analysis*, 2014, Article 236124.

Çakmak, A. F., & Başar, F. (2014b). On line and double integrals in the non-Newtonian sense. *AIP Conference Proceedings*, 1611(1), 415–423. https://doi.org/10.1063/1.4893841

Dehaene, S. (2003). The neural basis of the Weber–Fechner law: a logarithmic mental number line. *Trends in cognitive sciences*, *7*(4), 145-147.

Fechner, G. T. (1860). *Elemente der Psychophysik*. Breitkopf & Härtel. (English translation: *Elements of psychophysics*, Vol. 1, 1966)

Florack, L., & van Assen, H. (2012). Multiplicative calculus in biomedical image analysis. *Journal of Mathematical Imaging and Vision*, 42(1), 64–75. https://doi.org/10.1007/s10851-011-0282-0

Grossman, M. (1979). An introduction to non-Newtonian calculus. *International Journal of Mathematical Education in Science and Technology*, 10(4), 525–528.

Grossman, M. (1983). *Bigeometric calculus: A system with a scale-free derivative*. Archimedes Foundation.

Grossman, M., & Katz, R. (1972). *Non-Newtonian calculus*. Lee Press.

Hearnshaw, J. B. (1996). *The measurement of starlight: Two centuries of astronomical photometry*. Cambridge University Press.

Johnson, H. L., & Morgan, W. W. (1953). Fundamental stellar photometry for standards of spectral type on the revised system of the Yerkes spectral atlas. *The Astrophysical Journal*, 117, 313–352. https://doi.org/10.1086/145697

Oke, J. B., & Gunn, J. E. (1983). Secondary standard stars for absolute spectrophotometry. *The Astrophysical Journal*, 266, 713–717. https://doi.org/10.1086/160817

Pogson, N. (1856). Magnitudes of thirty-six of the minor planets. *Monthly Notices of the Royal Astronomical Society*, 17, 12–15.

Portugal, R. D., & Svaiter, B. F. (2011). Weber–Fechner law and the optimality of the logarithmic scale. *Minds and Machines*, 21(1), 73–81. https://doi.org/10.1007/s11023-010-9223-8

Ryden, B., & Peterson, B. M. (2010). *Foundations of astrophysics*. Pearson Education.

Stevens, S. S. (1957). On the psychophysical law. *Psychological Review*, 64(3), 153–181. https://doi.org/10.1037/h0046162

Weber, E. H. (1834). *De pulsu, resorptione, auditu et tactu*. Koehler.

CHAPTER 5

CANAL SURFACES IN ANTI-DE SITTER 4-SPACE: A DIFFERENTIAL GEOMETRIC APPROACH

1. FATMA ALMAZ[1]

1. Introduction

The Anti-de Sitter (AdS) space, which holds a significant place in differential geometry and theoretical physics, is defined as a Lorentzian manifold with a constant negative Riemannian curvature. This distinguishes it from the positive-curvature de Sitter space and the zero-curvature Minkowski space. In the context of general relativity, it represents vacuum solutions to Einstein's equations with a negative cosmological constant.

The importance of the AdS space stems particularly from its deep connections in theoretical physics. One of its best-known applications is the groundbreaking AdS/CFT (Anti-de Sitter/Conformal Field Theory) equivalence principle in string theory and quantum gravity. This equivalence establishes a strong link between a theory of gravity defined in an AdS space and a conformal field theory defined on one of its boundaries, allowing for the investigation of challenging quantum gravity problems through more understandable boundary theories. It is also used as a

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fundamental model space in fields such as supersymmetric field theories, black hole thermodynamics, and cosmology.

In particular, the 4-dimensional Anti-de Sitter space (AdS_3) can be directly related to certain models of physical spacetime. This means that many models in theoretical physics, especially a 4-dimensional conformal field theory, can be formulated as a theory of gravity in the 4-dimensional AdS_3 space. In this context, the geometric and topological properties of AdS_3 play a critical role in better understanding quantum gravity models and spacetime itself.

Given the AdS/CFT equivalence principle, the properties of submanifolds (including canal surfaces) in the AdS space can be vital for understanding the corresponding structures in limit theory. For example, these surfaces can be interpreted as 'branes' in the AdS space and play a significant role in string theory or cosmological models. Geodesics or minimal surface properties of channel surfaces can be used to model specific physical processes within gravity theory. Generalizing canal surfaces to AdS_3 expands the scope of current canal surface theory. This opens up new research on how canal surfaces can be classified for different centroids (time-like, space-like, null) and radius functions.

In [1, 2, 3], tube surfaces, another form of channel surface in different spatial forms, have been considered. In [4], the authors investigates pseudo-Riemannian manifolds that share a common family of geodesics and give some characterizations of the geometric properties and structures of such manifolds, exploring the conditions under which two or more distinct pseudo-Riemannian metrics can induce the same set of unparameterized geodesics. In [6, 7], These references focuses on the analysis of geodesics on various surfaces embedded in Minkowski 3-space, a fundamental setting in differential geometry with applications in relativistic physics. Specifically, one line of inquiry investigates surfaces that share common geodesics, aiming to characterize the geometric conditions and properties under which distinct surfaces can possess the same geodesic paths. Concurrently, the studies also delve into the geodesics of tubular surfaces within this Minkowski geometry, exploring how the unique structure of these surfaces influences their geodesic behavior. By employing techniques from differential geometry, these research efforts contribute to a deeper understanding of surface theory in pseudo-Euclidean spaces, offering new insights into the classification, characterization, and kinematic properties of surfaces based on their geodesic structures in Minkowski 3-space.

2. Preliminaries

$4-$dimensional pseudo-Euclidean space with signature $(2,4)$ provided with an indefinite flat metric given by

$$\langle , \rangle = -(d\lambda_1)^2 - (d\lambda_2)^2 + (d\lambda_3)^2(d\lambda_4)^2,$$

where $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is a standart rectangular coordinate system in pseudo-Euclidean 4-space.

Recall that an arbitrary vector $v \in E_2^4 \backslash \{0\}$ can have one of three characters: it can be spacelike if $g(v, v) > 0$ or $v = 0$, timelike if $g(v, v) < 0$ and null if $g(v, v) = 0$ and $v \neq 0$.

The norm of a vector v is given by $\parallel v \parallel = \sqrt{g(v, v)}$ and two vectors v and w are said to be ortogonal if $g(v, w) = 0$.

The pseudo-hyperbolic space $H_1^3(x_0, r)$ centered at $x_0 \in E_2^4$, with radius $r > 0$ of E_2^4 is defined by

$$H_1^3(x_0, r) = \{x \in E_2^4 : \langle x - x_0, x - x_0 \rangle = -r^2\}.$$

The pseudo-hyperbolic space $H_1^3(x_0, r)$ is diffeomorfic to $S^1 \times \mathbb{R}^2$. The hyperbolic space $H^3(x_0, r)$ is defined by

$$H^3(x_0, r) = \{x \in E_2^4 : \langle x - x_0, x - x_0 \rangle = -r^2, x_1 > 0\},$$

[5, 8, 9, 10, 11].

The 3-dimensional Anti-de Sitter space is a Lorentz manifold with constant negative sectional curvature. It is often described as a hypersurface in a 4-dimensional Minkowski space (E_2^4 or E_1^3).

A one-parameter anti-de Sitter space is given by the following equation

$$H_1^3(-\sinh^2\theta) = \{x \in E_2^4 : g(x, x) = -\sinh^2\theta.$$

Furthermore, let $\alpha : I \to H_1^3$ be a spacelike curve such that

$\langle\alpha'(t),\alpha'(t)\rangle > 0$ holds. Therefore, since the curve is spacelike, it can be parameterized at unit speed.

Furthermore, with $\langle t'(s),t'(s)\rangle \neq -1$, the unit vector $n(s) = \frac{t'(s)-\alpha(s)}{\|t'(s)-\alpha(s)\|}$ and the vector $e(s) = \alpha(s) \wedge t(s) \wedge n(s)$ are defined. Then, where $k_g(s) = \|t'(s) - \alpha(s)\|$ is the geodesic curvature, $\tau_g(s) = -k_g(s)^{-2}\det(\alpha(s),\alpha(s)',\alpha(s)'',\alpha(s)''')$, and $\delta = sign(n(s))$. Let $\{\alpha(s),t(s),n(s),e(s)\}$ be the non-null moving Frenet frame along a unit speed non-null curve α in ADS_3, consisting of the tangent, principal normal, first binormal and second binormal vector field, respectively. If α is a non-null curve with non-null vector fields, then $\{\alpha(s),t(s),n(s),e(s)\}$ is a pseudo-orthonormal frame and the Frenet equations gives

$$\begin{aligned} \alpha'(s) &= t(s) \\ t'(s) &= \alpha(s) + k_g(s)n(s) \\ n'(s) &= -\delta k_g(s)t(s) + \delta\tau_g(s)e(s) \\ e'(s) &= \delta\tau_g(s)n(s), \end{aligned} \tag{2.1}$$

[5, 8, 9, 10, 11]. If $\langle t'(s),t'(s)\rangle = -1$, then $k_g(s) = 0$ can be found. In this case, it can be said that the curve $\alpha(s)$ given in H_1^3 is a geodesic curve.

3. Characterization of canal surfaces created according to the geodesic frame in the anti-de Sitter 4-space

In this section, the canal surfaces generated by arbitrary curve are investigated according to mathematical approach. A canal surface is expressed as the envelope of a setting out sphere with exchanging radius, which is described by the orbit $\alpha(w(s))$ (spine curve) with its center and a radius function ρ in addition to its parametrized through Frenet frame of the spine curve $\alpha(w(s))$. If the radius function ρ is a constant, then the canal surface is called as a tube or tubular surface.

Let Υ be a canal surface in H_1^3. The curvature of the curve α is non-zero, and using the frame $\{\alpha(s),t(s),n(s),e(s)\}$ where $\Omega^1,\Omega^2,\Omega^3,\Omega^4 \in C^\infty$ are defined in the interval where the curve α is defined, the following equation can be written as

$$\Upsilon(s,\xi)-\alpha(s)=\Omega^1(s,\xi)\overleftarrow{\alpha}+\Omega^2(s,\xi)\overleftarrow{t}$$
$$+\Omega^3(s,\xi)\overleftarrow{n}+\Omega^4(s,\xi)\overleftarrow{e}. \quad (3.1)$$

Furthermore, for the vector $x(s)=\big(x_1(s),x_2(s),x_3(s),x_4(s)\big)$ where $x_i\in C^\infty, i\in\{1,2,3,4\}$, the following equation is written

$$\langle x(s),x(s)\rangle_{H_1^3}=x_1^2(s)+x_2^2(s)-x_3^2(s)-x_4^2(s). \quad (3.2)$$

Thus, the vector $\Upsilon(s,\xi)-\alpha(s)=(\Omega^1(s,\xi),\Omega^2(s,\xi),\Omega^3(s,\xi),\Omega^4(s,\xi))$ given in (3.1) is also considered in (3.2)

$$\Upsilon(s,\xi)-\alpha(s)=\big(\Omega^1(s,\xi)\big)^2+\big(\Omega^2(s,\xi)\big)^2-\ (\Omega^3(s,\xi))^2-(\Omega^4(s,\xi))^2 \quad (3.3)$$

Furthermore, from expression (3.3), we can say that the surface $\Upsilon(s,\xi)$ lies on the sphere with center $\alpha(s)$ and radius $d(s)$. Thus, the mathematical equations between the vector $\Upsilon(s,\xi)-\alpha(s)$, which is normal to the canal surface in ADS_3, and the vectors Υ_s and Υ_ξ, which are tangent to the sphere on which the surface lies, are given as

$$\langle\Upsilon(s,\xi)-\alpha(s),\Upsilon_s\rangle=0;\big\langle\Upsilon(s,\xi)-\alpha(s),\Upsilon_\xi\big\rangle=0.$$

In this case, let's examine the situations expressed by (3.3). First, using the metric given in expression (3.3), we get

$$(\Omega^1)^2+(\Omega^2)^2-(\Omega^3)^2-(\Omega^4)^2=d^2.$$

If this last expression is also derived with respect to the parameter s, we have

$$\Omega_s^1\Omega^1+\Omega_s^2\Omega^2-\Omega_s^3\Omega^2-\Omega_s^4\Omega^4=dd_s. \quad (3.4)$$

In this case, if the Frenet framework is used by taking the differential with respect to ξ in equation (3.1), we get

$$\Upsilon_\xi = \Omega^1_\xi(s,\xi)\overleftarrow{T} + \Omega^2_\xi(s,\xi)\overleftarrow{N} + \Omega^3_\xi(s,\xi)\overleftarrow{B_1} + \Omega^4_\xi(s,\xi)\overleftarrow{B_2}. \quad (3.5)$$

If this last equality and (3.1) is used in the equality $\langle \Upsilon(s,\xi) - \alpha(s), \Upsilon_\xi \rangle = 0$, we have

$$\Omega^1\Omega^1_\xi + \Omega^2\Omega^2_\xi - \Omega^3\Omega^3_\xi - \Omega^4\Omega^4_\xi = 0. \quad (3.6)$$

Furthermore, let's try to express the channel surface given in ADS_3 in a different way by finding the values of Ω^2, Ω^3 from equation (3.1). Thus, by finding the value of Υ_s and using the frame $\{\alpha(s), t(s), n(s), e(s)\}$,we have

$$\Upsilon(s,\xi) - \alpha(s) = \Omega^1(s,\xi)\overleftarrow{\alpha} + \Omega^2(s,\xi)\overleftarrow{t} + \Omega^3(s,\xi)\overleftarrow{n} + \Omega^4(s,\xi)\overleftarrow{e}$$

$$\Upsilon_s(s,\xi) - \alpha'(s) = \Omega^1_s\overleftarrow{\alpha} + \Omega^1\overleftarrow{\alpha'} + \Omega^2_s\overleftarrow{t} + \Omega^2\overleftarrow{t'} + \Omega^3_s\overleftarrow{n} + \Omega^3\overleftarrow{n'}$$
$$+\Omega^4_s\overleftarrow{e} + \Omega^4\overleftarrow{e'}$$

$$= \Omega^1_s\overleftarrow{\alpha} + \Omega^1\overleftarrow{t} + \Omega^2_s\overleftarrow{t} + \Omega^2\left(\overleftarrow{\alpha} + k_g\overleftarrow{n}\right) + \Omega^3_s\overleftarrow{n}$$
$$+\Omega^3\left(-\delta k_g\overleftarrow{t} + \delta\tau_g\overleftarrow{e}\right) + \Omega^4_s\overleftarrow{e} + \Omega^4\left(\delta\tau_g\overleftarrow{n}\right)$$

$$\Upsilon_s(s,\xi) = \overleftarrow{\alpha}(\Omega^1_s + \Omega^2) + \overleftarrow{t}\left(\Omega^1 + \Omega^2_s - \delta k_g\Omega^3 + 1\right)$$

$$+\overleftarrow{n}\left(k_g\Omega^2 + \Omega^3_s + \delta\tau_g\Omega^4\right) + \overleftarrow{e}\left(\delta\tau_g\Omega^3 + \Omega^4_s\right). \quad (3.7)$$

Thus, using the equality $\langle \Upsilon(s,\xi) - \alpha(s), \Upsilon_s \rangle = 0$, we have

$$0 = \Omega^1(\Omega^1_s + \Omega^2) + \Omega^2\left(\Omega^1 + \Omega^2_s - \delta k_g + 1\right)$$

$$-\Omega^3\left(k_g\Omega^2 + \Omega^3_s + \delta\tau_g\Omega^4\right) - \Omega^4\left(\delta\tau_g\Omega^3 + \Omega^4_s\right). \quad (3.8)$$

This can be written algebraically as follows

$$\Omega_s^1 + \Omega^2 = 0$$
$$\Omega^1 + \Omega_s^2 - \delta k_g \Omega^3 + 1 = 0$$
$$k_g \Omega^2 + \Omega_s^3 + \delta \tau_g \Omega^4 = 0$$
$$\delta \tau_g \Omega^3 + \Omega_s^4 = 0.$$

Also, using equations (3.8) and (3.4), we write

$$0 = \Omega^1\Omega_s^1 + \Omega^1\Omega^2 + \Omega^2\Omega^1 + \Omega^2\Omega_s^2 - \Omega^2\delta k_g + \Omega^2$$
$$-\Omega^3 k_g \Omega^2 - \Omega^3\Omega_s^3 - \delta\tau_g\Omega^4\Omega^3 - \Omega^4\delta\tau_g\Omega^3 - \Omega^4\Omega_s^4$$

$$0 = \Omega^1\Omega_s^1 + \Omega^2\Omega_s^2 - \Omega^3\Omega_s^3 - \Omega^4\Omega_s^4 + 2\Omega^1\Omega^2 - \Omega^2\delta k_g$$
$$+\Omega^2 - \Omega^3 k_g\Omega^2 - \delta\tau_g\Omega^4\Omega^3 - \Omega^4\delta\tau_g\Omega^3$$

$$0 = dd_s + \Omega^2\left(2\Omega^1 - \delta k_g + 1 - \Omega^3 k_g\right) - \delta\tau_g\Omega^4\Omega^3$$

$$dd_s = \delta\tau_g\Omega^4\Omega^3 - \Omega^2\left(2\Omega^1 - \delta k_g + 1 - \Omega^3 k_g\right).$$

Furthermore, if d =constant, then $k_g(s) = \|t'(s) - \alpha(s)\|$ is the geodesic curvature and $\tau_g = -{k_g}^{-2}\det(\alpha(s), \alpha(s)', \alpha(s)'', \alpha(s)''')$ is the geodesic torsion, we can write

$$\delta\tau_g\Omega^4\Omega^3 = \Omega^2\left(2\Omega^1 - k_g(\delta + \Omega^3) + 1\right);\ \delta = \mathrm{sign}(n(s))$$

$$k_g = \frac{\Omega^2(2\Omega^1 + 1) - \delta\tau_g\Omega^4\Omega^3}{\Omega^2(\delta + \Omega^3)},$$

$$k_g = \frac{\cosh\xi(2d\cosh\xi\cos b + 1) - \delta\tau_g d\sinh^2\xi\cos b}{\cosh\xi(\delta + d\sinh\xi\cos b)},$$

$$\tau_g = \frac{\Omega^2\left(2\Omega^1 - k_g(\delta + \Omega^3) + 1\right)}{\delta\Omega^4\Omega^3},$$

$$\tau_g = \frac{\cosh\xi\left(2d\cosh\xi\cos b - k_g(\delta + d\sinh\xi\cos b) + 1\right)}{\delta d\sinh^2\xi\cos b},$$

$$\det(\alpha(s), \alpha(s)', \alpha(s)'', \alpha(s)''') = -\frac{\Omega^2 {k_g}^2 \left(2\Omega^1 + 1 - k_g(\delta + \Omega^3)\right)}{\delta\Omega^4\Omega^3}$$

Also,

$$d^2 = (\Omega^1)^2 + (\Omega^2)^2 - (\Omega^3)^2 - (\Omega^4)^2$$

$$d^2 = (d\cosh\xi\cos b)^2 + (d\cosh\xi\sin b)^2 - (d\sinh\xi\cos b)^2 - (d\sinh\xi\sin b)^2.$$

This last statement can be written in the form of the following equation

$$\Omega^2 = d\cosh\xi\sin b; \Omega^3 = d\sinh\xi\cos b; \Omega^4 = d\sinh\xi\sin b$$

$$\Omega^1 = d\cosh\xi\cos b.$$

When we substitute the last given values (3.1), we create the surface as follows

$$\Upsilon(s,\xi) - \alpha(s) = \Omega^1(s,\xi)\overleftarrow{\alpha} + \Omega^2(s,\xi)\overleftarrow{t} + \Omega^3(s,\xi)\overleftarrow{n} + \Omega^4(s,\xi)\overleftarrow{e}$$

$$\Upsilon(s,\xi) = \alpha(s) + d\begin{pmatrix}\cosh\xi\cos b\overleftarrow{\alpha} + \cosh\xi\sin b\overleftarrow{t} \\ +\sinh\xi\cos b\overleftarrow{n} + \sinh\xi\sin b\overleftarrow{e}\end{pmatrix}; d, b \in \mathbb{R}.$$

Based on the information we have presented above, we can write the following theorem.

Theorem Let the center curve of the canal surface in H_1^3 be a unit speed curve $\alpha: I \to H_1^3$ with the geodesic curvature $k_g(s)$ and geodesic torsional curvature $\tau_g(s)$ in a one-parameter anti-de Sitter space. Then, the canal surface can be parametrized as follows

$$\Upsilon(s,\xi) = \alpha(s) + d\begin{pmatrix}\cosh\xi\cos b\overleftarrow{\alpha} + \cosh\xi\sin b\overleftarrow{t} \\ +\sinh\xi\cos b\overleftarrow{n} + \sinh\xi\sin b\overleftarrow{e}\end{pmatrix}; d, b \in \mathbb{R} \quad (3.9)$$

and the curvatures of the canal surface are given

$$k_g = \frac{\cosh\xi(2d\cosh\xi\cos b+1)-\delta\tau_g d\sinh^2\xi\cos b}{\cosh\xi(\delta+d\sinh\xi\cos b)}, \quad (3.10)$$

$$\tau_g = \frac{\cosh\xi\left(2d\cosh\xi\cos b-k_g(\delta+d\sinh\xi\cos b)+1\right)}{\delta d\sinh^2\xi\cos b}. \quad (3.11)$$

Example *Let the center curve of the canal surface in H_1^3 be a unit speed curve $\alpha(s) = (3coss, 3sins, 5s, 0)$ with the geodesic curvature k_g and geodesic torsional curvature τ_g in a one-parameter anti-de Sitter space. Then, the canal surface can be parametrized as follows*

$$\Upsilon(s,\xi) = \begin{pmatrix} 3\cos s + d\cosh\xi\cos b, \\ 3\sin s + d\cosh\xi\sin b, \\ 5s + d\sinh\xi\cos b, \\ d\sinh\xi\sin b \end{pmatrix}; d, b \in \mathbb{R}$$

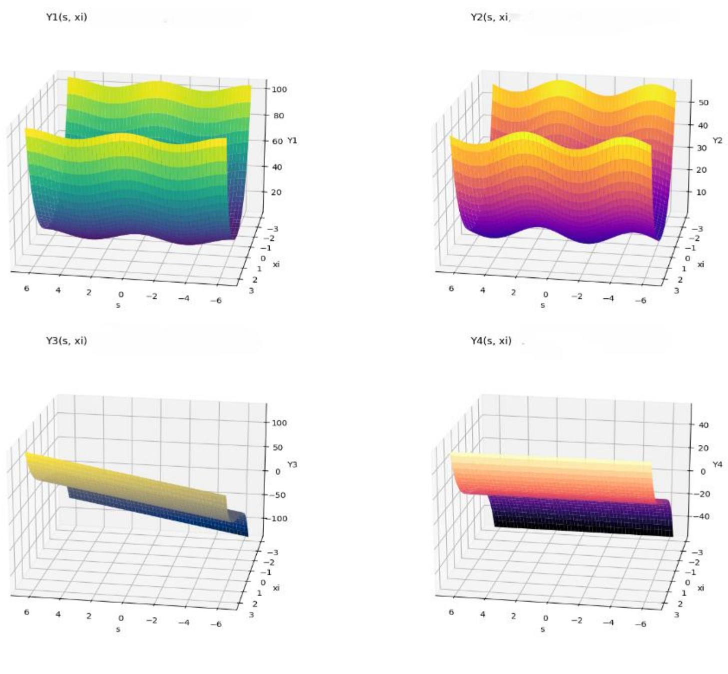

Figure 1: Component graphs of the canal surface.

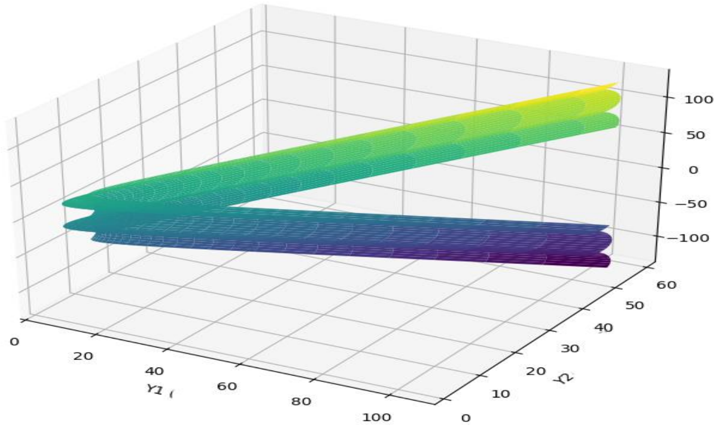

Figure 2: 3D projection graph of the canal surface over any three components

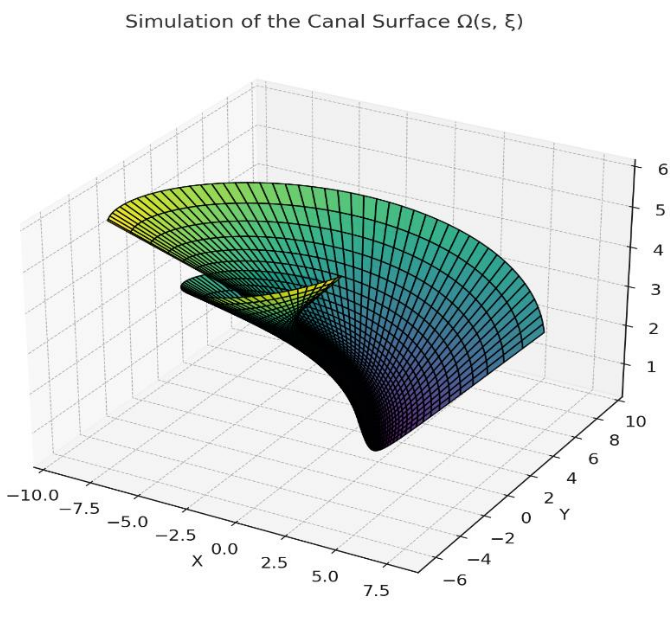

Figure 3: Canal surface in Antide sitter space generated by helix curve α.

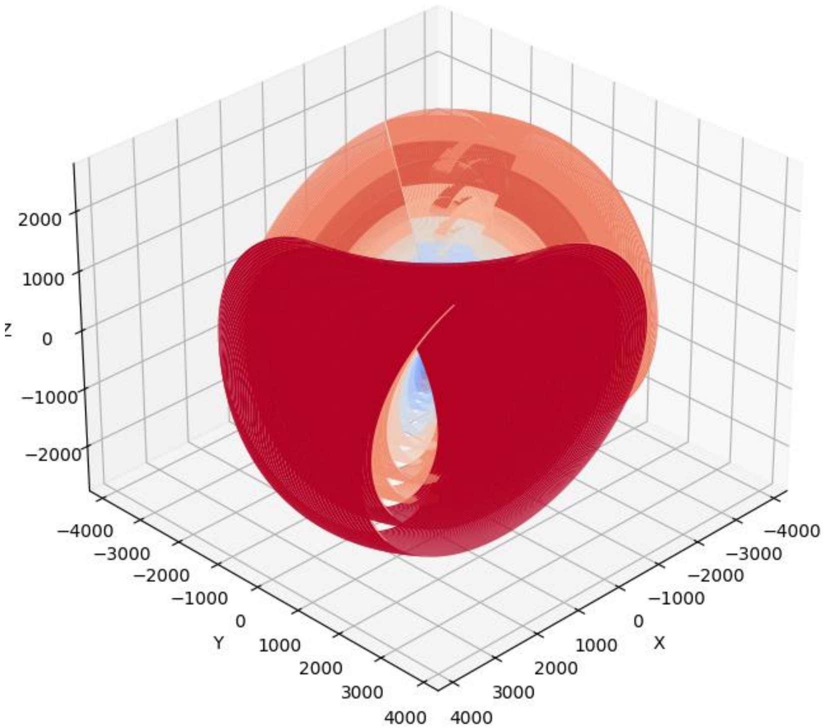

Figure 4: Rotational canal surface in Antide sitter space generated by the arbitrary curve α.

4. Conclusion

This study presents a detailed investigation of canal surfaces within the unique geometric structure of the 4-dimensional Anti-de Sitter 4-space. The findings reveal the definitions, parameterizations, and differential geometric properties of these surfaces. It has been shown that the constant negative curvature and Lorentzian characteristic of AdS_3 lead to canal surfaces exhibiting different behaviours than their counterparts in classical Euclidean space. These analyses have enabled new classifications of canal surfaces and will make significant contributions to the field of pseudo-Riemannian geometry, particularly to the development of submanifold theory in spaces with high-dimensional and special geometries. Furthermore, it is thought that this will provide a foundation for potential applications of such surfaces in theoretical physics, especially in the context of string theory and the AdS_3 equivalence principle. In our future studies, we will attempt to express some physical concepts using this surface form.

Acknowledgements

The authors wish to express their thanks to the authors of literatures for the supplied scientific aspects and idea for this study. Furthermore, we request to explain great thanks to reviewers for the constructive comments and inputs given to improve the quality of our work.

References

[1] Almaz F, Kulahci MA, A mathematical interpretation on special tube surfaces in Galilean 3-space, Hacettepe J. Mathematics and Statistics 2022: 51(3), 632-645.

[2] Almaz F, Kulahci M, A survey on tube surfaces in Galilean 3-space, Politeknik Dergisi 2020: 25(3), 1133-1142.

[3] Almaz F, Kulahci MA, The notes on the special tube surfaces generated by normal curves in E_2^4: Bol. Soc. Paran. Mat. 2025: 43, 1-20.

[4] Aminova AV, Pseudo-Riemannian manifolds with common geodesics, Uspekhi Mat. Nauk. 1993: 48: 107-164.

[5] O'Neill B, Semi-Riemannian Geometry with Applications to Relativity, Academic Press Inc., London 1983.

[6] Kasap E, Akyildiz FT, Surfaces with a Common Geodesic in Minkowski 3 −space, App. Math. Comp. 2006: 177(1): 260-270.

[7] Karacan MK, Yayli Y, On the geodesics of tubular surfaces in Minkowski 3space, Bull. Malays. Math. Sci. Soc. 2008: 31(1): 1-10.

[8] Kuhnel W, Differential Geometry Curves- Surfaces and Manifolds, Second Edition, Providence, RI, United States, American Math. Soc., 2006: 16, ISBN: 0-8218-3988-8.

[9] Pressley A, Elementary Differential Geometry, second edition. London, UK. Springer-Verlag London Limited, 2010.

[10] Röschel O, Die Geometrie des Galileischen Raumes, Forschungszentrum Graz Research Centre, Austria, 1986.

[11] Röschel O, Die Geometrie des Galileischen Raumes, Bericht der Mathematisch Statistischen Sektion in der Forschungs-Gesellschaft Joanneum, Bericht Nr. 256, Habilitationsschrift, Leoben, 1984.

CHAPTER 6

Quasi Hemi Slant Submanifolds of Generalized Kenmotsu Manifold

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1. Introduction

Globally framed metric f-manifolds, which are a generalization of almost contact manifolds, were first introduced by H. Nakagawa (Nakagawa, 1966) and developed by S. I. Goldberg and K. Yano in 1971 (Goldberg and Yano, 1971). In 1972 Vanzura (Vanzura, 1972) defined almost s-contact structures on f-manifolds. Vanlı and Sarı generalized Kenmotsu manifolds to almost s-contact structures and defined generalized Kenmotsu manifolds (Turgut Vanlı and Sarı, 2017). Vanlı and Sarı also showed that the generalized Kenmotsu manifold can be written as a warped product of the Kaehler manifold with $\mathbb{R}^s$. Moreover They studied invariant submanifolds of this manifold (Turgut Vanlı and Sarı, 2023).

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The geometry of slant submanifolds, a generalisation of invariant and anti-invariant submanifolds, has been studied since 1990 and continues to be studied. The subject of slant submanifolds of Hermitian manifolds was introduced by B. Y. Chen (Chen, 1990). Firstly, Lotta defined slant submanifolds of almost contact manifold (Lotta, 196). After, Cabrerizo et all. studied slant submanifolds of Sasakian manifold (Cabrerizzo, 2000). Atçeken and Dirik studied pseudo slant submanifolds of Kenmotsu manifold. Many authors investigation on submanifolds (Atçeken and Dirik, 2014). Sarı et all. Investigated skew semi-invariant submanifolds of Kenmotsu manifold (Sarı,Ünal and Aksoy Sarı, 2018).

Quasi hemi slant submanifolds were studied by Prasad in 2020 as a generalisation semi-invariant submanifolds, semi-slant submanifolds and pseudo-slant submanifolds (Prasad, 2020). In this book chapter, we study quasi hemi slant submanifolds of generalized Kenmotsu manifold.

2. Generalized Kenmotsu Manifolds

Let $\bar{B}$ be (2n+s)-dimensional differentiable manifold, φ is tensor field, $\{\xi_1, \dots, \xi_s\}$ are vector fields and $\{\eta^1, \dots, \eta^s\}$ are 1-forms. Then $\bar{B}$ is said to be almost s-contact metric manifold by

$$\varphi^2 = -I + \sum_{i=1}^{s} \eta^i \otimes \xi_i , \; \eta^i(\xi_j) = \delta_{ij} \qquad (1)$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y). \qquad (2)$$

Therefore, Φ is said to be the fundamental 2-form, $\Phi(X, Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(T\bar{B})$. Moreover, an almost contact metric manifold is normal if

$$[\varphi, \varphi] + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i = 0.$$

Theorem 1. Let $(\bar{B}, \varphi, \xi_i, \eta^i, g)$ be a normal almost s-contact metric manifold. Then $\bar{B}$ is generalized Kenmotsu manifold if and only if

$$(\overline{\nabla}_X \varphi)Y = \sum_{i=1}^{s} \{g(\varphi X, Y)\xi_i + \eta^i(Y)\varphi X\}. \tag{3}$$

Corollary 1. Let $(\bar{B}, \varphi, \xi_i, \eta^i, g)$ be a generalized Kenmotsu manifold. Then we have

$$\overline{\nabla}_X \xi_i = -\varphi^2 X. \tag{4}$$

3. Quasi Hemi Slant Submanifolds of Generalized Kenmotsu Manifolds

In this section, we define and study quasi hemi slant submanifolds of generalized Kenmotsu manifold. We investigate geometric properties of distributions.

Definition 1. Let B be submanifold of generalized Kenmotsu manifold $\bar{B}$. B is said to be quasi hemi-slant submanifold if

- $TB = D \oplus D^{\perp} \oplus D^{\theta} \oplus Sp\{\xi_1, \dots, \xi_s\}$,
- $\varphi D = D$,
- $\varphi D^{\perp} \subset TB^{\perp}$
- The angle θ betwen φX and the space D^{θ} is constant for $X \in \Gamma(D^{\theta})$, where $\{D, D^{\perp}, D^{\theta}\}$ is orthogonal distribution and ξ_i are tangent to $\bar{B}$.

Example 1. $(\mathbb{R}^{2n+s}, \varphi, \eta^i, \xi_i, g)$ will denote the manifold $\mathbb{R}^{2n+s}$ with its usual generalized Kenmotsu structure given by

$$\eta^i = \frac{1}{2}\left(dz_i - \sum_{i=1}^{n} y_i dx_i\right), \xi_i = 2\frac{\partial}{\partial z_i}$$

$$\varphi\left(\sum_{i=1}^{n}(X_i\frac{\partial}{\partial x_i}+Y_i\frac{\partial}{\partial y_i}\right)+\sum_{j=1}^{s}Z_j\frac{\partial}{\partial Z_j})=\sum_{i=1}^{n}(Y_i\frac{\partial}{\partial x_i}-X_i\frac{\partial}{\partial y_i})+\sum_{j=1}^{s}\sum_{i=1}^{n}Y_i y_i\frac{\partial}{\partial Z_j}$$

$$g=e^{2\sum_{j=1}^{s}z_j}(\sum_{i=1}^{n}(dx_i\otimes dx_i+dy_i\otimes dy_i)+\sum_{j=1}^{s}\eta^j\otimes\eta^j,$$

where $(x_1,\dots,x_n,y_1,\dots,y_n,z_1,\dots,z_s)$ denoting the Cartesian coordinates on R^{2n+s}. Let N be submanifold of $\mathbb{R}^{10}$ defined by

$$N=X(s,t,u,v,k,w,z_1,z_2)$$
$$=e^{-2\sum_{j=1}^{2}z_j}(s,0,u,t,v,k,cos\,w,sin\,w,z_1,z_2)$$

Then local frame of TN

$$E_1=e^{-2\sum_{j=1}^{2}z_j}\frac{\partial}{\partial x_1},\quad E_2=e^{-2\sum_{j=1}^{2}z_j}\frac{\partial}{\partial y_1},$$

$$E_3=e^{-2\sum_{j=1}^{2}z_j}\frac{\partial}{\partial x_3},\quad E_4=e^{-2\sum_{j=1}^{2}z_j}\frac{\partial}{\partial y_2},$$

$$E_5=e^{-2\sum_{j=1}^{2}z_j}\frac{\partial}{\partial x_4},\quad E_6=e^{-2\sum_{j=1}^{2}z_j}\left(sin\,w\frac{\partial}{\partial y_3}+cos\,w\frac{\partial}{\partial y_4}\right),$$

$$E_7=e^{-2\sum_{j=1}^{2}z_j}\frac{\partial}{\partial z_1},\quad E_8=e^{-2\sum_{j=1}^{2}z_j}\frac{\partial}{\partial z_2}$$

and

$$E_1{}^*=e^{-2\sum_{j=1}^{2}z_j}\frac{\partial}{\partial x_2},\quad E_2{}^*=e^{-2\sum_{j=1}^{2}z_j}\frac{\partial}{\partial y_3}$$

from a basis of $TN^\perp$. We determine $D_1=sp\{E_1,E_2\}$, $D_2=sp\{E_3,E_4\}$ and $D_3=sp\{E_5,E_6\}$, then D_1, D_2 and D_3 are invariant, anti-invariant and slant distribution, respectively. Therefore $TN=D_1\oplus D_2\oplus D_3\oplus Sp\{\xi_1,\xi_2\}$ is a quasi hemi-slant submanifold of $\mathbb{R}^{10}$.

Now we define Gauss and Weingarten formulas for submanifolds.

Let $\overline{\nabla}$ be the Levi-Civita connection of $\bar{B}$. Therefore Gauss and Weingarten equations are given by

$$\overline{\nabla}_X Y = \nabla^*{}_X Y - \sigma(X, Y) \tag{5}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^*{}^{\perp}_X Y \tag{6}$$

where, $X, Y \in \Gamma(TB)$, $V \in \Gamma(TB^{\perp})$, σ is the second fundamental from, $\nabla^{*\perp}$ is the connection in the normal bundle and A_V is the Weingarten endomorphism. Therefore we have

$$g(\sigma(X, Y), V) = g(A_V X, Y). \tag{7}$$

For every tangent vector field X on B we can write

$$\varphi X = TX + NX \tag{8}$$

where TX and NX denote the tangent and normal components of φX, respectively. For every normal vector field V, we can state

$$\varphi V = tV + nV \tag{9}$$

where tV is the tangent component of φV and nV is the normal one.

On the other hand , let B be a quasi hemi-slant submanifold of generalized Kenmotsu manifold $\bar{B}$. The projection morphisms of TB to the distributions $D, D^{\perp}$ and D^{θ} are denoted respectively by P, Q and R. Then for each $W \in \Gamma(TM)$ we have

$$X = PX + QX + RX + \eta(X)\xi. \tag{10}$$

Thus from (8) we get $TX = TPX + TRX$ and $NX = NPX + NQX$.

By using (5), (6) and (10) and several computations we obtaion following propositions.

Proposition 1. For all $Y \in \Gamma(TB)$ we have

$$g(PX, Y) = g(X, PY),\ for\ any\ X, Y \in \Gamma(D) \tag{11}$$

$$g(QX,Y) = g(X,QY), \quad for\ any\ X,Y \in \Gamma(D^{\perp}) \quad (12)$$

$$g(RX,Y) = g(X,RY), \quad for\ any\ W,Y \in \Gamma(D^{\theta}) \quad (13)$$

$$\nabla_X \xi = PX, \ h(X,\xi) = 0 \ \ for\ any\ W \in \Gamma(D) \quad (14)$$

$$\nabla_X \xi = 0, \ h(X,\xi) = QX \ \ for\ any\ X \in \Gamma(D^{\perp}) \quad (15)$$

$$\nabla_X \xi = \varphi TRX, \ h(X,\xi) = \varphi NRX \ \ for\ any\ X \in \Gamma(D^{\theta}). \quad (16)$$

Theorem 2. Let B be quasi hemi-slant submanifold of generalized Kenmotsu manifold $\bar{B}$. The distrbution D is not integrable.

Proof For all $X, Y \in \Gamma(D)$ we get

$$g([X,Y],\xi) = g(\nabla_X Y,\xi) - g(\nabla_Y X,\xi)$$
$$= -g(Y,\nabla_X \xi) + g(X,\nabla_Y \xi).$$

From equation (14), we have

$$g([X,Y],\xi) = g(Y,PX) - g(X,PY)$$
$$= 2g(X,PY).$$

Thus D is integrable if and only if $g(X,PY) = 0.$ From (11) the proof is completed.

Theorem 3. Let B be quasi hemi-slant submanifold of generalized Kenmotsu manifold $\bar{B}$. The distribution $D^{\perp}$ is always integrable.

Proof For all $X, Y \in \Gamma(D)$, we have

$$g(\nabla_Y X,\xi) = -g(X,\nabla_Y \xi).$$

On the other hand, for all $X, Y \in \Gamma(D)$, we have

$$g([Y, X], \xi) = g(\nabla_Y X, \xi) - g(\nabla_X Y, \xi)$$

or

$$g([Y, X], \xi) = -g(X, \nabla_Y \xi) + g(Y, \nabla_X \xi).$$

There proof follows from (15).

Theorem 4. Let B be quasi hemi-slant submanifold of generalized Kenmotsu manifold $\bar{B}$. The distribution D^θ is always integrable.

Proof For all $X, Y \in \Gamma(D^\theta)$, we have

$$g(\nabla_Y X, \xi) = -g(X, \nabla_Y \xi).$$

From equation (16), we get

$$g([X, Y], \xi) = -g(Y, \varphi TQW) + g(X, \varphi TQY).$$

Then we get following equation by (2)

$$g([X, Y], \xi) = g(\varphi QY, TQW) - g(\varphi QW, TQY).$$

After some calculations, we have

$$g([X, Y], \xi) = g(TQY, TQW) - g(TQW, TQY).$$

This completes the proof.

Theorem 5. Let B be quasi hemi-slant submanifold of generalized Kenmotsu manifold $\bar{B}$. The distrbution $D \oplus \{\xi\}$ is always integrable if and only if $h(X, \varphi Y) = h(Y, \varphi X)$.

Proof For all $X, Y \in \Gamma(D \oplus \{\xi\})$ we have

$$\begin{aligned} \varphi([X, Y]) &= \varphi \bar{\nabla}_X Y - \varphi \bar{\nabla}_Y X \\ &= \bar{\nabla}_X \varphi Y - (\bar{\nabla}_X \varphi) Y - \bar{\nabla}_Y \varphi X - (\bar{\nabla}_Y \varphi) X. \end{aligned}$$

Then we obtain following equation by (5) and (6)

$$\varphi([X,Y]) = \nabla_X \varphi Y - h(X,\varphi Y) - g(\varphi X, Y)\xi - \eta(Y)\varphi X$$
$$+\nabla_Y \varphi X - h(Y,\varphi X) - g(\varphi Y, X)\xi - \eta(X)\varphi Y.$$

Then we give $[X,Y] \in \Gamma(D \oplus \{\xi\})$ if and only if $h(X,\varphi Y) = h(Y,\varphi X)$, where $\varphi([X,Y])$ shows the component of $\nabla_X Y$ from ortogonal complementary distribution of $D \oplus \{\xi\}$ in B.

Corollary 2. Let B be quasi hemi-slant submanifold of generalized Kenmotsu manifold $\bar{B}$. The distribution $D^{\perp} \oplus \{\xi\}$ is always integrable if and only if $A_{\varphi Y}X = A_{\varphi X}Y$.

Theorem 6. Let B be quasi hemi-slant submanifold of generalized Kenmotsu manifold $\bar{B}$. The distribution $D \oplus D^{\perp}$ is not integrable.

Proof For all $X, Y \in \Gamma(D \oplus D^{\perp})$ we get

$$g([X,Y],\xi) = -g(Y,\nabla_X \xi) + g(X,\nabla_Y \xi).$$

From equation (11), we have

$$g([X,Y],\xi) = g(Y, PX + QX) - g(X, PY + QY)$$
$$= 2g(Y,PX) - 2g(X,QY).$$

Thus $D \oplus D^{\perp}$ is integrable if and only if $g(Y,PX) = g(X,QY)$. This completes the proof.

Corollary 3. Let B be quasi hemi-slant submanifold of generalized Kenmotsu manifold $\bar{B}$. The distribution $D \oplus D^{\theta}$ and $D^{\perp} \oplus D^{\theta}$ is not integrable.

4. Conclusion

Generalized Kenmotsu manifolds have potential for applications in many fields of mathematics and physics. Researchers have increased studies on this field from different areas in recent years. In this study, the geometric properties of distributions arising from the definition of quasi hemi slant submanifolds of generalized Kenmotsu manifold are examined. The works on this subject will be useful tools for the applications of quasi hemi slant submanifold with different manifolds.

References

Altın, M., Yoldaş, H.İ. & Ünal, İ. (2022). Some result on N(k)-contact metric manifolds. *Advanced Studies: Euro-Tbilisi Mathematical Journal, 10,* 83-97.

Atçeken, M. & Dirik, S. (2014). On the geometry of pseudo-slant submanifolds of a Kenmotsu manifold. *Gulf Journal of Mathematics*, *2(2),* 51-66.

Beyendi, S., Akyol, M.A. & Stankovic, M.S. (2023). Pointwise quasi hemi slant submanifolds. *Filomat, 37(1),* 127-138.

Cabrerizo, J. L., Carriazo, A., Fernandez, L. M. & Fernandez, M. (2000). Slant Submanifolds in Sasakian Manifolds. *Glasgow Mathematical Journal, 42,* 125-138.

Chen, B. Y. (1990). *Geometry of Slant Submanifolds*. Katholieke Universiteit, Leuven.

Khan V. A & Khan M. A. (2007). Pseudo slant submanifolds of a Sasakian Manifold., *Indian Journal Prue and Applied Mathematics, (38),* 31-42.

Lotta, A. (1996). Slant Submanifolds in Contact Geometry. *Bulletin Mathematique Societe Roumanie, 39,* 183-198.

Prasad, R., Shukla, S.S., Haseeb, A. & Kumar, S. (2020). Quasi hemi-slant submanifolds of Kaehler manifolds. *Honam Mathematical Journal, 42(4),* 795-810.

Prasad, R., Haseeb, A. & Gupta, P. (2022). Quasi hemi-slant submanifolds of Kenmotsu manifolds. *Journal of Applied Mathematics & Informatics, 40(3-4),* 475 – 490.

Sarı, R., Ünal, İ. & Aksoy Sarı, E. (2018). Skew semi-invariant submanifolds of para Kenmotsu manifold. *Gümüşhane University Journal of Science and Technology, CMES,*112-118.

Turgut Vanlı, A. & Sarı, R. (2016). Generalized Kenmotsu manifolds, *Communications in Mathematics and Applications, 7(4),* 311-328.

Turgut Vanlı, A. & Sarı, R. (2023). Some submanifolds of generalized Kenmotsu manifolds, *Gulf Journal of Mathematics, 14(1),* 114-124.

CHAPTER 7

A NOTE ON FIBRATION OF CROSSED SQUARES OVER PAIRS OF CROSSED MODULES OVER LIE ALGEBRAS

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1.Introduction

Crossed modules have long been used as models for low-dimensional homotopy types and have origins in the algebraic study of homotopy theory. The definition of a crossed module (Whitehead, 1949) initially given by Whitehead and offers an algebraic representation of homotopy 2-types. Its adaption to Lie algebras, which was initially proposed by Gerstenhaber (Gerstenhaber, 1964)

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and later developed by Ellis and Loday (Ellis, 1988, Ellis, 1993b), (Loday, 1982) offers an optimal setting for studying low-dimensional homotopical situations using Lie theoretic methods. The categorical aspects like limit, fibration, pullback on Lie algebras were studied in (Yılmaz et al., 2021, Taşbozan et al., 2022, Ulualan, 2007). Extensions to higher dimensions, including quadratic modules, crossed squares and 2-crossed modules serve as the algebraic model on homotopy 3-types and are known categorical equivalent under appropriate functors. Some related studies on commutative algebras could be seen in (Porter, 1987, Yılmaz et al., 2020).

Grothendieck's notion of a fibred category has proven helpful for the categorical analysis of algebraic structures defined over a fixed base. In particular, fibration applications in a category allow one to control how objects vary with respect to morphisms in an underlying category by means of pullbacks as cartesian morphisms. This viewpoint has already been successfully used to quadratic modules and crossed squares in several algebraic settings where fibration structures are given by forgetful functors to lower-dimensional structures.

The main purpouse of this work is to give a practical method to construct crossed squares over Lie algebras from appropriate pairs of crossed modules and to accurately prove that the resulting structures satisfy all axioms necessary for a crossed square over Lie algebras. After recalling the classical definitions and results due to Ellis we then introduce the categorical setting of pairs of crossed modules over Lie algebras, which serves as natural base of our construction. In this manner we define the associated crossed square and obtain a verification of its conditions.

2.Preliminaries

We will retain the definition of crossed modules over Lie algebra from (Ellis, 1993a).

Definition 2.1 Let C and R be two Lie k-algebras and R acts on C. The morphism

$$\partial: \mathrm{C} \rightarrow \mathrm{R}$$

of Lie k-algebra is called pre-crossed module over Lie algebra if

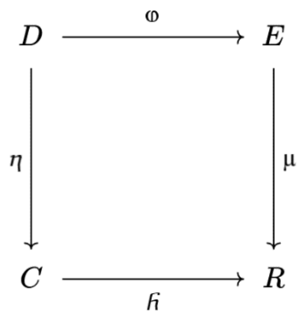
$$\partial(\mathrm{r}\Delta\mathrm{x}) = [\mathrm{r}, \partial(\mathrm{x})]$$

for x in C and r in R where $\Delta: \mathrm{R} \times \mathrm{C} \rightarrow \mathrm{C}$ is the Lie action of R on C. In addition if ∂ : C → R satisfy

$$\partial(x')\Delta x = [x, x']$$

∂: C → R is called crossed module of Lie algebras and denoted with (C, R, ∂).

Definition 2.2 Suppose we are given Lie algebras together with structure maps arranged so that the resulting diagram is



commutative and assume that R acts on the Lie algebras E, D, and C.

Let

$$h : C \times E \rightarrow D$$

be a bilinear map.

This data is crossed square over Lie algebras if the following conditions hold:

1. $\omega, \eta, \mu, \hbar$, and $\mu\omega = \hbar\eta$ are crossed modules over Lie algebras.

2. ω and η preserves the action of R.

3. h is k-bilinear:

$$h(kx, e) = h(x, ke) = k\, h(x, e)$$

4. h is linear in the first variable:

$$h([x, x'], e) = h(x, e) - h(x', e)$$

5. h is linear in the second variable:

$$h(x, [e\,, e']) = h(x, e) - h(x, e')$$

6. h is compatible with the action of R:

$$r \blacktriangleright h(x, e) = h(r\, \Delta x, e) = h(x, r \,\square\, e)$$

7. $\omega\, (h(x, e)) = x \cdot e$.

8. $\eta\, (h(x, e)) = e \cdot x$.

9. $h(x, \eta\,(d)) = x \cdot d$.

10. $h(\omega\,(d), e) = e \cdot d$.

where Δ: $R \times C \to C$, $\blacktriangleright : R \times D \to D$, $\square : R \times E \to E$ are the actions.

Such a structure form crossed square category over Lie algebras. We will show such a crossed square with

$$\begin{vmatrix} D & C \\ E & R \end{vmatrix}$$

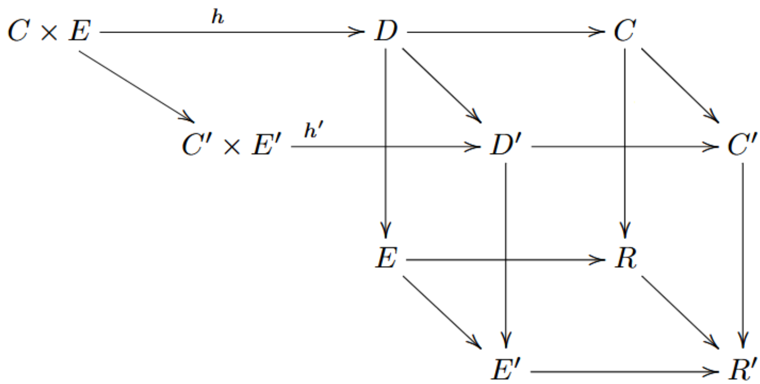
Let

$$\varepsilon : (D, E, C, R) \to (D', E', C', R')$$

be a morphism of crossed squares over Lie algebras. The morphisms

$$\varepsilon_D : D \to D', \varepsilon_E : E \to E'$$
$$\varepsilon_C : C \to C', \varepsilon_R : R \to R'$$

are crossed modules over Lie algebra making the diagram

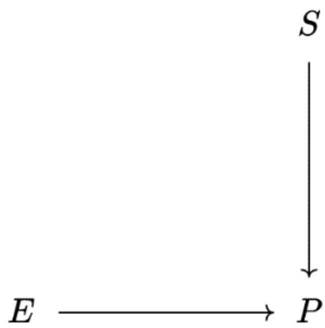


commutative and the homomorphisms ε_D, ε_C, ε_E are ε_R-equivariant. We will denote this category by **Crs**^{2}.

3. Crossed Squares over Lie Algebras from Pairs of Crossed Modules over Lie Algebras

In (Brown& Sivera, 2009) Brown and Sivera mentioned bifibration of crossed squares over pairs of crossed modules. In this section, we will give the notion of the category: pairs of crossed modules for Lie algebras.

Definition 3.1 Let $\tau : C \to P$ and $\omega : E \to P$ be crossed module over Lie algebra. The category, pairs of crossed modules over Lie algebra, $XMod^2$ consists of objects



and with the morphisms preserving the action of P on E and C. Shortly we will write (C, E, P, τ, ω) for a pair of crossed modules.

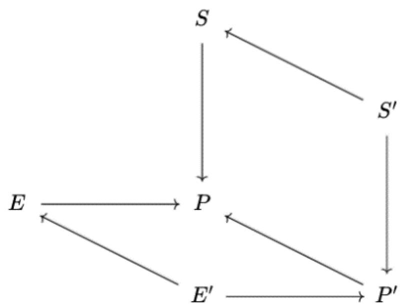
Let

$$\begin{vmatrix} D & S \\ E & P \end{vmatrix}$$

be a crossed square and the morphism

$$\alpha = (\alpha_1, \alpha_2, \alpha_3): (P', E', S', \tau', \omega') \to (P, E, S, \tau \ \ , \omega \ \ )$$

in $XMod^2$ as given by

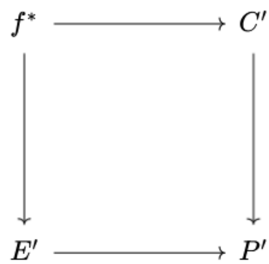


We define

$$\alpha^* = \{(x', r', s') \in E' \times P' \times S' : \omega'(x') = \tau'(s'), \alpha_2(x') = \eta(d), \alpha_3(s') = \omega(d)\}$$

and $\omega_1(x', s', d') = c', \omega_2(x', s', d') = x'$ to give the next proposition where $\mu: D \to S, \hbar: D \to E$.

Theorem 3.1 The diagram



is an object in Crs^2.

Proof:

1. From the definition τ' and ω' are crossed module over Lie algebras. First, let us obtain that ω_1 is a crossed module of Lie algebras.

$$\begin{aligned} \omega_1(x'' \Delta (e', x', d)) &= (\omega_1\tau'(x'') \cdot e', x''x', \alpha_3(x'') \cdot d) \\ &= [x'', x'] \\ &= [x'', \omega_1(e', x', d)] \end{aligned}$$

for $x'' \in C', (e', x', d) \in \alpha^*$ and

$$
\begin{aligned}
\omega_1\,((e',x',d))\,\Delta\,(e',x'',d') &= x'\,\Delta\,(e'',x'',d') \\
&= (\tau\,'(x')\,\cdot\,e'',x'x'',\alpha_3\,(x'')\,\cdot\,d) \\
&= (\omega\,'(e')\,\cdot\,e'',x'x'',\mu\,(d)\,\cdot\,d') \\
&= (e'e'',x'x'',dd') \\
&= [(e',x',d),(e'',x'',d')]
\end{aligned}
$$

for $(e',x',d),(e'',x'',d') \in \alpha^*$. Similar way ω_2 becomes a crossed moduleover Lie algebras. Since composition of two crossed modules $\tau'\omega_1, \omega'\omega_2$ are crossed modules and from the definition of α^* it is clear that $\tau'\omega_1 = \omega'\omega_2$.

2. ω_1 and ω_2 preserves the action for $r' \in R$ and $(e',x',d) \in \alpha^*$

$$
\begin{aligned}
\omega_2\big(r'\cdot(e',x',d)\big) &= \omega_2(r'\square e',r'\,\Delta\,x',\alpha_1(r')\,\cdot\,d) \\
&= r'\,\square\,e' \\
&= r'\,\square\,\omega_2(e',x',d)
\end{aligned}
$$

3. Define

$$h' : E' \times C' \to E' \times C' \times D$$

$$(e',x')' \to (\tau'(x')\,\square\,e',\omega'(e')\,\Delta\,x'',h(\alpha_2(e'),\alpha\,'\,(x'')))$$

where $h : E \times C \to D$ is the h-map of

$$\begin{vmatrix} \mathrm{D} & \mathrm{C} \\ \mathrm{E} & \mathrm{R} \end{vmatrix}$$

For $x' \in C', e' \in E'$ and $k \in k$ we have

$$
\begin{aligned}
&k.h'(e',x') \\
&= k.(\,\tau\,'(x')\,\square\,e',\omega\,'(e')\,\Delta\,x'',h(\alpha_2(e'),\alpha_3\,'(x''))) \\
&= (k.\tau\,'(x')\,\square\,e',k.\omega\,'(e')\,\Delta\,x'',k.h(\alpha_2(e'),\alpha_3\,'(x'')))
\end{aligned}
$$

$$= \quad (\tau'(x') \,\square\, ke', \omega'(ke') \,\Delta\, x'', h(k\alpha_2(e'), \alpha_3{}'(x'')))$$

$$= \quad (\tau'(x') \,\square\, ke', (ke')x'', h((ke'), \alpha_3{}'(x'')))$$

$$= \quad h'(ke', x')$$

$$k.h'(e', x') \;=\; k.(\,\tau'(x') \,\square\, e', \omega'(e') \,\Delta\, x'', h(\alpha_2(e'), \alpha_3{}'\,(x'')))$$

$$= \quad (k.\tau'(x') \,\square\, e', k.\omega'(e') \,\Delta\, x'', k.h(\alpha_2(e'), \alpha_3{}'(x'')))$$

$$= \quad (\mu'(kx') \,\square\, e', \omega'(ke') \,\Delta\, kx'', h(\alpha_2(e'), k\alpha_3{}'(x'')))$$

$$= \quad (\mu'(kx') \,\square\, e', \omega'(e') \,\Delta\, kx'', h(\alpha_2(e'), \alpha_3{}'(kx'')))$$

$$= \quad h'(e', kx')$$

4. For $x', x'' \in C'$ and $e' \in E'$

$$h'(e', [x', x''])$$

$$= \quad (\tau'([x', \mathrm{x}'']) \,\square\, e', \omega'(e')\Delta([\mathrm{x}', \mathrm{x}'']), h(\alpha_2(e'), \alpha_3{}'\,([\mathrm{x}', \mathrm{x}''])))$$

$$= \quad ([\tau'(\mathrm{x}')\square e', \tau'(\mathrm{x}'')\square e'], [\omega'(e')\Delta\mathrm{x}', \omega'(e')\Delta\,\mathrm{x}''],$$

$$h(\alpha_2(e'), [\alpha_3'(\mathrm{x}'), \alpha_3'(\mathrm{x}'')])$$

$$= \quad (\tau'(\mathrm{x}') \,\square\, e' \;+\; \tau'(\mathrm{x}'')\square e', \tau'(\mathrm{x}') \cdot e'))$$

$$+\,\omega'(e'', h((\alpha_2(e'), \alpha_3{}'\,(\mathrm{x}')), h(\alpha_2)(e')\alpha_3{}'\,(\mathrm{x}''))$$

$$= \quad (\tau'(\mathrm{x}') \cdot e', \tau'(\mathrm{x}') \cdot e', h(\alpha_2(e'), \alpha_3{}'\,(\mathrm{x}'))) + (\tau'(\mathrm{x}'') \cdot e', \omega'(e') \cdot \mathrm{x}'', h(\alpha_2(e')\alpha_3{}'\,(\mathrm{x}'')))$$

$$= \; h'(e', \mathrm{x}') \;+\; h'(e', \mathrm{x}'')$$

5. For $x' \in C'$ and $e', e'' \in E'$, it can be seen similarly.

6. For $x' \in C', e' \in E'$ and $r' \in R$,

$$r' \cdot h'(e', x') = \; r' \cdot (\tau'(x') \,\square\, e', \omega'(e') \,\Delta\, x'', h(\alpha_2(e'), \alpha_3{}'\,(x')))$$

$$= (\,r' \cdot (\tau'(x')\square e'), r' \cdot (\omega'(e')\Delta x''), r' \cdot h\big(\alpha_2(e'), \alpha_3'(x')\big))$$

$$= \left(\left(r' \cdot \tau'(\mathrm{x}')\right) \square\, e', \left(r' \cdot\ \omega'(e')\right) \Delta\, \mathrm{x}'', h\left(r' \square\, \alpha_2(e'), \alpha_3'(\mathrm{x}')\right)\right)$$

$$= \ (\tau'(\mathrm{x}') \ \cdot\ r' \ \cdot\ e', \omega'(r'e')\ \Delta\, \mathrm{x}'', h(\alpha_2(r' \square\, e'), \alpha_3{}'\ (\mathrm{x}')))$$

$$= \ (\tau'(\mathrm{x}') \ \cdot\ (r' \square\, e'), \omega'(r' \square\, e')\ \Delta\, \mathrm{x}'', h(\alpha_2(r' \square\, e'), \alpha_3{}'\ (\mathrm{x}')))$$

$$= \ h'(r' \square\, e', \mathrm{x}')$$

$$r' \ \cdot\ h'(e', \mathrm{x}') \ = \ r' \ \cdot\ (\tau'(\,\mathrm{x}') \square\, e', \omega'(e')\ \Delta\ \ \mathrm{x}'', h(\alpha_2(e'), \alpha_3{}'\ (\,\mathrm{x}')$$

$$= \ r' \ \cdot\ (\tau'(\,\mathrm{x}') \square\, e'), r' \ \cdot\ (\omega'(e')\ \Delta\ \ \mathrm{x}''), r'\ h(\alpha_2(e'), \alpha_3{}'\ (\,\mathrm{x}'))$$

$$= \ (r'\ \Delta\, \tau'(\mathrm{x}')) \ \cdot\ e', (r' \ \cdot\ \omega'(e'))\ \Delta\, \mathrm{x}'', h(r' \square \alpha_2(e'), \alpha_3{}'\ (\mathrm{x}'))$$

$$= \ \left(\tau'(r'\mathrm{x}') \cdot\ e', (\omega'(e') \cdot\ r')\Delta\, \mathrm{x}', h\left(\alpha_2(e'), \alpha_3'(r'\mathrm{x}')\right)\right)$$

$$= \ (\tau'(r'\ \Delta\, \mathrm{x}') \ \cdot\ e', \omega'(e') \ \cdot\ (r'\ \Delta \mathrm{x}'), h(\alpha_2(e'), \alpha_3{}'\ (r'\ \Delta \mathrm{x}')))$$

$$= \ h'(e', r'\ \Delta\, \mathrm{x}')$$

7. For $x' \ \in\ C', e' \ \in\ E'$ and $r' \ \in\ R$

$$r' \ \cdot\ h'(e', \mathrm{x}') = \ r' \ \cdot\ (\tau'(\mathrm{x}') \square\, e', \omega'(e')\ \Delta\, \mathrm{x}'', h(\alpha_2(e'), \alpha_3{}'\ (\mathrm{x}')))$$

$$= \ r' \ \cdot\ (\tau'(\mathrm{x}') \square\, e'), r' \ \cdot\ (\omega'(e')\ \Delta\, \mathrm{x}''), r' \ \cdot\ h(\alpha_2(e'), \alpha_3{}'\ (\mathrm{x}'))$$

$$= \ (r' \ \cdot\ \tau'(\mathrm{x}')) \ \cdot\ e', (r' \ \cdot\ \omega'(e'))\ \Delta\, \mathrm{x}'', h(r' \ \cdot\ \alpha_2(e'), \alpha_3{}'\ (\mathrm{x}'))$$

$$= \ (\tau'(\mathrm{x}') \ \cdot\ r' \ \cdot\ e', \omega'(r'e')\ \Delta\, \mathrm{x}'', h(\alpha_2(r' \square\, e'), \alpha_3{}'\ (\mathrm{x}')))$$

$$= \ (\tau'(\mathrm{x}') \ \cdot\ (r' \square\, e'), \omega'(r' \square\, e')\ \Delta\, \mathrm{x}'', h(\alpha_2(r' \square\, e'), \alpha_3{}'\ (\mathrm{x}')))$$

$$= \ h'(r' \square\, e', \mathrm{x}')$$

$$r' \ \cdot\ h'(e', \mathrm{x}') \ = \ r' \cdot\ (\tau'(\mathrm{x}') \square\, e', \omega'(e')\ \Delta\, \mathrm{x}'', h(\alpha_2(e'), \alpha_3{}'\ (\mathrm{x}')))$$

$$= \ r' \ \cdot\ (\tau'(\mathrm{x}') \square\, e'), r' \ \cdot\ (\omega'(e')\ \Delta\, \mathrm{x}''), r' \ \cdot\ h(\alpha_2(e'), \alpha_3{}'\ (\mathrm{x}'))$$

$$= \ (r' \ \cdot\ \tau'(\mathrm{x}')) \ \cdot\ e', (r' \ \cdot\ \omega'(e'))\ \Delta\, \mathrm{x}'', h(r' \ \cdot\ \alpha_2(e'), \alpha_3{}'\ (\mathrm{x}'))$$

$$= (\tau'(r'x') \cdot e', (\omega'(e') \cdot r') \Delta x', h(\alpha_2(e'), \alpha_3{}' (r'x')))$$

$$= (\tau'(r' \Delta x') \cdot e', \omega'(e') \cdot (r' \Delta x'), h(\alpha_2(e'), \alpha_3{}' (r' \Delta x')))$$

$$= h'(e', r' \Delta x')$$

8. For $x' \in C', e' \in E'$ and $r' \in R$;

$$r' \cdot h'(e', \mathrm{x}') = r' \cdot (\tau'(\mathrm{x}') \square e', \omega'(e') \Delta \mathrm{x}'', h(\alpha_2(e'), \alpha_3{}' (\mathrm{x}')))$$

$$= r' \cdot (\tau'(\mathrm{x}') \square e'), r' \cdot (\omega'(e') \Delta \mathrm{x}''), r' \cdot h(\alpha_2(e'), \alpha_3{}' (\mathrm{x}'))$$

$$= (r' \cdot \tau'(\mathrm{x}')) \square e', (r' \cdot \omega'(e')) \Delta \mathrm{x}'', h(r' \cdot \alpha_2(e'), \alpha_3{}' (\mathrm{x}'))$$

$$= (\tau'(\mathrm{x}') \cdot r' \cdot e', \omega'(r'e') \Delta \mathrm{x}'', h(\alpha_2(r' \square e'), \alpha_3{}' (\mathrm{x}')))$$

$$= (\tau'(\mathrm{x}') \cdot (r' \square e'), \omega'(r' \square e') \Delta \mathrm{x}'', h(\alpha_2(r' \square e'), \alpha_3{}' (\mathrm{x}')))$$

$$= h'(r' \square e', \mathrm{x}')$$

$$r' \cdot h'(e', \mathrm{x}') = r' \cdot (\tau'(\mathrm{x}') \square e', \omega'(e') \Delta \mathrm{x}'', h(\alpha_2(e'), \alpha_3{}' (\mathrm{x}')))$$

$$= r' \cdot (\tau'(\mathrm{x}') \square e'), r' \cdot (\omega'(e') \Delta \mathrm{x}''), r' \cdot h(\alpha_2(e'), \alpha_3{}' (\mathrm{x}'))$$

$$= (r' \cdot \tau'(\mathrm{x}')) \cdot e', (r' \cdot \omega'(e')) \Delta \mathrm{x}'', h(r' \cdot \alpha_2(e'), \alpha_3{}' (\mathrm{x}'))$$

$$= (\tau'(r'\mathrm{x}') \cdot e', (\omega'(e') \cdot r') \Delta \mathrm{x}', h(\alpha_2(e'), \alpha_3{}' (r'\mathrm{x}')))$$

$$= (\tau'(r' \cdot \mathrm{x}') \cdot e', \omega'(e') \cdot (r' \Delta \mathrm{x}'), h(\alpha_2(e'), \alpha_3{}' (r' \Delta \mathrm{x}')))$$

$$= h'(e', r' \Delta \mathrm{x}')$$

9. For $(e', x', d) \in \alpha *$ and $e'' \in E'$;

$$h'(e', \omega_1(e', \mathrm{x}', d)) = h'(e'', \mathrm{x}')$$

$$= (\tau'(\mathrm{x}') \square e'', \omega'(e'') \Delta \mathrm{x}', h(\alpha_2(e''), \alpha_3{}'(c')))$$

$$= (\omega'(e') \cdot e'', \omega'(e'') \Delta \mathrm{x}', h(\alpha_2(e''), \mu(d))$$

$$= (e' \cdot e'', e'' \cdot \mathrm{x}', \alpha_2(e'') \cdot d)$$

$$= e'' \cdot (e', \mathrm{x}', d)$$

10. For $(e', x', d) \in \alpha^*$ and $x'' \in C'$;

$$h'(\omega_2(e', \mathrm{x}', d), \mathrm{x}'') = h'(e', \mathrm{x}')$$

$$= (\tau'(\mathrm{x}'') \square e', \omega'(e') \Delta \mathrm{x}'', h(\alpha_2(e'), \alpha_3{}'(\mathrm{x}'')))$$

$$= (\mathrm{x}'' \cdot e', \tau'(\mathrm{x}') \cdot \mathrm{x}'', h(\hbar(d), \alpha_3{}'(\mathrm{x}'')))$$

$$= (\mathrm{x}'' \cdot e', \mathrm{x}' \cdot \mathrm{x}'', \alpha_3(\mathrm{x}'') \cdot d)$$

$$= \mathrm{x}'' \cdot (e', \mathrm{x}', d)$$

4. Conclusion

In this work, we examined how crossed squares of Lie algebras can be obtained from pairs of crossed modules by using the structural relations that link their actions and homomorphisms. Beginning with the classical definitions of crossed modules and the basic properties recalled in the preliminary section, we developed the categorical approach in which pairs of crossed module over Lie algebra are given. Within this manner, we give the construction and showed that it naturally yields a crossed square when the required conditions are hold.

The main result demonstrates that the data arising from a morphism in pairs of crossed modules maps to an object in Crs^2. Each of the axioms defining a crossed square from the pairs of crossed module conditions verified explicitly. Also some equivalent structures with crossed squares were worked in (Yılmaz, 2022, Soylu Yılmaz et al., 2022, Yılmaz et al., 2019).

The method presented here for producing crossed squares from pairs of crossed modules over Lie algebras. Such constructions strengthens the connections between crossed module notion and higher Lie-algebraic structures. These results may serve as a starting point for further results involving Lie 3-algebras, higher homotopical structures, and non-abelian cohomology, providing a way for future work in the non-abelian algebra of Lie algebras.

References

Brown, R. Sivera. R. (2009) Algebraic colimit calculations in homotopy theory using fibred and cofibred categories. *Theory and Application of Categories*, 222-251.

Ellis, G. J. (1988) Higher dimensional crossed modules of algebras. *Journal of Pure and Applied Algebra*, *52 (3)*, 277-282.

Ellis, G. (1993a) Crossed squares and combinatorial homotopy. *Mathematische Zeitschrift, 214 (1)*, 93-110.

Ellis, G. J. (1993b) Homotopical aspects of Lie algebras. *Journal of The Australian Mathematical Society, 54 (3)*, 393-419.

Gerstenhaber, M. (1964) A uniform cohomology theory for algebras. *Proceedings of the National Academy of Sciences*, *51*(4), 626-629.

Loday, J.-L. (1982) Spaces with finitely many nontrivial homotopy groups. *Journal of Pure and Applied Algebra*, *24*, 179-202.

Porter, T. (1987) Some Categorical Results in the theory of crossed modules in commutative algebras. *Journal of Algebra, 109*, 415-429.

Soylu Yılmaz, E., Yılmaz, K. (2022) On relations among quadratic modules. *Mathematical Methods in the Applied Sciences, 45(18)*, 12231-12244.

Taşbozan, H., Güzelkokar, A. (2022) Quadratic modules of Lie algebras fibred over nil(2)- modules of Lie algebras. *Journal of Universal Mathematics, 5(2)*, 68-75.

Ulualan, E. (2007) Braiding for categorical and crossed Lie algebras and simplicial Lie algebras. *Turkish Journal of Mathematics, 31(3)*, 239-255.

Whitehead, J.H.C. (1949) Combinatorial Homotopy II. *Bulletin American Mathematical Society, 55*, 453-496.

Yılmaz, E. S., Yılmaz, K. (2020) On crossed squares of commutative algebras. *Mathematical Sciences and Applications E-Notes, 8(2),* 32-41.

Yılmaz, K., Yılmaz, E. S. (2019) Baues cofibration for quadratic modules of Lie algebras. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 68(2)*, 1653-1663.

Yılmaz, K., Yılmaz, E. S., Güzelkokar, A. (2021) XModLie Fibred Over Lie Algebras. *Ikonion Journal of Mathematics, 3(2)*, 9-16.

YILMAZ, E. S. (2022) (Co) Limit calculations in the category of 2-crossed R-modules. *Turkish Journal of Mathematics, 46(7),* 2902-2915.

CHAPTER 8

A PERSPECTIVE ON ISOMORPHISM PROBLEMS THROUGH G-SET MODULES

MEHMET UC[1]

Introduction

Group rings have played a fundamental role in both the analysis of algebraic structures and the development of representation theory since the mid-20th century. The group ring RG, defined for a ring R and a group G, is a powerful tool for translating the structure of G to the algebraic plane. In particular, studies of integral group rings $\mathbb{Z}G$ have revealed profound problems concerning the extent to which a finite group can be determined solely from ring-level information.

Isomorphism problem in group rings is summarized in the literature by the following question: Does the isomorphism $RG \cong RH$ always yield $G \cong H$? This question has been answered in the affirmative in some important cases. Perlis and Walker (1950) showed that finite Abelian groups are determined by group rings over the rational numbers. Deskins (1956) obtained a similar determinability result for finite Abelian p-groups over fields with

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characteristic p. Higman (1940) gave important positive results for integral group rings in the context of abelian groups and Hamiltonian 2-groups. Results by researchers such as Sandling (1974, 1985), Whitcomb (1968), and Weiss (1991) showed that the result $\mathbb{Z}G \cong \mathbb{Z}H \Rightarrow G \cong H$ is valid for metabelian, nilpotent, and certain linear groups.

On the other hand, the examples given by Dade (1971) showed that the isomorphism problem can be answered negatively for all fields, so the strongest version, the complete group ring conjecture, came to the fore: $\mathbb{Z}G \cong \mathbb{Z}H \Rightarrow G \cong H$. This conjecture is not completely solved today, but it has been verified for a wide class of groups such as abelian, metabelian, nilpotent, Hamiltonian 2-groups, and simple groups (Sehgal and Milies, 2002).

Zassenhaus' conjectures (Sehgal, 1996) concerning the normalized unit group of $\mathbb{Z}G$, rational conjugacy of torsion units, behavior of finite subgroups, and structural constraints on automorphisms, generated a vast body of research. Significant progress has been made by Hughes–Pearson (1972), Milies (1973), Luthar and Passi (1989), Luthar and Trama (2007), Dokuchaev–Juriaans (1996), and many others.

The normalizing conjecture began with Coleman (1964) and was later expanded by Li, Parmenter and Sehgal (1999). The problem of the existence of free subgroups was studied by researchers such as Hartley and Pickel (1980), Gonçalves (1984), and Passman (1996) and revealed the complexity of unit groups of integral group rings.

The concept of a group ring has extensive generalizations, such as skew group rings, cross products, semigroup rings, loop rings, and partial group algebras. In particular, Goodaire and Milies (1988, 1989, 1996) obtained extensive results on isomorphism problems on alternating loop rings and on the validity of Zassenhaus conjectures.

The concept of a group module was first introduced by Koşan et al. (2014) and was subsequently developed and generalized in later studies Kosan (2020), Uc and Alkan (2017). Isomorphism problems arising in the context of group modules are a natural generalization of the classical group ring isomorphism problem. Given a ring R and an R-module M, the fundamental question is whether modules MG and MH are isomorphic, and whether this isomorphism yields $G \cong H$. It is known that, in general, $G \cong H \Rightarrow MG \cong MH$ always holds, while the converse is not true; that is, $G \ncong H$ can exist even though $MG \cong MH$. Therefore, determining all groups H such that $MG \cong MH$ for a given M is crucial for understanding the extent to which module-theoretic methods determine group structure. Uc and Mercan (2025) provided a fundamental starting point in this area by showing that $MH \cong MG$ for isomorphic groups H and G. Moreover, the question of whether two non-isomorphic groups can form isomorphic group modules under the same module M lies at the heart of isomorphism problems in group modules and reveals the deep relations between group theory and module theory.

The concept of a G-set module was first introduced in a systematic framework by Uc and Alkan (2023) and is considered a natural generalization of the classical group module approach. This structure allows us to examine the MS module, which results from the interaction of a group G on a G-set S combined with an R-module M. This allows for a richer analysis of the interaction between group influence and module structure. This chapter addresses isomorphism problems in G-set modules, specifically exploring how G-set isomorphisms are reflected in module isomorphisms, the conditions under which isomorphic modules necessitate G-set isomorphism, and the extent to which module-theoretic structures determine group influence. The main results of this chapter include the proof that a G-set isomorphism induces an RG-module isomorphism of the form $MS \rightarrow$ MT, the converse results obtained through characterization of modules, the application of character theory and the Maschke decomposition to G-set modules, and the proof of fundamental theorems concerning whether non-isomorphic groups can generate the same G-set module. These results provide a holistic approach to isomorphism problems in G-set modules by revealing both the structural properties of G-set modules and how group influence is encoded at the module level.

Preliminaries

In this section, we present a restructured formulation of the concept of a G-set module, inspired by earlier work, most notably Koşan et al. (2014), but rewritten to provide a broader, clearer, and more general framework suitable for this book chapter. Throughout, G denotes a finite group with identity element e, R is a commutative ring with unity 1, M is a left R-module, and RG is the corresponding group ring. The notation $H \leq G$ indicates that H is a subgroup of G, while S represents a G-set equipped with an action of G on S. Whenever N is an R-submodule of M, the notation $N_R \leq M_R$ will be used. Given a G-set S, the G-set module MS is defined as the collection of all formal finite sums of the form $\sum_{s\in S} m_s s$, where each coefficient m_s belongs to M and only finitely many coefficients are nonzero. Equality of two such expressions $\mu = \eta$ is interpreted as coefficientwise equality, meaning $m_s = n_s$ for all $s \in S$. The addition in MS is defined componentwise that is $\mu + \eta = \sum_{s\in S} m_s s + \sum_{s\in S} n_s s = \sum_{s\in S}(m_s + n_s)s$. The scalar multiplication by $r \in R$ is given by $r\mu = r(\sum_{s\in S} m_s s) = \sum_{s\in S}(r m_s)s$. Under this operation, MS becomes an R-module. If $\rho = \sum_{g\in G} r_g g$ is an element of the group ring RG, then its action on MS is defined by

$$\rho\mu = (\sum_{g\in G} r_g g)(\sum_{s\in S} m_s s) = \sum_{s\in S} (rm_s)(gs)$$

which makes MS a left RG-module.

This module will be denoted $(MS)_{RG}$, and in its purely R-linear form by $(MS)_R$. The structure $(MS)_{RG}$ is called the G-set module of S by M over RG. Since the action of G on S naturally extends to MS, the structure of a G-set is also inherited by MS. If S is an H-set for a subgroup $H \leq G$, then MS becomes an RH-module. Furthermore, if S is both a G-set and a group, and if $M = R$, then RS coincides with the classical group algebra. Likewise, when a group acts on itself by multiplication, we obtain $(MS)_{RG} = (MG)_{RG}$, showing that MG is the basic example of a G-set module.

Since the actions of G on a set S correspond bijectively to homomorphisms $G \rightarrow \Sigma_S$ (where Σ_S denotes the full permutation group on S), G-set modules form a wide and rich class of RG-modules. In this sense, MG, introduced in Kosan et al. (2014) in the case where G acts on itself, may be viewed as the earliest instance of a G-set module. Consequently, the structure MS provides a natural generalization of several classical constructions: group rings and group modules. The theory of G-set modules thus unifies these frameworks and extends the range of module-theoretic questions one can ask regarding RG-modules. This generalized perspective offers a broader and more flexible approach for studying the structural behavior of group

Main Results

Theorem 1. Let G be a group, R a ring, and M an R -module. Let S and T be G-sets which are isomorphic via a G-set isomorphism $\alpha : S \rightarrow T$. Then the associated G-set modules are isomorphic as R-modules equipped with a compatible G-action that is $MS \cong MT$.

Proof. $\alpha : S \to T$ is a G-set isomorphism; that is α is bijective, and $\alpha(g \cdot s) = g \cdot \alpha(s)$ for all $g \in G$ and $s \in S$. Our aim is to construct an isomorphism such that $\alpha^* : MS \to MT$ between the associated G-set modules. An element $x \in MS$ is a finite formal sum of the form $x = \sum_{s\in S} m_s s$, where each $m_s \in M$ and only finitely many coefficients m_s are nonzero. Similarly, an element $y \in MT$ may be written uniquely as $y = \sum_{t\in T} n_t t$, with $n_t \in M$.

Define the map $\alpha^* : MS \to M$, $\alpha^*(\sum_{s\in S} m_s s) = \sum_{s\in S} m_s\, \alpha(s)$. To prove α^* is well defined, we must show that if $\sum_{s\in S} m_s \cdot s = \sum_{s\in S} m'_s s$ as elements of MS, then α^*sends these two expressions to the same element of MT. Applying α^* we obtain $\alpha^*(\sum_{s\in S} m_s s) = \sum_{s\in S} m_s\, \alpha(s)$ and $\alpha^*(\sum_{s\in S} m'_s s) = \sum_{s\in S} m'_s\, \alpha(s)$. Since $m_s = m'_s$for all s, these two sums are equal in MT. Hence, α^* is well-defined.

Let x=$\sum_{s\in S} m_s s$ and $y = \sum_{s\in S} n_s s$ in MS. Then,

$$\alpha^*(x+y) = \alpha^*\left(\sum\nolimits_{s\in S} m_s s + \sum\nolimits_{s\in S} n_s s\right) = \alpha^*(\sum_{s\in S}(m_s + n_s)s$$
$$= \sum_{s\in S}(m_s + n_s)\alpha(s) = \sum_{s\in S} m_s\alpha(s) + \sum_{s\in S} n_s\alpha(s)$$
$$= \alpha^*(x) + \alpha^*(y).$$

Thus, α^* is additive.

Let $r \in R$, then $\alpha^*(r.x) = \alpha^*(r.\sum_{s\in S} m_s\, s) = \alpha^*(\sum_{s\in S}(rm_s)\, s) = r\sum_{s\in S} m_s\alpha\,(s) = r\alpha^*(x)$. So, α^* is an R-module homomorphism.

The G-action on MS is defined by $g \cdot (ms) = m(gs)$, extended linearly. Since α is a G-set isomorphism,

$$\alpha^*(gx) = \alpha^*\left(g\left(\sum\nolimits_{s\in S} m_s\, s\right)\right) = \sum\nolimits_{s\in S} m_s\alpha(g\, s)$$

$$= \sum_{s \in S} m_s g\alpha(s) = g(\sum_{s \in S} m_s \alpha(s)) = g\alpha^*(x).$$

Hence α^* commutes with the G-action.

Suppose $\alpha^*(x) = \alpha^*(\sum_{s \in S} m_s s) = \sum_{s \in S} m_s \alpha(s) = 0$. In the formal sum structure of MT, distinct basis labels $\alpha(s)$ impose that all coefficients must vanish. Thus $m_s = 0$ for every $s \in S$, which implies $x = 0$. Therefore α^* is injective.

Let $y = \sum_{t \in T} n_t t \in MT$. Because α is bijective, for every $t \in T$ there exists a unique $s \in S$ such that $t = \alpha(s)$. Write $y = \sum_{s \in S} n_{\alpha(s)} \alpha(s)$and define $\sum_{s \in S} n_{\alpha(s)} s \in MS$. Then $\alpha^*(x) = \alpha^*(\sum_{s \in S} n_{\alpha(s)} s) = \sum_{s \in S} n_{\alpha(s)} \alpha(s) = y$.Thus, α^* is surjective. Since the map α^* is well-defined, an RG-module homomorphism, injective and surjective, α^* is an RG-module isomorphism of G-set modules. Hence $MS \cong MT$.

Example 1. Let $G = S_3$, the symmetric group on three elements. We construct two G-sets S and T, each of which contains one nontrivial transitive orbit and two trivial orbits. Define $S = V \cup W = \{v_1, v_2, v_3\} \cup \{w_1, w_2\}$ and $T = A \cup B = \{a_1, a_2, a_3\} \cup \{b_1, b_2\}$. The action of S_3 on V is the standard permutation action that is $\sigma \cdot v_i = v_{\sigma(i)}$ for $\sigma \in S_3$ and $i = 1,2,3$. The action of S_3 on W is trivial that is $\sigma \cdot w_j = w_j$ for $\sigma \in S_3$ and $j = 1,2$. Similarly, $\sigma \cdot a_i = a_{\sigma(i)}$ and $\sigma \cdot b_j = b_j$. Thus, each G-set contains one transitive orbit (size 3), and two trivial orbits (each size 1).

Define α by $\alpha(v_1) = a_1, \alpha(v_2) = a_2, \alpha(v_3) = a_3, \alpha(w_1) = b_1, \alpha(w_2) = b_2$. This map is bijective, orbit-preserving (transitive orbit maps to transitive, trivial to trivial) and G-equivariant. The following verifies the equivariance for V and for W, respectively.

$$\alpha(\sigma \cdot v_i) = \alpha(v_{\sigma(i)}) = a_{\sigma(i)} = \sigma \cdot a_i = \sigma \cdot \alpha(v_i).$$

$$\alpha(\sigma \cdot w_j) = \alpha(w_j) = b_j = \sigma \cdot b_j = \sigma \cdot \alpha(w_j).$$

Hence, $\alpha(g \cdot s) = g \cdot \alpha(s)$ for all $g \in S_3$ and $s \in S$, proving α is a G-set isomorphism.

Let $R = \mathbb{Z}$ and $M = \mathbb{Z}$ as an $\mathbb{Z}$-module. Define $MS = \mathbb{Z}S = \{\sum_{s\in S} m_s s : m_s \in \mathbb{Z}\}$, and $MT = \mathbb{Z}T = \{\sum_{t\in T} n_t t : n_t \in \mathbb{Z}\}$. For example, $u = 4v_1 - v_3 + 2w_1 - 5w_2 \in \mathbb{Z}S$, and $v = 4a_1 - a_3 + 2b_1 - 5b_2 \in \mathbb{Z}T$. By Theorem 1, the induced map α^* is defined by $\alpha^*(\sum_{s\in S} m_s s) = \sum_{s\in S} m_s\, \alpha(s)$. Explicitly, $\alpha^*(v_i) = a_i, \alpha^*(w_j) = b_j$. For example, $\alpha^*(4v_1 - v_3 + 2w_1 - 5w_2) = 4a_1 - a_3 + 2b_1 - 5b_2$. We now verify each algebraic property in detail.

If $\sum_{s\in S} m_s s = \sum_{s\in S} m'_s s$, then, since $\{v_1, v_2, v_3, w_1, w_2\}$ form a formal basis, we have $m_s = m'_s$ for all $s \in S$. Applying α^*, we get $\sum_{s\in S} m_s\, \alpha(s) = \sum_{s\in S} m'_s\, \alpha(s)$. So, α^* is well-defined.

Let $x = \sum_{s\in S} m_s s, y = \sum_{s\in S} n_s s$. Then, $\alpha^*(x + y) = \alpha^*(\sum_{s\in S} m_s s + \sum_{s\in S} n_s s) = \alpha^*(\sum_{s\in S}(m_s + n_s)s = \sum_{s\in S}(m_s + n_s)\alpha(s) = \sum_{s\in S} m_s\alpha(s) + \sum_{s\in S} n_s\alpha(s) = \alpha^*(x) + \alpha^*(y)$. Hence, α^* preserves addition.

α^* is $\mathbb{Z}$-linear, because for any integer $k \in \mathbb{Z}$, $\alpha^*(kx) = \alpha^*(k\sum_{s\in S} m_s s) = \sum_{s\in S}(km_s)\alpha(s) = k\sum_{s\in S} m_s\alpha(s) = k\alpha^*(x)$.

Let $\sigma \in S_3$ and $x = \sum_{s\in S} m_s s$. Then, $\sigma \cdot x = \sum_{s\in S} m_s(\sigma \cdot s)$. Hence, $\alpha^*(\sigma \cdot x) = \alpha^*(\sum_{s\in S} m_s(\sigma \cdot s)) = \sum_{s\in S} m_s\alpha(\sigma \cdot s) = \sum_{s\in S} m_s(\sigma \cdot \alpha(s)) = \sigma \cdot \sum_{s\in S} m_s\alpha(s) = \sigma \cdot \alpha^*(x)$. Thus, α^* commutes with the G-action that is $\alpha^*(\sigma \cdot x) = \sigma \cdot \alpha^*(x)$.

Assume $\alpha^*(x) = 0$. Write $x = m_1v_1 + m_2v_2 + m_3v_3 + n_1w_1 + n_2w_2$. Then, $\alpha^*(x) = m_1a_1 + m_2a_2 + m_3a_3 + n_1b_1 + n_2b_2 = 0$. Since the elements a_1, a_2, a_3, b_1, b_2 are formally independent, their coefficients must all vanish that is $m_1 = m_2 = m_3 = 0; n_1 = n_2 = 0$. Thus, $x = 0$, proving α^* is injective.

Let $y = r_1a_1 + r_2a_2 + r_3a_3 + s_1b_1 + s_2b_2 \in \mathbb{Z}T$. Define $x = r_1v_1 + r_2v_2 + r_3v_3 + s_1w_1 + s_2w_2 \in \mathbb{Z}S$. Then, $\alpha^*(x) = \alpha^*(r_1v_1 + r_2v_2 + r_3v_3 + s_1w_1 + s_2w_2) = r_1\alpha(v_1) + r_2\alpha(v_2) + r_3\alpha(v_3) + s_1\alpha(w_1) + s_2\alpha(w_2) = r_1a_1 + r_2a_2 + r_3a_3 + s_1b_1 + s_2b_2 = y$. Hence, α^* is surjective.

We have verified that α^* is well-defined, additive, $\mathbb{Z}$-linear, G-equivariant, injective and surjective. Therefore, α^* is an isomorphism of G-set modules, and MS ≅ MT.

Theorem 2. Let R be a field, let G be a finite group, and let S and T be finite G-sets. Let M be a nonzero left R-module on which G acts trivially. Consider the corresponding G-set modules $MS := \bigoplus_{s\in S} M \cdot s$, $MT := \bigoplus_{t\in T} M \cdot t$ where the G-action is given by $g \cdot (m\, s) = m \cdot (g \cdot s)$ for all $m \in M$, $s \in S, g \in G$. If MS and MT are isomorphic as RG-modules, then S and T are isomorphic G-sets.

Thus, for any nonzero M with trivial G-action, the construction $S \mapsto MS$ is faithful on isomorphism classes of finite G-sets.

Proof. Since M is nonzero finite dimensional R-vector space on which G acts trivially, the RG-action on MS and MT arises solely from the permutation actions of G on S and T. This allows canonical identifications $M \otimes_R R(S)$ and $M \otimes_R R[T]$, where R(S) and R(T) denote the premutation modules with bases indexed by S and T.

If $\dim(M) = d > 0$, then$M \cong R^d$, and consequently $MS \cong (R(S))^d$ and $MT \cong (R(T))^d$ as RG-modules. Thus, $MS \cong MT$ implies $(R(S))^d \cong (R(T))^d$.

Since R has characteristic zero and G is finite, Maschke's theorem ensures that the group algebra RG is semisimple. Therefore, every finite-dimensional RG-module decomposes uniquely (up to ordering) as a direct sum of simple modules.

Write the simple decomposition of R(S) as $R(S) = V_1^{a_1} \oplus V_2^{a_2} \oplus \ldots \oplus V_k^{a_k}$, where V_i are non-isomorphic simple RG-modules and each $a_i \geq 0$. Similarly, write $R(T) = V_1^{b_1} \oplus V_2^{b_2} \oplus \ldots \oplus V_k^{b_k}$. Taking d copies, we obtain $(R(S))^d \cong V_1^{d.a_1} \oplus V_2^{d.a_2} \oplus \ldots \oplus V_k^{d.a_k}$ and $(R(T))^d \cong V_1^{d.b_1} \oplus V_2^{d.b_2} \oplus \ldots \oplus V_k^{d.b_k}$. The isomorphism $(R(S))^d \cong (R(T))^d$ forces $d.a_i = d.b_i$ for each i. Since $d > 0$, this yields $a_i = b_i$, and hence $R(S) \cong R(T)$ as RG-modules.

Over a characteristic-zero field, permutation modules are determined by their characters. The character χ_S of R(S) is given by $\chi_{S(g)} = \left|Fix_{S(g)}\right|$, the number of elements of S fixed by $g \in G$. Similarly, $\chi_{T(g)} = \left|Fix_{T(g)}\right|$.

Since $R(S) \cong R(T)$, their characters coincide such that $\chi_{S(g)} = \chi_{T(g)}$ for all $g \in G$. Therefore, each element of G fixes the same number of points in S and in T. The fixed-point data for all $g \in G$ determines the orbit decomposition of a finite G-set. Each transitive G-set is of the form G/H for some subgroup $H \leq G$, and two such sets G/H and G/K have identical permutation characters if and only if H and K are conjugate subgroups. Thus, S and T must consist of the same multiset of orbit types G/H, with identical multiplicities. Constructing a G-equivariant bijection orbit by orbit produces an explicit G-set isomorphism $S \cong T$.

Example 2. Let $R = \mathbb{C}$, the field of complex numbers. Let $G = S_3$, the symmetric group on three letters, which has six elements. Let $M = \mathbb{C}$ considered as a 1-dimensional $\mathbb{C}$-vector space with trivial G-action. We construct two G-sets S and T, show that they are isomorphic as G-sets, and then build the permutation RG-modules $\mathbb{C}S$ and $\mathbb{C}T$. Finally, we construct an explicit $\mathbb{C}G$-module isomorphism and explain why Theorem 1 guarantees that such an isomorphism force $S \cong T$ as G-sets.

Define $T = \{1, 2, 3\}$, equipped with the natural action of S_3. For any permutation $\sigma \in S_3$ and any element $i \in T$, the action is defined by $\sigma \cdot i = \sigma(i)$. This makes T a transitive G-set of size three. Next, construct another G-set S as the left coset space G/H, where $H = \{id, (12)\}$ is the subgroup of S_3 of order two. The left cosets of H in G are $H_1 = H$, $H_2 = (13)H$, $H_{13} = (23)$. Thus $S = \{H_1, H_2, H_3\}$ and the action of G on S is by left multiplication: $\sigma \cdot (gH) = (\sigma g)H$. This produces another transitive G-set of size three.

Define a bijection $\alpha : S \to T$ by setting $\alpha(H_1) = 1, \alpha(H_2) = 2, \alpha(H_3) = 3$. To verify that α is a G-set isomorphism, we must show that it is G-equivariant that is $\alpha(\sigma \cdot x) = \sigma \cdot \alpha(x)$ for all $\sigma \in S_3$ and all $x \in S$. Since both S and T are transitive G-sets of size three, the verification may be done using generators of S_3. For example, consider $\sigma = (123)$. Then, $(123) \cdot H_1 = (123)H = H_2$, so $\alpha((123) \cdot H_1) = \alpha(H_2) = 2$. On the other hand, $\alpha(H_1) = 1$ and $(123) \cdot 1 = 2$. Thus, the equivariance condition holds. A similar verification applies for $\sigma = (12)$, and since these elements generate S_3, α is fully G-equivariant. Therefore, α is a G-set isomorphism $S \cong T$.

The permutation module ℂS is the ℂ-vector space with basis $\{H_1, H_2, H_3\}$, with G acting by linear extension of its action on S. That is, for $\sigma \in S_3$ and basis element $x \in S$, we have $\sigma \cdot x =$ the unique coset in S obtained by left multiplying x by σ. Similarly, ℂT is the ℂ-vector space with basis $\{1, 2, 3\}$, with G acting by permutation of the basis elements $\sigma \cdot i = \sigma(i)$. Both ℂS and ℂT are ℂG-modules of dimension three.

Define $\alpha *: \mathbb{C}S \to \mathbb{C}T$ by linear extension of α. Explicitly, for any vector $v = \sum_{x \in S} a_x x \in \mathbb{C}S$, set $\alpha^*(v) = \sum_{x \in S} a_x \alpha(x)$. On basis elements, this means that $\alpha^*(H_1) = 1, \alpha^*(H_2) = 2, \alpha^*(H_3) = 3$. We now verify the properties that make α^* an isomorphism of $\mathbb{C}G$-modules.

Every element of $\mathbb{C}S$ has a unique expression as a finite linear combination of the basis elements H_1, H_2, H_3. Defining α^* on the basis elements and extending linearly ensures the map is well-defined. Its $\mathbb{C}$-linearity follows immediately from the linear extension.

To show α^* is a $\mathbb{C}G$-module homomorphism, we must verify that $\alpha^*(\sigma \cdot v) = \sigma \cdot \alpha^*(v)$ for all $\sigma \in G$ and all $v \in \mathbb{C}S$. It suffices to check this on basis elements. For any $x \in S$, we get $\alpha^*(\sigma \cdot x) = \alpha(\sigma \cdot x) = \sigma \cdot \alpha(x) = \sigma \cdot \alpha^*(x)$. By linearity, this identity holds for all vectors. Thus α^* is G-equivariant and hence a $\mathbb{C}G$-module homomorphism. Since α is a bijection of finite sets, its linear extension α^* is a bijection between finite-dimensional $\mathbb{C}$-vector spaces of the same dimension. The inverse is the linear extension of α^{-1}. Thus α^* is a $\mathbb{C}G$-module isomorphism.

In this example, S and T are isomorphic as G-sets, and the G-set isomorphism α gives rise to a $\mathbb{C}G$-module isomorphism $\alpha^*: \mathbb{C}S \to \mathbb{C}T$. The theorem discussed in the accompanying text states the converse under appropriate hypotheses, namely, that if $\mathbb{C}S$ and $\mathbb{C}T$ are isomorphic as $\mathbb{C}G$-modules, then S and T must be isomorphic as G-sets, provided R has characteristic zero and M is a nonzero finite-dimensional R-module with trivial G-action. Thus, the present example demonstrates both directions: a G-set isomorphism induces a module isomorphism, and, by the theorem, any module isomorphism of this form forces the underlying G-sets to be isomorphic.

Conclusion

This chapter systematically addresses isomorphism problems for G-set modules defined on G-sets. The paper first clarifies the fundamental structure of the G-set module concept with a preliminary section reorganized from previous literature (especially Uc and Alkan, 2023). It is shown that G-set modules generalize the concepts of group ring and group module, thus unifying both the group action on the set and the module structure.

The main contributions of the chapter are the detailed proof of two fundamental isomorphism results. First, it is proven that a G-set isomorphism between a G-set S and T translates into an RG-module isomorphism between the corresponding G-set modules. This result demonstrates that the G-set structure is fully reproducible at the module level.

Second, it has been shown that modules are also inversely deterministic under certain conditions: in particular, for a non-zero module M with trivial G-action defined on a characteristic zero field, the isomorphism MS $\cong$ MT necessitates a G-set isomorphism S $\cong$ T. This result obtained using Maschke's theorem, character theory, and permutation module decompositions, clearly demonstrates the power of G-set modules to distinguish isomorphism classes.

The chapter also included illustrative examples that clarify how G-set isomorphisms correspond to module isomorphisms, thereby complementing the theoretical results with explicit computations and constructions.

In conclusion, this work presents both advanced theoretical methods and structural characterizations for isomorphism problems on G-set modules, and it appears that this new class of modules makes important contributions to the relations between group theory, representation theory, and module theory.

References

Coleman, D. B. (1964). On the modular group ring of a p-group. *Proceedings of the American Mathematical Society*, *15, 4*, 511-514.

Dade, E. C. (1971). Deux groups finis ayant la meme algebre de group sur tout corps. *Mathematische Zeitschrift, 119*, 345-348.

Dokuchaev, M. A., & Juriaans, S. O. (1996). Finite Subgroups in Integral Group Rings. *Canadian Journal of Mathematics*, *48*(6), 1170-1179.

Deskins, W. E. (1956). Finite abelian groups with isomorphic group algebras. *Duke Mathematical Journal, 23*, 35-40.

Gonçalves, J. Z. (1984). Free subgroups of units in group rings. *Canadian Mathematical Bulletin*, *27,* 365-370.

Gonçalves, J. Z., & Passman, D. S. (1996). Construction of free subgroups in the group of units of modular group algebras. *Communications in Algebra*, *24, 13*, 4211-4215.

Goodaire, E. G., & Milies, C. P. (1988). Isomorphisms of Integral Alternative Loop Rings. *Rendiconti del Circolo Matematico di Palermo*, *38*, 126-135.

Goodaire, E. G., & Milies, C. P. (1989). Torsion Units in Alternative Loop Rings. *Proceedings of the American Mathematical Society*, *107*, 7-15.

Goodaire, E. G., & Milies, C. P. (1996). Finite subloops of Units in an Alternative Loop Ring. *Proceedings of the American Mathematical Society*, *124, 4*, 995-1002.

Harley, B., & Pickel, P. F. (1980). Free subgroups in the unit groups of integral group rings. *Canadian Journal of Mathematics*, *32, 6,* 1342-1352.

Hughes, I., & Pearson, K. R. (1972). The group of units of the integral group ring $\mathbb{Z}\boldsymbol{S_3}$. *Canadian Mathematical Bulletin*, *15*, 529-534.

Higman, G. (1940). The units of group rings. *Proceedings of the London Mathematical Society*, *2, 46*, 231-248.

Jespers, E., Leal, G., & Milies, C. P. (1995). Classifying indecomposable RA loops. *Journal of Algebra*, *176,* 5057-5076.

Kosan, M. T., Lee, T., & Zhou, Y. (2014). On modules over group rings. *Algebras and Representation Theory, 17* (1), 87-102.

Kosan, M. T., & Zemlicka, J. (2020). On group modules. *Journal of Algebra and Its Applications, 19* (7), 11.

Luthar, I. S., & Passi, I. B. S. (1989). Zassenhaus conjecture for A_5. *Proceedings of the Indian Academy of Sciences – Mathematical Sciences, 99* (1), 1-5.

Luthar, I. S., & Trama, P. (1991). Zassenhaus conjecture for S5. *Communications in Algebra*, *19*(8), 2353-2362.

Li, Y., Parmenter, M. M., & Sehgal, S. K. (1999). On the normalizer property for integral group rings. *Communications in Algebra*, a aparecer.

Milies, C. P. (1973). The Group of Units of the Integral Group Ring $\mathbb{Z}\mathrm{D}_4$. Boletim da Sociedade Brasileira de Matemática, 4, 2, 85-92.

Perlis, S., & Walker, G. (1950). Abelian group algebras of finite order. *Transactions of the American Mathematical Society, 68*, 420-426.

Sandling, R. (1974). Group rings of circle and unit groups. *Mathematische Zeitschrift, 140*, 195-202.

Sandling, R. (1985). The isomorphismproblem for group rings: a survery, in Orders and their Applications (Oberwolfach 1984). *Lecture Notes in Mathematics Springer-Verlag,* Berlin, pp., *1142*, 256-288.

Sehgal, S. K. (1996). *Units in Integral Group Rings*. Longman Scientific and Technical, New York.

Sehgal, S. K., & Milies, C. P. (2002). *An introduction to group rings*. Kluwer Academic Publishers.

Uc, M., & Alkan, M. (2017). On submodule characterization and decomposition of modules over group rings. *AIP Conference Proceedings, AIP Publishing LLC.* 1863, 1, pp. 300026.

Uc, M., & Alkan, M. (2023). On modules over G-sets. *Journal of Mathematics and Statistical Studies*, 4(4), 47–55.

Uc, M., & Mercan, H. T. (2025). İzomorfizma problemleri üzerine. Abstract book of 6th international conference on innovative academic studies ICIAS 2025, (17). Konya.

Weiss, A. (1991). Torsion units in integral group rings. *Journal für die reine und angewandte Mathematik, 415*, 175-187.

Whitcomb, A. (1968). *The group ring problem*. Ph.D. Thesis, University of Chicago.

CHAPTER 9

A MULTIDISCIPLINARY SURVEY OF LINEAR WEINGARTEN SURFACES

FERAY BAYAR[1]

Introduction

The classification of surfaces in three-dimensional space constitutes one of the most enduring challenges in differential geometry. From the foundational work of Euler and Monge to the modern era of discrete differential geometry, mathematicians have sought to characterize shapes not merely by their visual appearance, but by the intrinsic and extrinsic properties of their curvature. Among the myriad classes of surfaces defined over the last two centuries, Weingarten surfaces occupy a distinguished position (López, 2008).

Defined initially by Julius Weingarten in 1861, these surfaces are characterized by a functional relationship between their principal curvatures, κ_1 and κ_2 (Weingarten, 1861). This implies that the Jacobian of the curvature map vanishes everywhere, or equivalently,

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that the lines of curvature on such surfaces are intimately tied to the variation of the surface normal (Hopf, 1951).

While the general Weingarten condition $W(\kappa_1, \kappa_2) = 0$ allows for arbitrary complexity, a specific subset known as Linear Weingarten (LW) surfaces has emerged as the focal point of contemporary research. These are defined by a linear relation between the Mean curvature (H) and Gaussian curvature (K) :

$$2aH + bK = c \tag{1}$$

Despite the simplicity of this linear constraint, the resulting shapes are geometrically rich. They generalize the classical theories of minimal surfaces ($H = 0$), Constant Mean Curvature (CMC) surfaces ($H = c$), and developable surfaces ($K = 0$).

In recent decades, the study of LW surfaces has transcended theoretical mathematics. In architectural geometry, they act as a "Holy Grail" for rationalization, enabling the fabrication of complex double-curved skins using standardized panels (Pellis et al., 2021). Simultaneously, in biophysics, the Helfrich spontaneous-curvature model, which governs the shape of lipid bilayers and vesicles, reduces to linear Weingarten conditions under specific symmetry constraints (Helfrich, 1973).

Mathematical Foundations and Classification Results

The mathematical study of Linear Weingarten (LW) surfaces is rooted in the differential geometry of principal curvatures. Let S be a smooth, oriented surface in Euclidean space $\mathbb{R}^3$, and let κ_1, κ_2 denote its principal curvatures. A surface is termed a Weingarten surface if there exists a smooth, non-trivial function W such that $W(\kappa_1, \kappa_2) = 0$.

The specific class of Linear Weingarten surfaces arises when this functional dependence is linear with respect to the mean curvature $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ and the Gaussian curvature $K = \kappa_1\kappa_2$. The general defining equation is:

$$2aH + bK = c \quad (2)$$

where $a, b, c \in \mathbb{R}$ are constant not all zero.

The geometric behavior of LW surfaces is governed by the algebraic structure of Equation (2). By substituting the definitions of H and K, the relation describes a quadratic curve in the (κ_1, κ_2) phase plane:

$$b\kappa_1\kappa_2 + a(\kappa_1 + \kappa_2) - c = 0 \quad (3)$$

Following López (2008), the classification depends on the discriminant $\Delta = a^2 + bc$:

- Elliptic Type ($\Delta > 0$): The principal curvatures lie on a hyperbola in the phase plane. This class includes surfaces of constant mean curvature ($b = 0$) and surfaces of constant positive Gaussian curvature ($a = 0, c/b > 0$). Compact, strictly convex LW surfaces in this regime are necessarily spheres (Hopf, 1951).
- Hyperbolic Type ($\Delta < 0$): This regime allows for surfaces with negative Gaussian curvature. López (2009) demonstrated that complete surfaces of this type in hyperbolic space $\mathbb{H}^3$ exhibit complex branch behaviors, distinct from the classical Euclidean pseudosphere.
- Parabolic Type ($\Delta = 0$) : This often corresponds to degenerating cases or surfaces where one principal curvature is linearly related to the other, leading to developable surfaces or specific channel surfaces.

Rotational Linear Weingarten Surfaces

Rotational symmetry simplifies the partial differential equation (PDE) of the LW condition into a solvable ordinary differential equation (ODE). Consider a surface of revolution parametrized by:

$$\mathbf{x}(u,v) = (f(u)\cos\, v, f(u)\sin\, v, g(u)) \tag{4}$$

where u is the arc length of the profile curve $\gamma(u) = (f(u), 0, g(u))$. The principal curvatures are given by the meridional curvature $\kappa_1 = f'g'' - f''g'$ and the parallel curvature $\kappa_2 = g'/f$. Substituting these into Equation (2) yields a second-order non-linear ODE. Aydin (2022) utilized the variational characterization of these profiles to show that they correspond to the extremals of curvature-dependent energy functionals. Specifically, when $b \neq 0$, the profile curves can be expressed in terms of elliptic integrals, generalizing the classical Delaunay unduloids and nodoids found in CMC theory.

Ruled Weingarten Surfaces

A ruled surface is generated by a line moving along a curve, parametrized as $\mathbf{x}(u,v) = \alpha(u) + v\beta(u)$. The classical theorem of Beltrami implies that the only ruled Weingarten surfaces are developable surfaces (where $K = 0$) and ruled helicoids.

In the context of the linear relation, Öztürk et al. (2013) provided a definitive classification in Euclidean 3-space. They proved that a non-developable ruled surface satisfying $2aH + bK = c$ must be a helicoid. If $b \neq 0$, the relationship forces the helicoid to be part of a restricted family where the pitch relates to the coefficients a and c. In Minkowski 3-space, Dillen and Kühnel

(1999) extended this to show that ruled LW surfaces can also include specific Lorentzian cylinders and cones.

Tubular Surfaces and Spacelike Tubes

Tubular surfaces (or canal surfaces with constant radius) provide a rigid geometric setting. Geometrically, a tube of radius r around a spine curve γ has one constant principal curvature, say $\kappa_1 = -1/r$. Substituting $\kappa_1 = \text{const}$ into the linear relation $2aH + bK = c$ immediately constrains κ_2 to be constant as well. Since κ_2 for a tube depends on the curvature of the spine γ, this implies the spine itself must have constant curvature.

Thus, as shown by Pulov et al. (2018), LW tubular surfaces are restricted to:

- Tubes over straight lines (cylinders),
- Tubes over circles (tori),
- Tubes over helices.

Non-Euclidean Geometries and Singularities

Modern research extends LW theory to Riemannian space forms (Hyperbolic space $\mathbb{H}^3$) and Lorentzian space forms (de Sitter space $\mathbb{S}_1^2$).

Hyperbolic Space and Bryant Surfaces

In Hyperbolic 3 -space $\mathbb{H}^3(-1)$, the LW condition is often adapted to the background geometry, typically formulated as $\alpha(K - 1) + 2\beta(H - 1) + \gamma = 0$.

A celebrated class of surfaces in $\mathbb{H}^3$ are those with constant mean curvature $H = 1$, known as Bryant surfaces. These surfaces are the hyperbolic cousins of Euclidean minimal surfaces. Bryant

(1987) established a Weierstrass-type representation for these surfaces. Just as Euclidean minimal surfaces are generated by holomorphic data (g, ω), a Bryant surface $f: \Sigma \to \mathbb{H}^3$ can be constructed from a holomorphic null immersion $F: \Sigma \to SL(2, \mathbb{C})$.

Theorem (Lawson Correspondence). There exists an isometric correspondence between CMC surfaces in different space forms. Locally, a Bryant surface in $\mathbb{H}^3$ corresponds to a minimal surface in $\mathbb{R}^3$.

This correspondence allows for the construction of "cousins" of classical minimal surfaces (like the Enneper surface or Catenoid) in the hyperbolic setting. Hauswirth et al. (2002) utilized this to solve the asymptotic plateau problem, proving the existence of complete Bryant surfaces bounding specific curves at infinity.

Lorentzian Geometry and Singularities

In Lorentzian space forms, such as de Sitter space $\mathbb{S}^2_1$, the metric is indefinite $(+, +, -)$. This fundamentally alters the surface theory, as surfaces can be spacelike, timelike, or lightlike.

A crucial distinction in this setting is the generic presence of singularities. In Riemannian geometry, singularities are often defects. In Lorentzian geometry, they are intrinsic features known as wave fronts. Yasumoto and Rossman (2020) studied "Bianchi-type" LW surfaces in de Sitter space and classified their singularities:

- Cuspidal Edges: A singularity where the surface folds back, equivalent to the caustic of a light front.
- Swallowtails: A generic higher-order singularity appearing in the evolution of wave fronts.

The study of these singularities links LW theory to singularity theory and the topology of caustics (Saji et al., 2009).

Discrete Differential Geometry and Computation

Translating LW theory into algorithms for architecture and engineering requires Discrete Differential Geometry (DDG).

Discrete Curvature and Nets

On a discrete mesh $M = (V, E, F)$, classical definitions of curvature involving derivatives are unavailable. DDG defines curvature via integrated quantities:

- Discrete Gaussian Curvature (K_v) : Defined by the angle defect at a vertex v :

$$K_v = 2\pi - \sum_{f \ni v} \theta_f \tag{5}$$

- Discrete Mean Curvature (H_v) : Defined via the Steiner formula or the variation of surface area. For a mesh edge e_{ij}, it is often related to the edge dihedral angle.

Discrete LW surfaces are characterized as special parallel nets. A key property utilized in computation is that for any Weingarten surface, the gradients ∇K and ∇H are parallel. In the discrete setting, this implies a specific relationship between the offset meshes. Pellis et al. (2021) formulated discrete LW surfaces using the framework of isotropic geometry, where the condition linearizes effectively.

The Guided Projection Algorithm

The state-of-the-art method for rationalizing a freeform design into an LW surface is the Guided Projection Algorithm (Tang et al., 2014). This is an iterative optimization method.

Given a target surface $\mathcal{S}_{\text{target}}$, the algorithm seeks a mesh $\mathcal{M}$ that minimizes the distance to $\mathcal{S}_{\text{target}}$ while satisfying the LW constraint.

- Constraint Formulation: Direct enforcement of $aK + bH + c = 0$ is numerically unstable due to the rational nature of curvature formulas.
- Implicit Constraint: The algorithm instead enforces the collinearity of curvature gradients:

$$\det(\nabla K, \nabla H) = 0 \quad (6)$$

- Fairness: High-degree B-splines are often fitted to the resulting mesh to ensure "Class A" surface quality, characterized by the smooth flow of reflection lines (Pellis et al., 2020).

Applications in Architecture and Physics

Architectural Geometry: The Economics of Mold Re-use

Modern architecture favors freeform skins, but the cost of unique molds is prohibitive. LW surfaces provide a rigorous geometric solution to this economic problem.

On a general freeform surface, the curvature at any point is a pair (κ_1, κ_2). The image of the surface in the $\kappa_1 - \kappa_2$ plane is a 2D region. This means every panel has a unique intrinsic shape. However, for a Weingarten surface, the relation $W(\kappa_1, \kappa_2) = 0$ implies that the curvature image collapses to a 1D curve.

Theorem (Mold Re-use Principle): If a surface satisfies the Linear Weingarten condition, its local patches fall into a one-parameter family of isometries. Consequently, a facade of N panels can be manufactured using approximately $O(\sqrt{N})$ molds, rather than $O(N)$ (Pellis et al., 2020).

Specifically, Gavriil et al. (2020) demonstrated that for cold-bent glass, aligning panel strips with the asymptotic lines of a hyperbolic LW surface minimizes the stress forming, allowing for safe fabrication of complex double-curved facades.

Grid shells and Structural Mechanics

Grid shells are lightweight lattice structures that resist loads through membrane action. Tellier et al. (2019) identified Isotropic Linear Weingarten (iLW) surfaces as a structural optimum.

- Funicularity: A surface is funicular for a vertical load if the stress state is purely axial (compression/tension) with no bending. Tellier proved that iLW surfaces are the only surfaces that are funicular for a uniform projected vertical load while admitting a conjugate net discretization (Tellier, 2020).
- Chadstone Shopping Centre: The roof of this structure in Melbourne was designed using dynamic relaxation. The resulting equilibrium shape is a discrete approximation of an iLW surface, balancing structural efficiency with the planarity of the glazing panels (Chadwick et al., 2017).

Soft Matter Physics: Helfrich Energy

In biophysics, the shape of lipid bilayers is determined by the Helfrich curvature energy (Helfrich, 1973):

$$E = \int_S \left[\frac{1}{2}k_c(2H - c_0)^2 + \bar{k}K\right] dA \qquad (7)$$

where c_0 is the spontaneous curvature induced by lipid asymmetry or proteins. The Euler-Lagrange equation for this energy is a complex fourth-order PDE. However, for specific geometries crucial to cellular function, such as membrane tethers and tubules, the geometry simplifies. Pulov et al. (2018) showed that for axially symmetric membranes, the shape equation reduces to a first-order integral that is equivalent to the Linear Weingarten condition. Here, the "linearity" constants a, b, c are determined by the membrane's physical moduli $(k_c, \bar{k})$ and the internal pressure difference. This reduction allows physicists to analytically predict the radius and stability of membrane tubes pulled from cells.

References

Aydin, M. E. (2022). Rotational Weingarten surfaces and their classification. Mathematics, 10(4), 578.

Bryant, R. L. (1987). Surfaces of mean curvature one in hyperbolic space. Astérisque, 154-155, 321-347.

Chadwick, A., Matthews, C., Villafane, M. E., & Williams, C. J. K. (2017). PLATO Gridshell design method for Chadstone shopping mall roof. IABSE Conference, Bath 2017.

Dillen, F., & Kühnel, W. (1999). Ruled Weingarten surfaces in Minkowski 3-space. Bulletin of the Belgian Mathematical Society, 6(2), 301-312.

Gálvez, J. A., Martínez, A., & Milán, F. (2003). Linear Weingarten surfaces in $\mathbb{R}^3$. Monatshefte für Mathematik, 138(2), 133-144.

Gavriil, K., Guseinov, R., Perez, J., Pellis, D., Henderson, P., Rist, F., Pottmann, H., & Bickel, B. (2020). Computational design of cold bent glass facades. ACM Transactions on Graphics, 39(6), 208:1-208:16.

Hauswirth, L., Roitman, P., & Rosenberg, H. (2002). The Geometry of Finite Topology Bryant Surfaces Quasi-Embedded in Hyperbolic Space. Journal of Differential Geometry, 60, 2001-2002.

Helfrich, W. (1973). Elastic properties of lipid bilayers: theory and possible experiments. Zeitschrift für Naturforschung C, 28(11-12), 693-703.

Hertrich-Jeromin, U., & Pember, M. (2018). Channel linear Weingarten surfaces in space forms. Geometry, Integrability and Quantization, 19, 168-187.

Hopf, H. (1951). Über Flächen mit einer Relation zwischen den Hauptkrümmungen. Mathematische Nachrichten, 4, 232-249.

López, R. (2008). On Linear Weingarten Surfaces. International Journal of Mathematics, 19(4), 439-448.

López, R. (2009). Linear Weingarten surfaces in Euclidean and hyperbolic space. Matematica Contemporanea, 35.

Öztürk, G., Bulca, B., Bayram, B. K., & Arslan, K. (2013). Ruled Linear Weingarten surfaces in Euclidean 3-space. Mathematical Physics, Analysis and Geometry.

Pellis, D., Kilian, M., Wang, H., Jiang, C., Müller, C., & Pottmann, H. (2020). Architectural freeform surfaces designed for cost-effective paneling through mold re-use. Advances in Architectural Geometry 2020.

Pellis, D., Kilian, M., Pottmann, H., & Pauly, M. (2021). Computational Design of Weingarten Surfaces. ACM Transactions on Graphics, 40(4).

Pulov, V. I., Hadzhilazova, M. T., & Mladenov, I. M. (2018). On a Class of Linear Weingarten Surfaces. Geometry, Integrability and Quantization.

Saji, K., Umehara, M., & Yamada, K. (2009). The geometry of fronts. Annals of Mathematics, 169(2), 491-529.

Tang, C., Sun, X., Gomes, A., Wallner, J., & Pottmann, H. (2014). Form-finding with Polyhedral Meshes Made Simple. ACM Transactions on Graphics, 33(4).

Tellier, X., Douthe, C., Hauswirth, L., & Baverel, O. (2019). Linear Weingarten surfaces for conceptual design of double-curvature envelopes. Proceedings of the International Fib Symposium.

Tellier, X. (2020). Morphogenesis of curved structural envelopes under fabrication constraints (PhD Thesis). Université Paris-Est.

Weingarten, J. (1861). Über eine Klasse auf einander abwickelbarer Flächen. Journal für die reine und angewandte Mathematik, 59, 382-393.

Yasumoto, M., & Rossman, W. (2020). Semi-discrete linear Weingarten surfaces with Weierstrass-type representations and their singularities. Osaka Journal of Mathematics, 57(1), 169-185.

Yoon, D. W. (2003). Ruled linear Weingarten surfaces in Minkowski 3-space. Taiwanese Journal of Mathematics, 7(4).

CHAPTER 10

ALPHA-COSYMPLECTIC PSEUDO-METRIC MANIFOLDS ADMITTING RICCI SOLITONS

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Introduction

A systematic study of contact structures satisfying an associated pseudo-Riemann metric was introduced by Calvaruso and Perrone (Calvaruso & Perrone, 2010). This structure was first undertaken by Takahashi in Sasakian structures (Takashi, 1969). Contact pseudo metric structures (η, g) where η is a contact 1-form, and g is a pseudo Riemann metric associated with it. These structures are inherently generalizations of contact metric structures.

The class of almost contact metric manifolds, known as Kenmotsu manifolds, was first introduced by Kenmotsu (Kenmotsu, 1972). It is well known that Kenmotsu manifolds can be characterized through their Levi-Civita connection. Kenmotsu defined a structure closely related to the warped product, which was characterized by tensor equations.

A comprehensive investigation of almost Kenmotsu pseudo-metric manifolds remains outstanding in contemporary literature. Wang and Liu initiated the study of the geometry of almost Kenmotsu pseudo-metric manifolds (Wang & Liu, 2016). Their work highlights the analogies and distinctions with respect to the

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Riemannian metric tensor and they investigate classification results concerning local symmetry and nullity conditions.

Local symmetry is a substantial restriction for Kenmotsu manifolds. Furthermore, if Kenmotsu's structure satisfies Nomizu's condition (Nomizu, 1968), i.e., $R \cdot R = 0$ then it has negative constant curvature. If the Kenmotsu manifold is conformally flat, then the manifold is a space of constant negative curvature -1 for dimensions greater than 3. The tensor product $R \cdot R = 0$ defines the notion of a semi-symmetric manifold. For all vector fields on, where acts as a derivation on (Nomizu, 1968). Such a space is called a "semi-symmetric space" since the curvature tensor at a point is the same as the curvature tensor of a symmetric space (which can change with the point p). Thus, locally symmetric spaces are obviously semi-symmetric, but the converse is not true (Calvaruso & Perrone, 2002). Ogawa obtained that if a compact Kaehler manifold is semi-symmetric, then it is locally symmetric (Ogawa, 1977). These spaces were studied in the sense of a complete intrinsic classification by Szabó (Szabó, 1982).

We have a contact metric manifold (M, η, ξ, g) with the contact distribution

$$D = ker\eta \subset TM, dimD = 2n \tag{1}$$

Then we give the D-conformal curvature tensor B defined as follows:

Definition 1. Let (M, g) be a $(2n+1)$-dimensional Riemannian manifold $(n \geq 2)$. Then the D-conformal curvature tensor field on M defined as follows:

$$B(X,Y)Z = R(X,Y)Z \tag{2}$$

$$+\frac{1}{2n-2}[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX$$

$$-S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX]$$

$$-\frac{k-2}{2n-2}[g(X,Z)Y - g(Y,Z)X]$$

$$+\frac{k}{2n-2}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y\eta(Y)\eta(Z)X]$$

which is designed to measure conformal curvature only along D, essentially ignoring the Reeb direction ξ in a conformally invariant way. Here, $k = \frac{r+4n}{2n-1}$ and r is a scalar curvature of M (Chuman, 1983).

Instead of asking whether the whole manifold is conformally flat, we ask: Is the induced conformal structure on D flat? B is the contact metric adaptation of the usual Weyl tensor for this restricted question. k adjusts the coefficients so that B becomes conformally invariant with respect to contact-metric conformal transformations (those preserving the contact structure up to a conformal factor). D-conformally flat means the contact distribution D carries an induced conformal structure. That conformal structure is flat (locally conformally equivalent to IR^{2n} with the standard conformal structure). No condition is imposed on the reeb direction. Then the manifold may still have curvature in the vertical direction.

In differential geometry, evolution equations that deform Riemannian metrics according to their curvature often reveal deep insights into the structure and classification of manifolds. The best known instance is Hamilton's Ricci flow, which was introduced in 1982. This flow evaluates the Riemannian metric $g(t)$ on the manifold M by the partial differential equation

$$\frac{\partial}{\partial t}\big(g(t)\big) + 2S\big(g(t)\big) = 0,\ \ g(0) = g_0 \qquad (3)$$

where $g(t)$ is a one-parameter family of Riemannian metrics on a smooth manifold M, and S denotes the Ricci curvature tensor of the metric $g(t)$. The Ricci flow deforms an initial metric g_0 in the direction of its Ricci curvature, analogous to heat diffusion smoothing irregularities in temperature distributions. This curvature-driven flow tends to drive the metric toward a more uniform, canonical geometry, thereby serving as a potent instrument for probing the topological and geometric structure of manifolds. Its most renowned application lies in Grigori Perelman's proof of the Poincaré Conjecture, wherein the flow is employed as a dynamical system to examine three-dimensional manifolds. The Ricci flow functions as a natural geometric partial differential equation, whose evolution encodes profound information about the underlying manifold, with singularities developing at locations of concentrated curvature that subsequently reflect topological characteristics. Ricci solitons are a key concept within the investigation of the Ricci flow. They correspond to natural solutions whose structure is not affected by anything other than the diffeomorphism and scaling, suggesting that it is the fundamental geometric content of these solutions. Ricci solitons are important for a number of reasons. As indicated by Hamilton and stated in the precise form by Perelman, high-curvature regions of a Ricci flow with singularities, when rescaled uniformly (parabolic scaling) so as to "blow up" these regions, converge to Ricci solitons. In order to understand singularity formation and thus the subsequent surgery procedure in geometric analysis, having good understanding of classification of solitons (especially shrinking solitons) is tantamount.

In this context, the study of Ricci solitons including their existence, classification, uniqueness, rigidity, and stability constitutes a fundamental and dynamic area of research in contemporary geometry. Exploring their properties or investigating their topological consequences under different curvature conditions (conformal flat, Weyl conformal tensor, D-conformal flat, or within specific manifold classes) provides a deep insight into the relationship between curvature, topology, and geometric evolution equations.

The study is organized as follows: In introduction section, we shall give the short literature information of the study title. In preliminaries section, we shall present the concepts of the manifold theory and the next section is devoted to describe the basic formulas and some propositions of alpha-cosymplectic pseudo-metric manifolds. The last section contains the main results of the study. We shall give some results of alpha-cosymplectic pseudo-metric manifolds satisfying certain curvature tensor conditions. In particular, D-conformal semi-symmetric, Ricci D-conformal semi-symmetric, and D-conformal flat cases are considered on alpha-cosymplectic pseudo-metric manifolds admitting Ricci solitons.

Preliminaries

Let M be a $(2n+1)$-dimensional smooth manifold endowed with a triple (φ, ξ, η), where φ is a type of (1,1) tensor field, ξ is a vector field, η is a 1-form on M such that

$$\eta(\xi) = 1, \varphi^2 = -I + \eta \otimes \xi, \varphi(\xi) = 0$$
$$\eta \circ \phi = 0, rank(\varphi) = 2n \tag{4}$$

If M admits a Riemannian metric g, defined by

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$
$$\eta(X) = g(X, \xi) \tag{5}$$

then M is called almost contact structure (φ, ξ, η, g). Also, the fundamental 2-form Φ of M is defined by

$$\Phi(X, Y) = g(X, \varphi Y) \tag{6}$$

(Yano & Kon, 1984). If the Nijenhuis tensor vanishes, defined by

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y]$$

$$-\varphi[X,\varphi Y]+\varphi^2[X,Y]+2d\eta(X,Y)\xi \quad (7)$$

then (M,φ,ξ,η) is said to be normal (Blair, 1976). It is obvious that a normal almost Kenmotsu manifold is said to be Kenmotsu manifold. Let (M,g) be an n-dimensional Riemannian manifold. We denote by ∇ the covariant differentiation with respect to the Riemann metric g. Then we have

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z \quad (8)$$

The Ricci tensor of M is defined a

$$S(X,Y)\ =\sum_{i=1}^{n} R(X,e_i,Y,e_i) \quad (9)$$

where $\{e_1,e_2,\dots,e_n\}$ is a local orthonormal basis. Also, the Ricci operator Q is a tensor field of type (1,1) on M defined by

$$g(QX,Y)=S(X,Y) \quad (10)$$

for any vector fields (Blair, 1976). Almost contact metric manifolds such that η and Φ are closed called almost cosymplectic manifolds. Also, an almost contact metric manifold such that $d\eta=0$ and $d\Phi=2\eta\wedge\Phi$ is said to be an almost Kenmotsu manifold (Kenmotsu, 1972). An almost contact metric manifold is said to be an almost alpha-cosymplectic manifold if

$$d\eta=0,d\Phi=2\alpha(\eta\wedge\Phi) \quad (11)$$

Here, α is a real constant (Kim & Pak, 2005). It is obvious that a normal almost alpha-cosymplectic manifold is said to be an alpha-cosymplectic manifold.

Alpha-Cosymplectic Pseudo-Metric Manifolds

This section is devoted to give fundamental concepts of alpha-cosymplectic pseudo-metric manifolds. In particular, basic curvature properties of alpha-cosymplectic pseudo-metric manifolds are presented. Here, α is given by a smooth function on M such that $d\alpha\wedge\eta=0$.

A pseudo Riemannian metric g on M is said to be compatible with the almost contact structure (φ, ξ, η) if $g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y)$ where $\varepsilon = \pm 1$. A smooth manifold M furnished with an almost contact structure (φ, ξ, η) and a compatible pseudo Riemannian metric g is called an almost contact pseudo metric manifold which is denoted by $(M, \varphi, \xi, \eta, g)$. It is obvious that $g(\varphi X, Y) = -g(X, \varphi Y), \eta(X) = \varepsilon g(X, \xi),\ g(\xi, \xi) = \varepsilon$. An almost contact pseudo metric manifold satisfying Eq. (11) is said to be an almost alpha-cosymplectic pseudo-metric manifold for $\alpha \in IR$. When an almost alpha-cosymplectic pseudo-metric manifold M has a normal almost contact structure, we can say that it is an alpha-cosymplectic pseudo-metric manifold.

Proposition 1. Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact metric manifold. If M is an alpha-cosymplectic pseudo-metric manifold, then we have

$$\nabla_X \xi = -\alpha\varphi^2 X = \alpha[X - \eta(X)\xi] \tag{12}$$

$$(\nabla_X \varphi)Y = \alpha[\varepsilon g(\varphi X, Y)\xi - \eta(Y)\varphi X] \tag{13}$$

for $X, Y \in \chi(M)$ (Öztürk, 2020).

Proposition 2. Let $(M, \varphi, \xi, \eta, g)$ be an alpha-cosymplectic pseudo-metric manifold. Then we have

$$R(X, Y)\xi = [\alpha^2 + \xi(\alpha)][\eta(X)Y - \eta(Y)X] \tag{14}$$

$$R(X, \xi)Y = -[\alpha^2 + \xi(\alpha)][-\varepsilon g(Y, X)\xi + \eta(Y)X] \tag{15}$$

$$R(X, \xi)\xi - \varphi R(\varphi X, \xi)\xi = 2[\alpha^2 + \xi(\alpha)][-X + \eta(X)\xi] \tag{16}$$

$$\eta(R(X, Y)Z) = \varepsilon[\alpha^2 + \xi(\alpha)]$$

$$[-\eta(X)g(Y, Z) + \eta(Y)g(X, Z)] \tag{17}$$

$$S(X, \xi) = -2n[\alpha^2 + \xi(\alpha)]\eta(X) \tag{18}$$

$$Q\xi = -2n\varepsilon\,[\alpha^2 + \xi(\alpha)] \tag{19}$$

$$(\nabla_X \eta)Y = \alpha[\varepsilon g(X, Y) - \eta(X)\eta(Y)] \tag{20}$$

$$S(\varphi X, \varphi Y) = [\alpha^2 + \xi(\alpha)]$$
$$(\varepsilon S(X,Y) - 2n[g(X,Y) - \varepsilon\eta(X)\eta(Y)]). \qquad (21)$$

Here, α is defined by a smooth function such that $d\alpha \wedge \eta = 0$ and $\varepsilon = g(\xi, \xi)$, (Öztürk, 2021).

Definition 2. Let $(M, \varphi, \xi, \eta, g)$ be an alpha-cosymplectic pseudo-metric manifold. If the following condition holds

$$S(X,Y) = \lambda g(X,Y) + \varepsilon\mu\eta(X)\eta(Y) \qquad (22)$$

for any $X, Y \in \chi(M)$, then M is said to be an η-Einstein alpha-cosymlectic pseudo-metric manifold. Here, λ and μ are the arbitrary functions on M. In particular, M becomes an Einstein manifold when $\mu = 0$ (Blair, 1976).

Ricci Solitons

This section contains the fundamental concepts and basic curvature properties that will be used in the main results section.

Definition 3. Let (M, g_0) be a n-dimensional Riemannian manifold. Then, the Ricci flow that evolves the metric tensor g in the partial differential equation given by Eq. (3) is called the Ricci flow (Hamilton, 1982). Here, t is the time parameter.

The Ricci flow is an extraordinary technique that entered the history of mathematics due to Grigori Perelman's critical role in solving the Poincaré Conjecture in 2002. At its origin, it can be viewed as a partial differential equation governing the evolution of a manifold's metric structure over time. In other words, this flow reshapes the geometry of space according to Ricci curvature, much like a sculptor shaping marble. This equation governs the time-dependent change of the g metric tensor in accordance with the curvature properties of the manifold. The fundamental philosophy of the process is to smooth out the uneven curvature distribution of space, much like an iron. Interestingly, this smoothing process behaves differently depending on the sign of the curvature. Regions with positive curvature (spherical structures) shrink over time, much like a balloon shrinks when exposed to hot air. On the other hand, regions with negative curvature (hyperbolic structures) expand over time, much like the wrinkles in a piece of paper unfolding.

Definition 4. Let (M, g) be a complete Riemannian manifold. The metric g is called a Ricci soliton if there exists a smooth vector field X on M and a real constant λ such that the following equation holds

$$(L_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \qquad (23)$$

Here, the vector field V is the potential vector field of the Ricci soliton, and $L_V g$ is the Lie derivative of the metric g in the direction of V. In this case, the Ricci soliton is denoted by (M, g, V, λ). For the Ricci soliton (M, g, V, λ), the cases where $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$ are called, respectively, the shrinking, steady, and expanding Ricci solitons (Hamilton, 1988).

Ricci solitons, which hold a special role in Ricci flow theory, are the rigid form solutions of this dynamic process. For example, like a wave that travels through the ocean while maintaining its shape, Ricci solitons also preserve their fundamental geometric character throughout the flow. That is, they either remain completely unchanged or undergo only a change of scale. These structures play a key role in analyzing the long-term behavior of Ricci flow. The term "soliton" was first used in wave mechanics to describe self-preserving, localized solutions. Geometrically, a Ricci soliton reflects the self-similarity property of the metric tensor under the Ricci flow.

Definition 5. Let (M, g) be a n-dimensional Riemannian manifold. The Lie derivative associated with the metric g in the V direction is defined by

$$(L_V g)(X, Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V) \tag{24}$$

(Yano & Kon, 1984).

Definition 6. Let (M, g, V, λ) be a Ricci soliton. If the potential vector field V is a Killing vector field $(L_V g = 0)$, then (M, g, V, λ) is said to be a trivial Ricci soliton (Chen 2015).

Definition 7. Let $(M, \varphi, \xi, \eta, g)$ be an alpha-cosymplectic pseudo-metric manifold. If there exists a Ricci soliton (g, V, λ) on M, then (M, g, V, λ) is called an alpha-cosimplectic pseudo-metric manifold admitting a Ricci soliton (Hamilton, 1988), (Kenmotsu, 1972).

Proposition 3. Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional alpha-cosymplectic pseudo-metric manifold. Then the Ricci curvature tensor field holds

$$S(X, Y) = -(\lambda + \alpha) g(X, Y) + \varepsilon \alpha \eta(X) \eta(Y) \tag{25}$$

on (M, g, ξ, λ) Ricci soliton, where α is assumed to be parallel along the characteristic vector field ξ.

Proof. In view of Proposition 2, Eq. (23) and Eq. (24), the proof is obvious.

Proposition 4. Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional alpha-cosymplectic pseudo-metric manifold. Then the following curvature properties of (M, g, ξ, λ) Ricci soliton are held

$$S(X, \xi) = -[\varepsilon(\lambda + \alpha) - \alpha] \eta(X) \tag{26}$$

$$QX = -(\alpha + \lambda) X + \alpha \eta(X) \xi \tag{27}$$

$$Q\xi = -\lambda \xi - \alpha(1 - \varepsilon) \xi \tag{28}$$

$$S(\xi, \xi) = -\varepsilon \lambda \tag{29}$$

$$r = \varepsilon\alpha - (2n+\varepsilon)(\alpha+\lambda). \quad (30)$$

Here, α is assumed to be parallel along the characteristic vector field ξ.

Proof. As a result of Proposition 3, the proofs can be obtained by simple calculations. These are left entirely to the readers.

Main Results

In this section, some curvature tensor fields are studied on alpha-cosymplectic pseudo-metric manifolds admitting Ricci solitons. In particular, several results are obtained using the D-conformal curvature tensor field. Thus we state the following results:

Theorem 1. Let (M,φ,ξ,η,g) be a $(2n+1)$-dimensional alpha-cosymplectic pseudo-metric manifold $(n \geq 2)$. If the D-conformally semi-symmetric tensor product holds on (M,g,ξ,λ) Ricci soliton and α is parallel along the characteristic vector field ξ then the following statements satisfy:

(a) If ξ is space-like, then no Ricci soliton exists on M,

(b) If $\alpha = 0$ and ξ is time-like, the Ricci soliton behaves on M as follows:

$(\boldsymbol{b_1})$ $r = 0 \Rightarrow (g,\xi,\lambda)$ is expanding,

$(\boldsymbol{b_2})$ $r > 0 \Rightarrow (g,\xi,\lambda)$ is expanding if $0 < r \leq \frac{4n^2+14n-6}{4n^2-6n+1}$,

$(\boldsymbol{b_3})$ $r > 0 \Rightarrow (g,\xi,\lambda)$ is shrinking if $r > \frac{4n^2+14n-6}{4n^2-6n+1}$,

$(\boldsymbol{b_4})$ $r < 0 \Rightarrow (g,\xi,\lambda)$ is expanding.

(c) If $\alpha \neq 0$ and ξ is time-like, the Ricci soliton behaves on M as follows:

$(\boldsymbol{c_1})$ $r = 0 \Rightarrow (g,\xi,\lambda)$ is expanding,

$(\boldsymbol{c_2})$ $r > 0$ and $\alpha > 0 \Rightarrow (g,\xi,\lambda)$ is expanding, shrinking or steady,

(c_3) $r > 0$ and $\alpha < 0 \Rightarrow (g, \xi, \lambda)$ is expanding, shrinking or steady,

(c_4) $r < 0 \Rightarrow (g, \xi, \lambda)$ is expanding.

Proof. According to the hypothesis, we suppose that M is an alpha-cosymplectic D-conformally semi-symmetric pseudo-metric manifold. Now, let us introduce the D-conformal curvature tensor field B. If the D-conformal curvature tensor field B holds

$$R(X, Y) \cdot B = 0 \tag{31}$$

then M is said to be a D-conformal semi-symmetric manifold ($n \geq 2$) (Taleshian et al., 2011). In other words, we have

$$(R(X, Y) \cdot B)(Z, U)V = 0. \tag{32}$$

Then making use of Eqs. (31) and (32) we get

$$R(X, Y)B(Z, U)V - B(R(X, Y)Z, U)V$$
$$-B(Z, R(X, Y)U)V - B(Z, U)R(X, Y)V = 0. \tag{33}$$

With the help of Eqs. (14) and (25) by $X = \xi$, we deduce

$$[\alpha^2 + \xi(\alpha)][\eta(B(Z, U)V)Y - \varepsilon g(B(Z, U)V, Y)\xi - \eta(Z)B(Y, U)V]$$
$$+[\alpha^2 + \xi(\alpha)][\varepsilon g(Y, Z)B(\xi, U)V - \eta(U)B(Z, Y)V + \varepsilon g(Y, U)B(Z, \xi)V]$$
$$+[\alpha^2 + \xi(\alpha)][-\eta(V)B(Z, U)Y + \varepsilon g(V, Y)B(Z, U)\xi] = 0. \tag{34}$$

Then putting the inner product of both sides of Eq. (34) with respect to ξ, we have

$$[\alpha^2 + \xi(\alpha)][\varepsilon\eta(B(W, V)U)\eta(Z) - g(B(W, V)U, Z) - \varepsilon\eta(W)\eta(B(Z, V)U)] \tag{35}$$
$$+[\alpha^2 + \xi(\alpha)][g(Z, W)\eta(B(\xi, V)U) - \varepsilon\eta(V)\eta(B(W, Z)U) + g(Z, V)\eta(B(W, \xi)U)]$$
$$+[\alpha^2 + \xi(\alpha)][-\varepsilon\eta(U)\eta(B(W, V)Z) + g(U, Z)\eta(B(W, V)\xi)] = 0.$$

Taking $Y = Z$ and $\xi(\alpha) = 0$ in Eq. (35), it yields

$$\varepsilon\eta(B(Z,U)V)\eta(Z) - g(B(Z,U)V,Z) - \varepsilon\eta(Z)\eta(B(Z,U)V)$$
$$+g(Z,Z)\eta(B(\xi,U)V) - \varepsilon\eta(U)\eta(B(Z,Z)V)$$
$$+ g(Z,U)\eta(B(Z,\xi)V)$$
$$-\varepsilon\eta(V)\eta(B(Z,U)Z) + g(V,Z)\eta(B(Z,U)\xi) = 0. \quad (36)$$

Furthermore, using Eq. (2) it follows that

$$\eta(B(Z,U)V = L[g(Z,V)\eta(U) - g(U,V)\eta(Z)] \quad (37)$$

where L is defined by

$$L = \frac{1+\varepsilon\alpha^2(n-2)}{2(n-1)}. \quad (38)$$

Taking into account of Eqs. (36) and (37), we find

$$g(B(Z,U)V,Z)$$
$$= \varepsilon g(Z,Z)[L\eta(U)\eta(V) - Lg(U,V)]$$
$$+ \varepsilon g(Z,U)[Lg(Z,V) - L\eta(Z)\eta(V)]$$
$$-\varepsilon\eta(V)[L\eta(U)g(Z,Z) - L\eta(Z)g(Z,V)]. \quad (39)$$

In view of Eq. (39) we deduce

$$g(B(Z,U)V,Z) = \varepsilon L[g(Z,U)g(Z,V) - g(Z,Z)g(U,V)]. \quad (40)$$

Let $E_j = \{e_1, \cdots, e_n, \varphi e_1, \cdots, \varphi e_n, \xi\}$, $j = 1, \cdots, n$ be a local orthonormal φ-basis. Then taking contraction in Eq. (40) with respect to $Z = E_j$, we get

$$\sum_{j=1}^{2n+1} g(B\big(E_j, U\big)V, E_j) = \varepsilon F(1 - \varepsilon(2n + 1))g(U,V). \quad (41)$$

Also, using Eq. (2), it follows that

$$\sum_{j=1}^{2n+1} g(B\big(E_j, U\big)V, E_j) = S(U,V) \quad (42)$$
$$+\frac{1}{2n-2}\big[2S(U,V) - \varepsilon(2n+1)S(U,V) - rg(V,U) + \varepsilon S(U,V) +$$
$$r\eta(V)\eta(U) + 2n\alpha^2\eta(V)\eta(U)[\varepsilon + 1]\big] - \frac{k-2}{2n-2}[g(V,U) -$$
$$\varepsilon(2n+1)g(V,U)] + \frac{k}{2n-2}[-\varepsilon g(V,U) + (1 - 2n\varepsilon)\eta(V)\eta(U)].$$

By the help of Eqs. (41) and (42), we have

$$S(U,V) = Fg(U,V) + G\eta(U)\eta(V) \tag{43}$$

where F and G are defined as

$$F = \frac{-2r(n-1)+2(1-\varepsilon)+4n\varepsilon(k-1)-k(1+2\varepsilon)}{2n(\varepsilon-1)} + \frac{(\varepsilon-1)(n\alpha^2-2\alpha^2-1)+2n(1-n\alpha^2\varepsilon)}{2n(\varepsilon-1)} \tag{44}$$

and

$$G = \frac{r+2n\alpha^2(\varepsilon+1)+k(1-2n\varepsilon)}{2n(\varepsilon-1)}, \tag{45}$$

respectively.

Additionally, from Eq. (43) with $V = \xi$, we get

$$S(U,\xi) = F\varepsilon\eta(U) + G\varepsilon\eta(U) \tag{46}$$

and Eq. (26) can be written as

$$S(U,\xi) = -[\varepsilon(\lambda + \alpha) - \alpha]\eta(U). \tag{47}$$

So if we consider Eqs. (46) and (47) together, we have

$$\lambda = -\varepsilon(1-\alpha) - (F + G). \tag{48}$$

We note that $k = \frac{r+4n}{2n-1}$, $n \geq 2$, $\alpha, \lambda \in IR$. Taking account of Eqs. (44), (45) and (48), we obtain

$$\lambda = 1 - \alpha - \frac{1}{2n(\varepsilon-1)}[r(-2n+3) + 2n + 3 + (n+2)\alpha^2] - \frac{\varepsilon\left[-3-4n+\frac{2(n-1)(r+4n)}{2n-1}+(-2n^2+3n-2)\alpha^2\right]}{2n(\varepsilon-1)}. \tag{49}$$

From Eq. (49), we can be easily seen that when $\varepsilon = 1$ (ξ is space-like) the real solution for λ can not be calculated. Therefore, the first part of the theorem is proved. Now, let us consider Eq. (49) for

$\alpha = 0$. The solutions of Eq. (49) depending on the sign of r. So, there are three cases. If we choose ξ to be time-like we have

$$\lambda = 1 + \frac{1}{4n}\left[\frac{4n^2+14n-6}{2n-1}\right] \tag{50}$$

for $r = 0$. This means that $\lambda > 0$. Also, if $r > 0$, using Eq. (49) becomes

$$\lambda = 1 + \frac{1}{4n}\left[\frac{(-4n^2+6n-1)r+(4n^2+14n-6)}{2n-1}\right]. \tag{51}$$

Then the solutions of λ are as follows:

$$\begin{aligned} \lambda > 0 \quad & if \quad 0 < r < \frac{4n^2+14n-6}{4n^2-6n+1} \\ \lambda = 1 \quad & if \quad r = \frac{4n^2+14n-6}{4n^2-6n+1} \\ \lambda < 0 \quad & if \quad r > \frac{4n^2+14n-6}{4n^2-6n+1} \end{aligned} \tag{52}$$

Under the condition of $r < 0$, with the help of Eq. (49) it follows that $\lambda > 0$. Thus, it completes the second part of the proof, as shown in Eqs. (50) and (52). Finally, we investigate the case of $\alpha \neq 0$ and ξ is time-like depending on r. If $r = 0$, then $\lambda > 0$. In a similar way, when $\alpha > 0$ (or $\alpha < 0$) and $r < 0$, the λ values are still positive. However, when $r > 0$, the case becomes more complicated. For example, if $\alpha < 0$ and $r > 0$, we have the following cases. If $r \leq r_1 \Rightarrow \lambda > 0, \forall \alpha < 0$. Also, if $r > r_1$, then the sign of λ depends on α as follows:

$$\begin{aligned} \alpha < \alpha_1 \Rightarrow \quad & \lambda > 0 \\ \alpha = \alpha_1 \Rightarrow \quad & \lambda = 0 \\ \alpha_1 < \alpha < 0 \Rightarrow \quad & \lambda < 0. \end{aligned} \tag{53}$$

Here, $r_1 = \frac{4n^2+14n-6}{4n^2-6n+1}$ and α_1 is the negative root of $\lambda(\alpha) = 0$. The analogy holds for the case where $\alpha > 0$. Thus, the proof ends.

Theorem 2. Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional alpha-cosymplectic pseudo-metric manifold $(n \geq 2)$ and $\nabla_\xi \alpha = 0$. If the

Ricci D-conformal semi-symmetric tensor product holds on (M, g, ξ, λ) Ricci soliton, then the following statements satisfy:

(a) If $\alpha = 0$ and ξ is space-like, (g, ξ, λ) is shrinking,

(b) If $\alpha = 0$ and ξ is time-like, (g, ξ, λ) is expanding,

(c) If $\alpha \neq 0$ and ξ is space-like (or time-like), (g, ξ, λ) is expanding, shrinking or steady depending on α.

Proof. Assume that M is a Ricci D-conformal semi-symmetric pseudo-metric manifold which means

$$B(X, Y) \cdot S(Z, U) = 0 \tag{54}$$

for $n \geq 2$. Thus Eq. (54) becomes

$$S(B(X, Y)Z, U) + S(Z, B(X, Y)U) = 0. \tag{55}$$

Then taking $X = U = \xi$ in Eq. (55) we have

$$S(B(\xi, Y)Z, \xi) + S(Z, B(\xi, Y)\xi) = 0. \tag{56}$$

where $\xi(\alpha) = 0$. Furthermore, from Eq. (37) we deduce

$$B(\xi, Z)U = L[\varepsilon\eta(U)Z - \xi g(Z, U)] \tag{57}$$

and

$$B(\xi, Z)\xi = L[Z - \varepsilon\eta(Z)\xi]. \tag{58}$$

Taking into account of Eqs. (57) and (58) in Eq. (56) we get

$$S(Y, Z) = -2n\alpha^2\varepsilon g(Y, Z). \tag{59}$$

On the other hand, using Eq. (59) with $Z = \xi$, we have

$$S(Y, \xi) = -2n\alpha^2\eta(Y) \tag{60}$$

and Eq. (26) can be written as

$$S(Y, \xi) = -[\varepsilon(\lambda + \alpha) - \alpha]\eta(Y). \tag{61}$$

So from Eqs. (60) and (61), we obtain

$$\lambda = \varepsilon[-1 + \alpha(1 + 2n\alpha)]. \tag{62}$$

Then if $\alpha = 0$ and ξ is space-like in Eq. (62), we get

$$\lambda < 0, \qquad \lambda = -1,$$

and if $\alpha = 0$ and ξ is time-like in Eq. (62) we have

$$\lambda > 0, \qquad \lambda = 1,$$

where λ is defined by

$$\lambda = -2n\alpha^2 - \alpha + 1.$$

Thus, the proof of the first two cases are obvious. When $\alpha \neq 0$ and ξ are space-like (or time-like), the last case depends on α. If we apply the same methodology as in the last case of Theorem 1, we obtain $\lambda > 0$ or $\lambda < 0$. It completes the proof.

Theorem 3. Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional D-conformally flat alpha-cosymplectic pseudo-metric manifold $(n \geq 2)$. If α is parallel along the characteristic vector field ξ then the following statements satisfy on (M, g, ξ, λ) Ricci soliton:

(a) If ξ is space-like, then no Ricci soliton exists on M,

(b) If ξ is time-like, the Ricci soliton behaves on M as follows:

($\boldsymbol{b_1}$) $\alpha < 1 \Rightarrow (g, \xi, \lambda)$ is expanding,

($\boldsymbol{b_2}$) $\alpha = 0 \Rightarrow (g, \xi, \lambda)$ is expanding,

($\boldsymbol{b_3}$) $\alpha = 1 \Rightarrow (g, \xi, \lambda)$ is steady,

($\boldsymbol{b_4}$) $\alpha > 1 \Rightarrow (g, \xi, \lambda)$ is shrinking.

Proof. According to the hypothesis, let us assume that M is a D-conformally flat alpha-cosymplectic pseudo-metric manifold. Namely, we have

$$B(X, Y)Z = 0. \tag{63}$$

By the help of Eqs. (2) and (63), it yields

$$R(X,Y)Z = -\frac{1}{2n-2}[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY - g(Y,Z)QX - S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX] + \frac{k-2}{2n-2}[g(X,Z)Y - g(Y,Z)X]$$
$$-\frac{k}{2n-2}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X]. \quad (64)$$

Then taking the inner product on both sides of Eq.(64) with respect to U, we get

$$g(R(X,Y)Z,U) = -\frac{1}{2n-2}[S(X,Z)g(Y,U) - S(Y,Z)g(X,U) + g(X,Z)g(QY,U) \quad (65)$$
$$-g(Y,Z)g(QX,U) - \varepsilon S(X,Z)\eta(Y)\eta(U) + \varepsilon S(Y,Z)\eta(X)\eta(U) - \eta(X)\eta(Z)g(QY,U) + \eta(Y)\eta(Z)g(QX,U)]$$
$$+\frac{k-2}{2n-2}[g(X,Z)g(Y,U) - g(Y,Z)g(X,U)]$$
$$-\frac{k}{2n-2}[\varepsilon g(X,Z)\eta(Y)\eta(U) - \varepsilon g(Y,Z)\eta(X)\eta(U) + \eta(X)\eta(Z)g(Y,U) - \eta(Y)\eta(Z)g(X,U)]$$

where $R(X,Y,Z,U) = g(R(X,Y)Z,U)$. Taking into account of Eqs.(17) and (18), Eq. (46) provides

$$\eta(R(X,Y)Z) = -\varepsilon[\alpha^2 + \xi(\alpha)][\eta(X)g(Y,Z) + \eta(Y)g(X,Z)]$$
$$=-\frac{\varepsilon}{2n-2}\{[(\varepsilon-1)\eta(Y)S(X,Z) + (1-\varepsilon)\eta(X)S(Y,Z)$$
$$-2n[\alpha^2 + \xi(\alpha)][\eta(Y)S(X,Z) - \eta(X)S(Y,Z)]\}$$
$$+\frac{k-2}{2n-2}[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]$$
$$-\frac{\varepsilon k}{2n-2}[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)] \quad (66)$$

for $U = \xi$. Putting $Y = \xi$ in Eq. (66) we have

$$-\frac{\varepsilon}{2n-2}[(\varepsilon-1)\varepsilon S(X,Z)$$
$$-2n[\alpha^2 + \xi(\alpha)](\eta(X)\eta(Z) + \varepsilon g(X,Z))] \quad (67)$$
$$+\frac{(k-2)}{2n-2}\varepsilon[g(X,Z) - \eta(X)\eta(Z)] - \frac{k}{2n-2}[g(X,Z) - \eta(X)\eta(Z)]$$

$$= [\alpha^2 + \xi(\alpha)][g(X,Z) - \eta(X)\eta(Z)].$$

Next, simplifying Eq. (48) for $\xi(\alpha) = 0$, we obtain

$$S(X,Z) = \left(\frac{2n-2}{\varepsilon-1}\right)\left[\frac{\varepsilon(k-2)-k+2n\alpha^2}{2n-2} - \alpha^2\right] g(X,Z)$$
$$+\left(\frac{2n-2}{\varepsilon-1}\right)\left[\frac{-\varepsilon(k-2)+k+2n\varepsilon\alpha^2}{2n-2} + \alpha^2\right]\eta(X)\eta(Z). \qquad (68)$$

So Eq. (68) turns into

$$S(X,Z) = \left[\frac{2(\alpha^2-\varepsilon)}{\varepsilon-1} + k\right] g(X,Z)$$
$$+\left[\frac{2\left(\varepsilon-\alpha^2+n\alpha^2(\varepsilon+1)\right)}{\varepsilon-1} - k\right]\eta(X)\eta(Z). \qquad (69)$$

Here, we defined by A and B

$$A = \frac{2(\alpha^2-\varepsilon)}{\varepsilon-1} + k \ , B = \frac{2\left(\varepsilon-\alpha^2+n\alpha^2(\varepsilon+1)\right)}{\varepsilon-1} - k$$

respectively. On the other hand, using Eq. (69) with $Z = \xi$, we have

$$S(X,\xi) = \varepsilon[A + B]\eta(X) \qquad (70)$$

and Eq. (26) holds

$$S(X,\xi) = -[\varepsilon(\lambda + \alpha) - \alpha]\eta(X). \qquad (71)$$

In view of Eqs. (70) and (71), we obtain

$$\lambda = -\varepsilon(1-\alpha) - \frac{2n\alpha^2(\varepsilon+1)}{\varepsilon-1}. \qquad (72)$$

Then from Eq. (72), when $\varepsilon = 1$, there is no Ricci soliton on M. If we choose ξ to be time-like, we obtain

$$\lambda = 1 - \alpha \qquad (73)$$

Thus the proof is clear using Eq. (73) as follows:

$$\begin{aligned} \lambda > 0 &\quad \Leftrightarrow \quad \alpha < 1 \\ \lambda = 0 &\quad \Leftrightarrow \quad \alpha = 1 \\ \lambda < 0 &\quad \Leftrightarrow \quad \alpha > 1. \end{aligned} \tag{74}$$

Theorem 4. Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional φ-D-conformally flat alpha-cosymplectic pseudo-metric manifold ($n \geq 2$). If α is parallel along the characteristic vector field ξ then the following statements satisfy on (M, g, ξ, λ) Ricci soliton:

(a) If $\alpha = 0$ and ξ is space-like, (g, ξ, λ) is shrinking,

(b) If $\alpha = 0$ and ξ is time-like, no Ricci soliton exists on M,

(c) If $\alpha \neq 0$ and ξ is space-like (or time-like), (g, ξ, λ) is expanding or shrinking depending on α.

Proof. Let us suppose that M is an alpha-cosymplectic pseudo-metric manifold satisfying the φ-D-conformally flat condition. Then M holds

$$g(B(\varphi X, \varphi Y)\varphi Z, \varphi V) = 0. \tag{75}$$

Making use of Eqs.(2) and (75), we get

$$\begin{aligned} g(R(\varphi X, \varphi Y)\varphi Z, \varphi V) + \tfrac{1}{2n-2}[S(\varphi X, \varphi Z)g(\varphi Y, \varphi V) - \\ S(\varphi Y, \varphi Z)g(\varphi X, \varphi V) \\ +S(\varphi Y, \varphi V)g(\varphi X, \varphi Z) - S(\varphi X, \varphi V)g(\varphi Y, \varphi\phi Z)] \\ -\tfrac{k-2}{2n-2}[g(\varphi X, \varphi Z)g(\varphi Y, \varphi V) - g(\varphi Y, \varphi Z)g(\varphi X, \varphi V)] = 0. \end{aligned} \tag{76}$$

Taking into account of Eqs.(17) and (21), Eq. (76) takes the form

$$\begin{aligned} \varepsilon[\alpha^2 + \xi(\alpha)][g(\varphi X, \varphi W)g(\varphi Y, \varphi\phi U) - g(\varphi Y, \varphi W)g(\varphi X, \varphi U)] \\ + \tfrac{[\alpha^2+\xi(\alpha)]}{2n-2}[\varepsilon S(X, Z)g(Y, V) - 2ng(X, Z)g(Y, V) + \\ 2n\varepsilon\eta(X)\eta(Z)g(Y, V) \\ -S(X, Z)\eta(Y)\eta(V) + 2n\varepsilon\eta(Y)\eta(V)g(X, Z) - S(Y, Z)g(X, V) \\ +2ng(X, V)g(Y, Z) - 2n\varepsilon\eta(Y)\eta(Z)g(X, V) + S(Y, Z)\eta(X)\eta(V) \\ -2n\varepsilon\eta(X)\eta(V)g(Y, Z) + \varepsilon S(Y, V)g(X, Z) - 2ng(X, Z)g(Y, V) \end{aligned}$$

$$+2n\varepsilon\eta(Y)\eta(V)g(X,Z) - S(Y,V)\eta(X)\eta(Z) + 2n\varepsilon\eta(X)\eta(Z)g(Y,V)$$

$$-\varepsilon S(X,V)g(Y,Z) + 2ng(X,V)g(Y,Z) - 2n\varepsilon\eta(X)\eta(V)g(Y,Z) + S(X,V)\eta(Y)\eta(Z) + 2n\varepsilon\eta(Y)\eta(Z)g(X,V)]$$

$$-\frac{\varepsilon(k-2)}{2n-2}[\varepsilon g(X,Z)g(Y,V) - g(X,Z)\eta(Y)\eta(V) - g(Y,V)\eta(X)\eta(Z)$$

$$-\varepsilon g(Y,Z)g(X,V) + g(Y,Z)\eta(X)\eta(V) + g(X,V)\eta(Y)\eta(Z)] = 0. \qquad (77)$$

Let $E_j = \{e_1, \cdots, e_n, \varphi e_1, \cdots, \varphi e_n, \xi\}$, $j = 1, \cdots, n$ be a local orthonormal φ-basis on M. Then taking contraction in Eq. (77) with respect to $X = V = E_j$ and $\xi(\alpha) = 0$, we obtain

$$S(Y,Z) = -\frac{E_1}{E_3}g(Y,Z) - \frac{E_2}{E_3}\eta(Y)\eta(Z) \qquad (78)$$

Here, the functions as shown in Eq. (78) are as follows:

$$E_1 = \alpha^2[\varepsilon + 2(n+1)] + b(2n\varepsilon - 1) - a\alpha^2(6n + \varepsilon r) + 2na\alpha^2\varepsilon(4n+1)$$

$$E_2 = \alpha^2(2n\varepsilon - 1) - b(2n-1) + a[\alpha^2 r - 4n\alpha^4(2n-1) + 2n\alpha^2(\varepsilon+1)]$$

$$E_3 = \left[\frac{\alpha^2(\varepsilon-n)}{n-1}\right],\ a = \frac{1}{2n-2},\ b = \frac{k-2}{2n-2},\ k = \frac{r+4n}{2n-1}.$$

On the other hand, make use of Eq. (78) with $Z = \xi$, we find

$$S(Y,\xi) = -\frac{\varepsilon}{E_3}[E_1 + E_2]\eta(Y) \qquad (79)$$

and Eq. (26) satisfies

$$S(Y,\xi) = -[\varepsilon(\lambda+\alpha) - \alpha]\eta(Y). \qquad (80)$$

In view of Eqs. (79) and (80), we have

$$\lambda = \frac{n-1}{\varepsilon-n}\left[(2n+1)(\varepsilon+1) + \frac{-4n+r+\varepsilon(8n^2+4n-r)}{2(n-1)}\right]$$

$$+\frac{n-1}{\varepsilon-n}\left[\frac{n(r+2)(\varepsilon-1)}{(n-1)(2n-1)\alpha^2} - \frac{2n(2n-1)}{n-1}\alpha^2\right] - \varepsilon + \frac{\alpha}{\varepsilon}. \qquad (81)$$

Then from Eq. (81), if $\varepsilon = 1$ and $\alpha = 0$, we obtain $\lambda < 0$. Also, when $\varepsilon = -1$ and $\alpha = 0$, there are no Ricci solitons because λ is undefined. If $\varepsilon = 1$ and $\alpha \neq 0$, then we can write

$$\lambda = -4n - 3 + \alpha + \frac{-4n^2 + (4n^2 - 2n)\alpha^2}{n-1} \tag{82}$$

The solutions of Eq. (82) depends on the value of α. In this case, the following relations are held:

$$\begin{aligned} &\alpha < \alpha_1 \; or \; \alpha > \alpha_2 \Rightarrow \quad \lambda > 0 \\ &\alpha_1 < \alpha < \alpha_2 \qquad\quad \Rightarrow \quad \lambda < 0 \end{aligned}$$

where $\alpha_1 < 0 < \alpha_2$ are roots of

$$2n(2n-1)\alpha^2 + (n-1)\alpha + (-8n^2 + n + 3) = 0.$$

Thus completes the proof.

References

Begawadi, C. S. & Kumar, E. G. & Venkatesha, V. (2005). On irrational D-conformal curvature tensor. *Novi Sad Journal of Mathematics, 35* (2), 85-92.

Besse, A. L. (1987). *Einstein manifolds*. NY: Springer Science and Business Media.

Blair, D. E. (1976). *Contact manifolds in Riemannian geometry*. NY: Lecture Notes in Mathematics, Vol. 509, Springer-Verlag, Berlin-Heidelberg Press.

Calvaruso, G. & Perrone, D. (2002). Semi-symmetric contact metric three-manifolds. *Yokohama Mathematical Journal, 49,* 149-161.

Calvaruso, G. & Perrone, D. (2010). Contact pseudo-metric manifolds. *Differential Geometry and its Applications, 28,* 615-634.

Chen, B. (2015). Some results on concircular vector fields and their applications to Ricci solitons. *Bulletin of the Korean Mathematical Society*, *52(5)*, 1535-1547.

Chuman, G. (1983). On the D-conformal curvature tensor. *Tensor*, *40* (2), 125-134.

Dileo, G. & Pastore A. M. (2007). Almost Kenmotsu manifolds and local symmetry. *Bulletin of the Belgian Mathematical Society-Simon Stevin, 14,* 343-354.

Hamilton, R. S. (1982). Three-manifolds with positive Ricci curvature. *Journal of Differential Geometry, 17(2)*, 255-306.

Hamilton, R. S. (1988). The Ricci flow on surfaces, Mathematics and general relativity (Santa Cruz, CA, 1986). *Contemporary Mathematics*, *71*, 301-307.

Janssens, D. & Vanhecke, L. (1981). Almost contact structures and curvature tensors. *Kodai Mathematical Journal, 4,* 1-27.

Kenmotsu, K. (1972). A class of contact Riemannian manifold. *Tohoku Mathematical Journal, 24,* 93-103.

Kim, T. W. & Pak, H. K. (2005). Canonical foliations of certain classes of almost contact metric structures. *Acta Mathematica Sinica, 21,* 841-846.

Nomizu, K. (1968). On hypersurfaces satisfying a certain condition on the curvature tensor. *Tôhoku Mathematical Journal, 20,* 46-69.

Ogawa, Y. (1977). A condition for a compact Kaehlerian space to be locally symmetric. *Natural Science Report of the Ochanomizu University, 28,* 21-23.

Olszak, Z. (1981). On almost cosymplectic manifolds. *Kodai Mathematical Journal*, 4, 239-250.

O'Neil, B. (1983). *Semi-Riemannian geometry with applications to relativity*. New York: Academic Press.

Öztürk, H. (2017). On α-Kenmotsu manifolds satisfying semi-symmetric conditions. *Konuralp Journal of Mathematics, 5,* 192-206.

Öztürk, H., Mısırlı, İ. & Öztürk, S. (2017). Almost α-cosymplectic manifolds with η-parallel tensor fields, *Academic Journal of Science, 7(3),* 605-612.

Öztürk, S. & Öztürk, H. (2020). On α-Kenmotsu pseudo metric manifolds. *Afyon Kocatepe University Journal of Science and Engineering, 20,* 975-982.

Öztürk, S. & Öztürk, H. (2021). Almost α-cosymplectic pseudo metric manifolds. *Journal of Mathematics, 2021,* Article ID 4106025, 1-10.

Öztürk, S. (2022). A study on α-Kenmotsu manifolds. Canan Demir (Ed.), *Science and mathematics sciences, theory, current researches and new trends 5* (pp. 45-58). Montenegro: Ivpe.

Öztürk, S. & Öztürk, H. (2023). Three-dimensional semi-symmetric almost α-cosymplectic manifolds. *Symmetry 2023, 15, 2022,* 1-17.

Öztürk, H. & Öztürk, S. (2022). Certain curvature tensor fields on α-Kenmotsu Pseudo Metric Manifolds. Hikmet Y. Çoğun & İshak Parlar (Eds.), *Current Debates on Natural and Engineering Sciences 5* (pp. 384-395). Ankara: BIDGE Publication.

Öztürk, H. (2024). Some results on α-Kenmotsu manifolds admitting Ricci Solitons. İlker Eryılmaz (Ed.), *Unified perspectives in mathematics and geometry* (pp. 119-141). Ankara: BIDGE Publication.

Perelman, G. (2002). The entropy formula for the Ricci flow and its geometric applications. *arXiv preprint math/0211159.*

Szabó, Z. I. (1982). Structure theorem on Riemannian spaces satisfying $R \cdot R = 0$. *Journal of Differential Geometry, 17,* 531-582.

Takahashi, T. (1969). Sasakian manifold with pseudo-Riemannian metric. *Tôhoku Mathematical Journal, 21,* 271-290.

Taleshian, A. & Hosseinzadeh, A. A. & Khaniani, F. (2011). On D-conformal curvature tensor Sasakian manifolds. *Journal of Nonlinear Analysis and Application, 2011,* 1-5.

Wang, Y. & Liu, X. (2016). Almost Kenmotsu pseudo-metric manifolds. *Analele Stiintifice ale Universitatii Al I Cuza din Iasi-Matematica, 62,* 241-256.

Yano, K. & Kon, M. (1984). *Structures on manifolds*. Singapore: World Scientific Publishing Co.

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