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İÇİNDEKİLER

FROM SIMPLICIAL R-ALGEBROIDS TO 2-CROSSED MODULES OF R-ALGEBROIDS

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Introduction

Whitehead introduced crossed modules of groups for the first time in (Whitehead ,1941:409), (Whitehead, 1946: 806). Group crossed modules are equivalent to simplicial groups with Moore complex of length one (Conduche, 1984:155) and similarly for groupoid crossed modules (Mutlu & Porter, 1998: 174). Conduche addressed the idea of a group 2-crossed module and shown in (Conduche, 1984:155) that the category of group 2-crossed modules is equal to the category of simplicial groups with a two-length Moore complex. Arvasi and Ulualan investigated the relationships between simplicial groups with a length of two Moore complex, crossed squares, quadratic modules, and 2-crossed modules in (Arvasi & Ulualan, 2006:1). The

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definitions of algebra crossed and 2-crossed modules (Arvasi & Porter, 1996 : 426), (Arvasi & Porter, 1998: 455), (Doncel & Grandjean, 1992: 131), (Porter, 1986: 458) are similar to those of the group case, actions by the automorphisms is replaced by the actions by the multipliers, this algebra action is discussed in (Ege Arslan & Hürmetli 2021:72), (Ege Arslan, 2023: 1), (Arvasi & Ege 2003: 478) and its different properties are examined. In (Gülsün Akay, 2025: 565), it is shown that homotopy relation is an equivalence relation on morphisms between free simplicial algebras.

Algebra 2-crossed modules and simplicial algebras are closely related, just like in the group case case (Conduche, 1984:155), (Mutlu & Porter, 1998: 174), (Mutlu & Porter, 1998: 148). A 2-crossed module can be obtained from a simplicial algebra if it has a Moore complex of length two. Equivalence from category of simplicial algebra with a two-length Moore complex to category algebra crossed module is given in (Porter, 1986: 458), (Arvasi 1997:160), (Grandjean, M.J. Vale, 1986). Also in (Ege Arslan & ark., 2019: 5293), (Özel, Ege Arslan &Akça, 2024), (Ege Arslan & Kaplan, 2022: 17), (Ege Arslan, 2019:150) a higher-dimensional categorical perspective on 2-crossed modules, fibrations of 2-crossed modules and functorial relations are examined. As a more broadly, Mitchell in (Mitchell, 1972:1), (Mitchell, 1985: 96.) and Amgott in (Amgott, 1986: 1) specifically studied R-algebroids, where R is a commutative ring. R-algebroids were defined categorically by Mitchell. Later, Mosa introduced crossed modules of R-algebroids as a generalization of crossed modules of associative R-algebras and demonstrated in his thesis (Mosa,1986) that they are equivalent to special double R-algebroids with connections. Additionally, it was mentioned in (Gürmen & Ulualan, 2020: 113) that there was a close relationship between the category of simplicial R-algebroids with the length one Moore complex and the internal categories in the category of R-algebroids. Subsequent investigations by Akça and Avcıoğlu

(Avcıoğlu & Akça, 2017: 37), (Avcıoğlu & Akça, 2018: 2863), (Avcıoğlu & Akça, 2017), (Avcıoğlu & Akça, 2017), (Akça & Avcıoğlu, 2022) delve deeper into crossed modules of R-algebroids, unraveling intricate connections and properties. As a generalization of the 2-crossed module of commutative algebras, we present the 2-crossed module of R-algebroids in this work. Next, we construct a functor from the category of simplicial R-algebroids with Moore complex of length two to the category of 2-crossed modules of R-algebroids. This chapter produced from Ph. D. Thesis of I. Doğanay Yalğın, (Doğanay Yalğın, 2024).

Preliminaries

Most of the following data, can be found in (Mitchell, 1972:1), (Mitchell, 1978: 867), (Mitchell, 1985: 96), (Amgott, 1986: 1) and (Mosa, 1986).

Let R be a commutative ring. An R-category is a category where composition is R-bilinear and all homsets possess R-module structures. This framework enables the exploration of categorical concepts and constructions within the realm of R-modules, offering a robust foundation for algebraic and categorical inquiries.

An R-algebroid is a small R-category. R- algebroids can be non identity. A set of functions $s, t : Mor(U) \rightarrow Ob(U)$, the source and target functions, respectively, and an object set Ob(U)=U_0, a morphism set Mor(U), are included with an R-algebroid U.

A single object R-algebroid corresponds to an associative R-algebra. Let U and V be R-algebroids and $U_0 = V_0$, if the family of maps

$$\begin{array}{ccc} V(a,b) \times V(b,c) & \rightarrow & V(a,c) \\ (v,u) & \rightarrow & v^u \end{array}$$

satisfies the following conditions

1) $v^{u_1+u_2} = v^{u_1} + v^{u_2}$

2) $(v_1 + v_2)^u = v_1^u + v_2^u$

3) $(v^u)^{u'} = v^{uu'}$

4) $v'v^u = v'v^u$

5) $r \cdot v^u = (r \cdot v)^u = v^{r \cdot u}$

6) $v^{1_{tv}} = v$

for all $a, b, c \in U_0$ and $u, u', u_1, u_2 \in Mor(U), v, v', v_1, v_2 \in Mor(V)$ such that $t(v') = s(v), t(u) = s(u'), t(v) = t(v_1) = t(v_2) = s(u) = s(u_1) = s(u_2)$, $r \in R$, it is called the right action of *U* on *V* .

The left action of *U* on *V* similarly defined. While *U* has right and left action on *V* if the condition $({}^u v)^{u'} = {}^u(v^{u'})$ is satisfied for all $a, b, c, d \in U_0, v \in V(a, b),\ u \in U(d, a)$ and $u' \in U(b, c)$ then *U* has an associative action on *V* .

An R-functor is an R-linear functor between two R-categories, and an R-algebroid morphism is an R-functor between two R-algebroids.

In category *Alg*(*R*), all R-algebroids and their morphisms are included.

Let *R* is an commutative ring *U* and *V* be two R-algebroids of the same object set U_0 and *V* has an associative action on *U* . For the set

$$U \rtimes V = \{(u, v): u \in U, v \in V\},$$

if the following conditions are satisfied

1) $(u, v) + (u', v') = (u + u', v + v')$

2) ${}^r(u, v) = ({}^r u, {}^r v)$

3) $(u, v)(u'', v'') = (uu'' + uv'' + vu'', vv'')$

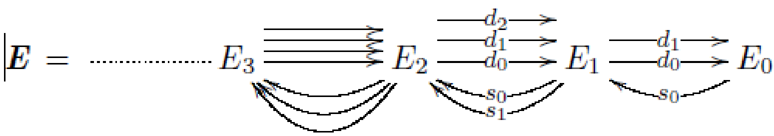
$U \rtimes V$ is an R-algebroid, where for all $(u, v) \in U \rtimes V$ and $r \in R$, $s(u, v) = s(u) = s(v), t(u, v) = t(u) = t(v)$, $(u, v), (u', v), (u', v'), (u'', v'') \in U \rtimes V$, $s(u,v) = s(u', v')$, $t(u, v) = t(u', v')$, $t(u, v) = s(u'', v'')$.

This R-algebroid is called the semi-direct product R-algebroid of U and V.

A simplicial R-algebroid is a sequence of R-algebroids $E = \{E_0, E_1, \cdots, E_n, \cdots\}$ together with homomorphisms $d_i^n : E_n \to E_{n-1}$, $0 \le i \le n \neq 0$ and $s_j^n : E_n \to E_{n+1}, 0 \le j \le n$ such that identity on object set, this homomorphisms are required to satisfy the simplicial identities

1) $d_i^{n-1} d_j^n = d_{j-1}^{n-1} d_j^n \quad , \quad 0 \le i < j \le n$
2) $s_i^{n+1} s_j^n = s_{j+1}^{n+1} s_i^n \quad , \quad 0 \le i \le j \le n$
3) $d_i^{n+1} s_j^n = s_{j-1}^{n-1} d_i^n \quad , \quad 0 \le i < j \le n$
4) $d_i^{n+1} s_j^n = Id \quad , \quad i = j \text{ or } i = j+1$
5) $d_i^{n+1} s_j^n = s_j^{n-1} d_{i-1}^n \quad , \quad 0 \le j < i-1 \le n$

We denote this simplicial R-algebroid with $\boldsymbol{E} = (E_n, d_i^n, s_j^n)$.



Let $\mathbf{E} = (E_n, d^n_i, s^n_j)$ and $\mathbf{F} = (F_n, \delta_i^n, \sigma_j^n)$ be R-algebroids. A simplicial map $\mathbf{f} = \{f_n : n \in N\} : \mathbf{E} \to \mathbf{F}$ is a family of homomorphisms $f_n = E_n \to F_n$ satisfying $\delta_i^n f_n = f_{n-1} d_i^n$ and $f_n s_j^{n-1} = \partial_j^{n-1} f_{n-1}$ for all $n \in$ N. We have thus defined category of simplicial R-algebroids, which we will denote by **Simp.R-Alg.**. Let **E** be a simplicial R-algebroid. The Moore complex (NE, ∂) of **E** is the chain complex defined by $NE_n = \bigcap_{i=0}^{n-1} ker d_i^n$ with $\partial_n : NE_n \to NE_{n-1}$ induced from d_n^n by restriction.

$$... \to NE_2 \xrightarrow{d_2^2} NE_1 \xrightarrow{d_1^1} E_0 = E_0$$

We say that the Moore complex (NE,∂) of **E** is of length k if $NE_n = 0$ for all $n \geq k + 1$. We denote category of simplicial R-algebroids with Moore complex of lenght k by **Simp.R-Alg.**$_{\leq k}$.

2-Crossed Modules R-Algebroids

As a generalization of the 2-crossed module of commutative algebras, we present the 2-crossed module of R-algebroids in this section.

A 2-crossed module of R-algebroids ($J,K,L,\partial_1,\partial_2,\{.\}_{1,2}$) is given by a chain complex of R-algebroids with same object set U_0 together with associative actions of L on K and J such that ∂_1 and ∂_2 R-algebroid morphisms such that identity on U_0, where L act on itself by composition. We also have an R-bilinear functions (the Peiffer liftings)

$$\{.\}_1 : K \times K \to J$$

$$\{.\}_2 : K \times K \to J$$

satisfying the following axioms for $k \in K(a,b)$, $k' \in K(b,c)$, $k'' \in K(c,d)$, $j \in J(a,b)$, $j' \in J(b,c)$ and $l \in L(a,b)$,$q \in L(d,e)$, $a,b,c,d,e \in P_0$

$$
\begin{array}{lll}
CM1) & \partial_2\{k,k'\}_1 & = kk' - k^{\partial_1(k')}, \\
 & \partial_2\{k,k'\}_2 & = kk' - {}^{\partial_1(k)}k', \\
CM2) & \{\partial_2(j),\partial_2(j')\}_{1,2} & = jj', \\
CM3) & \{k,k'k''\}_1 & = \{kk',k''\}_1 + \{k,k'\}_1^{\partial_1(k'')}, \\
 & \{k,k'k''\}_2 & = \{kk',k''\}_2 - {}^{\partial_1(k)}\{k',k''\}_2, \\
CM4) & \{\partial_2(j),k'\}_1 & = j^{k'} - j^{\partial_1(k')}, \\
 & \{\partial_2(j),k'\}_2 & = j^{k'}, \\
CM5) & \{k,\partial_2(j')\}_1 & = {}^{k}j' \\
 & \{k,\partial_2(j')\}_2 & = {}^{k}j' - {}^{\partial_1(k)}j' \\
CM6) & {}^{l}\{k',k''\}_{1,2} & = \{{}^{l}k',k''\}_{1,2} \\
 & \{k',k''\}_{1,2}^{p} & = \{k',k''^{p}\}_{1,2}
\end{array}
$$

Note that $\partial_2 : J \rightarrow K$ is a crossed module with action of K on J. Let $\boldsymbol{C} = (J, K, L, \partial_1, \partial_2, \{.\}_{1,2})$ and $\boldsymbol{C}' = (J', K', L', \partial'_1, \partial'_2, \{.\}'_{1,2})$ be 2-crossed modules, a 2-crossed module map $f = (f_2, f_1, f_0) : C \rightarrow C'$ consists of algebroid maps $f_0 : L \rightarrow L'$, $f_1 : K \rightarrow K'$ and $f_2 : J \rightarrow J'$ making the diagram

$$
\begin{array}{ccccc}
J & \xrightarrow{\partial_2} & K & \xrightarrow{\partial_1} & L \\
\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
J' & \xrightarrow{\partial'_2} & K' & \xrightarrow{\partial'_1} & L'
\end{array}
$$

commutative and preserving all actions and Peiffer liftings

1) $f_1({}^{l}k) = {}^{f_0(l)}f_1(k)$

$f_1(k^{l'}) = f_1(k)^{f_0(l')}$

2) $f_2({}^{l}j) = {}^{f_0(l)}f_2(j)$

$f_2(j^{l'}) = f_2(j)^{f_0(l')}$

3) $f_2\{k,k'\}_{1,2} = \{f_1(k), f_1(k')\}_{1,2}$

for $j \in J(b,c), k \in K(b,c), k' \in K(c,d), l \in L(a,b), l' \in L(b,c)$,

$a,b,c,d \in U_0$. Note from the definition that if $f = (f_2,f_1,f_0)$ is a 2-crossed module morphism then f_2,f_1 and f_0 are equal to each other on object set.

Thus, all R-algebroid 2-crossed modules and their morphisms form a category denoted by **2XMod**.

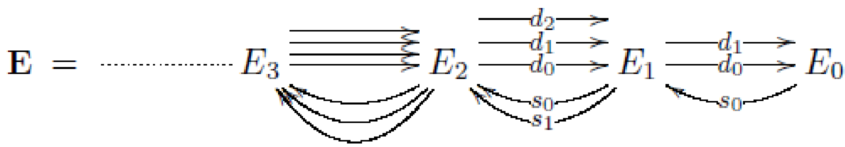
From Simp.R − Alg.$\leq_2$ to 2XMod

In this section, we shall construct a functor from **Simp**.**R** − **Alg**.$\leq_2$ to **2XMod**.

Construction of the functor

Given a simplicial R-algebroid $\mathbf{E} = (E_n, d^n{}_i, s^n{}_j)$ with Moore complex of lenght 2, we obtain a 2-crossed module of R-algebroids.

Let $\mathbf{E} = (E_n, d^n{}_i, s^n{}_j)$ be a simplicial R-algebroid with Moore complex of lenght 2. Then for the simplicial R-algebroid



its Moore complex is as follows,

$$... \to 0 \to 0 \to NE_2 \xrightarrow{d_2^2} NE_1 \xrightarrow{d_1^1} NE_0 = E_0$$

where $NE_2 = kerd_0^2 \cap kerd_1^2, NE_1 = kerd_0^1$ and $NE_0 = E_0$. Let

$L = NE_2, M = NE_1, P = E_0$ $=NE_2$, $M = NE_1$, $P = E_0$ and $\partial_1 = d_1^1|NE_1, \partial_2 = d_2^2|NE_2$

L acts on K and J as ;

- $$\begin{array}{ccc} L \times K & \to & K \\ (l,k) & \mapsto & {}^{l}k = s_0^0(l)k \end{array}$$
- $$\begin{array}{ccc} K \times L & \to & K \\ (k,l') & \mapsto & k^{l'} = ks_0^0(l') \end{array}$$
- $$\begin{array}{ccc} L \times J & \to & J \\ (l,j) & \mapsto & {}^{l}j = s_0^1 s_0^0(l)j \end{array}$$
- $$\begin{array}{ccc} J \times L & \to & J \\ (l,j') & \mapsto & j^{l'} = \text{j}s_0^1 s_0^0(l') \end{array}$$

for $k \in K(b,c), j \in J(b,c)$ and $l \in L(a,b), l' \in L(c,d)$, $a,b,c,d \in U_0$. For $k \in K(a,b)$ and $k' \in K(b,c)$ set

$$\begin{array}{rccl} \{.\}_1: & K \times K & \to & J \\ & (k,k') & \mapsto & \{k,k'\}_1 = s_1^1(k)[s_1^1(k') - s_0^1(k')] \end{array}$$

$$\begin{array}{rccl} \{.\}_2: & K \times K & \to & J \\ & (k,k') & \mapsto & \{k,k'\}_2 = [s_1^1(k) - s_0^1(k)]s_1^1(k') \end{array}$$

Thus

$$F_E = J \overset{\partial_2}{\to} K \overset{\partial_1}{\to} L$$

is a 2-crossed module.

$$
\begin{aligned}
CM1) \quad \bullet \quad \partial_2\{k,k'\}_1 &= d_2^2\left(s_1^1(k)[s_1^1(k')-s_0^1(k')]\right)\\
&= d_2^2 s_1^1(k)\left(d_2^2 s_1^1(k') - d_2^2 s_0^1(k')\right)\\
&= k\left(k' - s_0^0 d_1^1(k')\right)\\
&= kk' - k s_0^0\left(d_1^1(k')\right)\\
&= kk' - k^{\partial_1(k')}
\end{aligned}
$$

$$
\begin{aligned}
\bullet \quad \partial_2\{k,k'\}_2 &= d_2^2\left([s_1^1(k)-s_0^1(k)]\right)s_1^1(k')\\
&= [d_2^2 s_1^1(k) - d_2^2 s_0^1(k)] d_2^2 s_1^1(k')\\
&= \left(k - s_0^0 d_1^1(k)\right)k'\\
&= kk' - s_0^0\left(d_1^1(k)\right)k'\\
&= kk' - {}^{\partial_1(k)}k'
\end{aligned}
$$

$$
\begin{aligned}
CM2) \quad \bullet \quad \{\partial_2(j),\partial_2(j')\}_1 &= s_1^1 d_2^2(j)\left[s_1^1 d_2^2(j') - s_0^1 d_2^2(j')\right]\\
&= d_3^3 s_1^2(j)\left[d_3^3 s_1^2(j') - d_3^3 s_0^2(j')\right] - jj' + jj'\\
&= d_3^3 s_1^2(j)\left[d_3^3 s_1^2(j') - d_3^3 s_0^2(j')\right] - d_3^3 s_2^2(jj') + jj'\\
&= d_3^3 \underbrace{\left[s_1^2(j)\left(s_1^2(j') - s_0^2(j')\right) - s_2^2(jj')\right]}_{\in NE_3} + jj'\\
&= 0 + jj'\\
&= jj'
\end{aligned}
$$

$$
\begin{aligned}
CM3)\quad \bullet\ \{k,k'k''\}_1 &= s_1^1(k)\left[s_1^1(k'k'')-s_0^1(k'k'')\right]\\
&= s_1^1(k)\left[s_1^1(k')s_1^1(k'')-s_0^1(k')s_0^1(k'')\right]\\
&= s_1^1(k)[s_1^1(k')s_1^1(k'')-s_1^1(k')s_0^1(k'')+s_1^1(k')s_0^1(k'')\\
&\quad -s_0^1(k')s_0^1(k'')]\\
&= s_1^1(k)\left[s_1^1(k')\left(s_1^1(k'')-s_0^1(k'')\right)+\left(s_1^1(k')-s_0^1(k')\right)s_0^1(k'')\right]\\
&= s_1^1(k)s_1^1(k')\left[s_1^1(k'')-s_0^1(k'')\right]+s_1^1(k)\left[s_1^1(k')-s_0^1(k')\right]s_0^1(k'')\\
&= \{kk',k''\}_1+\{k,k'\}_1 s_0^1(k'')\\
&= \{kk',k''\}_1+\{k,k'\}_1 s_0^1(k'')\\
&\quad -\{k,k'\}_1 s_1^1 s_0^0 d_1^1(k'')+\{k,k'\}_1 s_1^1 s_0^0 d_1^1(k'')\\
&= \{kk',k''\}_1+\{k,k'\}_1\left[s_0^1(k'')-s_1^1 s_0^0 d_1^1(k'')\right]+\{k,k'\}_1^{\partial_1(k'')}\\
&= \{kk',k''\}_1+s_1^1(k)\left[s_1^1(k')-s_0^1(k')\right]\left[s_0^1(k'')-s_1^1 s_0^0 d_1^1(k'')\right]\\
&\quad +\{k,k'\}_1^{\partial_1(k'')}\\
&= \{kk',k''\}_1+s_1^1(k)\left[s_1^1(k')-s_0^1(k')\right]\left[s_0^1(k'')-d_3^3 s_1^2 s_0^1(k'')\right]\\
&\quad +\{k,k'\}_1^{\partial_1(k'')}\\
&= \{kk',k''\}_1+d_3^3\underbrace{\left[s_2^2 s_1^1(k)[s_2^2 s_1^1(k')-s_2^2 s_0^1(k')][s_2^2 s_0^1(k'')-s_1^2 s_0^1(k'')]\right]}_{\in NE_3}\\
&\quad +\{k,k'\}_1^{\partial_1(k'')}\\
&= \{kk',k''\}_1+\{k,k'\}_1^{\partial_1(k'')}
\end{aligned}
$$

$$
\begin{aligned}
\bullet\ \{k,k'k''\}_2 &= \left[s_1^1(k)-s_0^1(k)\right]s_1^1(k')s_1^1(k'')\\
&= \left[s_1^1(k)s_1^1(k')-s_0^1(k)s_1^1(k')\right]s_1^1(k'')\\
&= \left[s_1^1(k)s_1^1(k')-s_0^1(k)s_0^1(k')+s_0^1(k)s_0^1(k')-s_0^1(k)s_1^1(k')\right]s_1^1(k'')\\
&= \left[\left[s_1^1(kk')-s_0^1(kk')\right]+s_0^1(k)\left[s_0^1(k')-s_1^1(k')\right]\right]s_1^1(k'')\\
&= \left[s_1^1(kk')-s_0^1(kk')\right]s_1^1(k'')+s_0^1(k)\left[s_0^1(k')-s_1^1(k')\right]s_1^1(k'')\\
&= \{kk',k''\}_2-s_0^1(k)\left[s_1^1(k')-s_0^1(k')\right]s_1^1(k'')\\
&= \{kk',k''\}_2-s_0^1(k)\{k',k''\}_2\\
&= \{kk',k''\}_2-s_0^1(k)\{k',k''\}_2+s_0^1 s_0^0 d_1^1(k)\{k',k''\}_2\\
&\quad -s_0^1 s_0^0 d_1^1(k)\{k',k''\}_2\\
&= \{kk',k''\}_2\left(-s_0^1(k)+s_0^1 s_0^0 d_1^1(k)\right)\{k',k''\}_2-{}^{\partial_1(k)}\{k',k''\}_2\\
&= \{kk',k''\}_2\left(-s_0^1(k)+s_0^1 s_0^0 d_1^1(k)\right)[s_1^1(k')-s_0^1(k')]s_1^1(k'')\\
&\quad -{}^{\partial_1(k)}\{k',k''\}_2\\
&= \{kk',k''\}_2\left(-s_0^1(k)+d_3^3 s_0^2 s_0^1(k)\right)[s_1^1(k')-s_0^1(k')]s_1^1(k'')\\
&\quad -{}^{\partial_1(k)}\{k',k''\}_2\\
&= \{kk',k''\}_2 d_3^3\underbrace{\left[(-s_2^2 s_0^1(k)+s_0^2 s_0^1(k))[s_2^2 s_1^1(k')-s_2^2 s_0^1(k')]s_2^2 s_1^1(k'')\right]}_{\in NE_3}\\
&\quad -{}^{\partial_1(k)}\{k',k''\}_2\\
&= \{kk',k''\}_2-{}^{\partial_1(k)}\{k',k''\}_2
\end{aligned}
$$

$$
\begin{aligned}
CM4) \quad \bullet \{k, \partial_2(j)\}_1 &= s_1^1(k)\left[s_1^1 d_2^2(j) - s_0^1 d_2^2(j)\right] \\
&= s_1^1(k)\left[d_3^3 s_1^2(j) - d_3^3 s_0^2(k)\right] \\
&= d_3^3 s_2^2 s_1^1(k)\left[d_3^3 s_1^2(j) - d_3^3 s_0^2(j) - d_3^3 s_2^2(j) + j\right] \\
&= d_3^3 s_2^2 s_1^1(k)\left[d_3^3 s_1^2(j) - d_3^3 s_0^2(j) - d_3^3 s_2^2(j)\right] + d_3^3 s_2^2 s_1^1(k) j \\
&= d_3^3 \underbrace{\left[s_2^2 s_1^1(k)\left[s_1^2(j) - s_0^2(j) - s_2^2(j)\right]\right]}_{\in NE_3} + d_3^3 s_2^2 s_1^1(k) j \\
&= 0 + s_1^1(k) j \\
&= {}^{k}j
\end{aligned}
$$

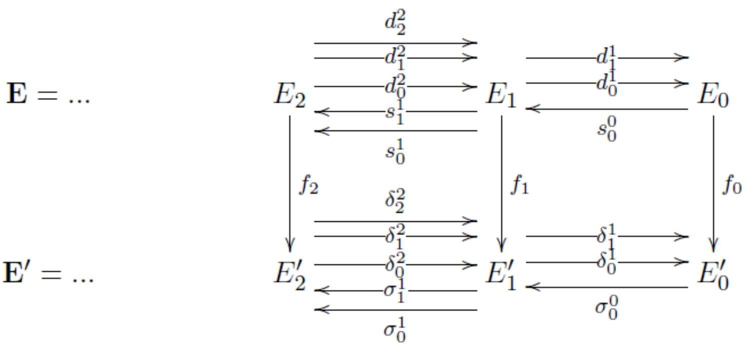
$$
\begin{aligned}
\bullet \{k, \partial_2(j)\}_2 &= \left[s_1^1(k) - s_0^1(k)\right] s_1^1 d_2^2(j) \\
&= \left[s_1^1(k) - s_0^1(k)\right] s_1^1 d_2^2(j) - s_1^1(k) j + s_1^1(k) j \\
&\quad - s_0^1 s_0^0 d_1^1(k) j + s_0^1 s_0^0 d_1^1(k) j \\
&= \left[d_3^3 s_2^2 s_1^1(k) - d_3^3 s_2^2 s_0^1(k)\right] d_3^3 s_1^2(j) - d_3^3 s_2^2 s_1^1(k) d_3^3 s_2^2(j) + s_1^1(k)(j) \\
&\quad - s_0^1 s_0^0 d_1^1(k) j + d_3^3 s_0^2 s_0^1(k) d_3^3 s_2^2(j) \\
&= d_3^3 \underbrace{\left[\left[s_2^2 s_1^1(k) - s_2^2 s_0^1(k)\right] s_1^2(j) - s_2^2 s_1^1(k) s_2^2(j) + s_0^2 s_0^1(k) s_2^2(j)\right]}_{\in NE_3} \\
&\quad + s_1^1(k) j - s_0^1 s_0^0 d_1^1(k) j \\
&= 0 + {}^{k}j - {}^{\partial_1(k)}j \\
&= {}^{k}j - {}^{\partial_1(k)}j
\end{aligned}
$$

$$
\begin{aligned}
CM5) \quad \bullet \{\partial_2(j), k'\}_1 &= s_1^1 d_2^2(j)\left[s_1^1(k') - s_0^1(k')\right] \\
&= s_1^1 d_2^2(j)\left[s_1^1(k') - s_0^1(k')\right] - j s_1^1(k') + j s_1^1(k') \\
&\quad + j s_0^1 s_0^0 d_1^1(k') - j s_0^1 s_0^0 d_1^1(k') \\
&= d_3^3 s_1^2(j)\left[d_3^3 s_2^2 s_1^1(k') - d_3^3 s_2^2 s_0^1(k')\right] - d_3^3 s_2^2(j) d_3^3 s_2^2 s_1^1(k') + j s_1^1(k') \\
&\quad + d_3^3 s_2^2(j) d_3^3 s_0^2 s_0^1(k') - j s_0^1 s_0^0 d_1^1(k') \\
&= d_3^3 \underbrace{\left[s_1^2(j)\left[s_2^2 s_1^1(k') - s_2^2 s_0^1(k')\right] - s_2^2(j) s_2^2 s_1^1(k') + s_2^2(j) s_0^2 s_0^1(k')\right]}_{\in NE_3} \\
&\quad + j s_1^1(k') - j s_0^1 s_0^0 d_1^1(k') \\
&= 0 + j^{k'} - j^{\partial_1(k')} \\
&= l^{k'} - j^{\partial_1(k')}
\end{aligned}
$$

$$
\begin{aligned}
\bullet \{\partial_2(j), k'\}_2 &= \left[s_1^1 d_2^2(j) - s_0^1 d_2^2(j)\right] s_1^1(k') \\
&= \left[s_1^1 d_2^2(j) - s_0^1 d_2^2(j) - j + j\right] s_1^1(k') \\
&= \left[d_3^3 s_1^2(j) - d_3^3 s_0^2(j) - d_3^3 s_2^2(j) + j\right] d_3^3 s_2^2 s_1^1(k') \\
&= d_3^3 \underbrace{\left[\left[s_1^2(j) - s_0^2(j) - s_2^2(j)\right] s_2^2 s_1^1(k')\right]}_{\in NE_3} + j s_1^1(k') \\
&= 0 + j^{k'} \\
&= j^{k'}
\end{aligned}
$$

$$
\begin{aligned}
CM6)\quad \bullet\ {}^{l}\{k,k'\}_1 &= s_0^1 s_0^0(l) s_1^1(k)[s_1^1(k') - s_0^1(k')] \\
&= s_1^1 s_0^0(l) s_1^1(k)[s_1^1(k') - s_0^1(k')] \\
&= s_1^1\left(s_0^0(l)(k)\right)[s_1^1(k') - s_0^1(k')] \\
&= s_1^1({}^{l}k')[s_1^1(k') - s_0^1(k')] \\
&= \{{}^{l}k,k'\}_1 \\
\bullet\ \{k,k'\}_2^q &= [s_1^1(k) - s_0^1(k)] s_1^1(k') s_0^1 s_0^0(q) \\
&= [s_1^1(kk) - s_0^1(k)] s_1^1(k') s_1^1 s_0^0(q) \\
&= [s_1^1(k) - s_0^1(k)] s_1^1\left((k') s_0^0(q)\right) \\
&= [s_1^1(k) - s_0^1(k)] s_1^1(k'^q) \\
&= \{k,k'^q\}_2
\end{aligned}
$$

Let $\boldsymbol{E} = (E_n, d_i^n, s_j^n)$ and $\mathbf{E}' = (E''_n, \delta_i^n, \sigma_i^n)$ be simplicial R-algebroids with Moore complex of lenght 2.



Also let **f** = (...,f_2,f_1,f_0) be a simplicial R-algebroid morphism from $\mathbf{E} = (E_n, d_i^n, s_i^n)$ to $\mathbf{E}' = (E''_n, \delta_i^n, \sigma_i^n)$. If $F_E = (L, M, P, \partial_1, \partial_2, \{.\}_{1,2})$ and $F_{E'} = (L', M', P', \partial'_1, \partial'_2, \{.\}'_{1,2})$ are 2-

crossed modules then we obtain a 2-crossed module morphism from F_E to $F_{E'}$ by using $\mathbf{f} = (\ldots, f_2, f_1, f_0)$

$$
\begin{array}{ccccc}
L & \xrightarrow{\partial_2} & M & \xrightarrow{\partial_1} & P \\
\downarrow{\scriptstyle \overline{f}_2} & & \downarrow{\scriptstyle \overline{f}_1} & & \downarrow{\scriptstyle \overline{f}_0} \\
L' & \xrightarrow{\partial'_2} & M' & \xrightarrow{\partial'_1} & P'
\end{array}
$$

we define as $\overline{f_0} = f_0$, $\overline{f_1} = f_1|_{NE_1}$, $\overline{f_2} = f_2|_{NE_2}$. We show that $(\overline{f_2}, \overline{f_1}, \overline{f_0})$ is a 2-crossed module morphism from F_E to $F_{E'}$

1) $$\begin{aligned} \overline{f_1}({}^{l}k) &= f_1(s_0^0(l)k) \\ &= f_1(s_0^0(l)f_1(k) \\ &= \sigma_0^0 f_0(l) f_1(k) \\ &= {}^{\overline{f_0}(l)}\overline{f_1}(k) \end{aligned}$$

$$\begin{aligned} \overline{f_1}(k^{l'}) &= f_1(ks_0^0(l')) \\ &= f_1(k)f_1(s_0^0(l') \\ &= f_1(k)\sigma_0^0 f_0(l') \\ &= \overline{f_1}(k)^{f_0(l')} \end{aligned}$$

2) $$\begin{aligned} \overline{f_2}({}^{l}j) &= f_2(s_0^1 s_0^0(l)k) \\ &= f_2(s_0^1 s_0^0(l)f_2(k) \\ &= \sigma_1^0 f_1 \sigma_0^0(l) f_2(j) \\ &= \sigma_1^0 \sigma_0^0 f_0(l) f_2(j) \\ &= {}^{\sigma f_0(l)}\overline{f_2}(j) \end{aligned}$$

$$\begin{aligned}\overline{f_2}(j^{l'}) &= f_2(js_0^1s_0^0(l'))\\ &= f_2(j)f_2(s_0^1s_0^0(l')\\ &= f_2(j)\sigma_1^0 f_1\sigma_0^0\,(l')\\ &= f_2(j)\sigma_1^0\sigma_0^0 f_{\,0}\,(l')\\ &= \overline{f_2}(j)^{\overline{f_0}(l')}\end{aligned}$$

3) $$\begin{aligned}\overline{f_2}\,\{k,k'\}_1 &= \mathrm{f}_2(s_1^1(k)[s_1^1(k')-s_0^1(k')])\\ &= \mathrm{f}_2 s_1^1(k)[f_2 s_1^1(k')-\mathrm{f}_2 s_0^1(k')]\\ &= \sigma_1^1 f_1(k)[\sigma_1^1 f_1(k') - \sigma_1^0 f_1(k')]\\ &= \left\{\overline{f_1}(k),\overline{f_1}(k')\right\}_1\end{aligned}$$

$$\begin{aligned}\overline{f_2}\,\{k,k'\}_2 &= \mathrm{f}_2([s_1^1(k)-\mathrm{s}_0^1(k)]s_1^1(k'))\\ &= [\mathrm{f}_2 s_1^1(k) - f_2\mathrm{s}_0^1(k)]\mathrm{f}_2 s_1^1(k'))\\ &= [\sigma_1^1 f_1(k)-\sigma_0^1 f_1(k)]\sigma_1^1 f_1(k')\\ &= \left\{\overline{f_1}(k),\overline{f_1}(k')\right\}_2\end{aligned}$$

for $j \in J(b,c), k \in K(b,c), k' \in K(c,d), l \in L(a,b), l' \in L(b,c), a,b,c,d \in U_0$.

Therefore (f_2,f_1,f_0) is a 2-crossed module morphism. Then, a direct calculation proves to following proposition: The assignment

$\mathbf{F} : \mathbf{Simp.R-Alg.}_{\leq 2} \rightarrow \mathbf{2XMod}$ defined by $\mathbf{F(E)} = F_E$ on objects and by $\boldsymbol{F}(\boldsymbol{f}) = (\overline{f_2},\overline{f_1},\overline{f_0})$ on morphisms is a functor.

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FROM CROSSED SQUARES OF R-ALGEBROIDS TO SIMPLICIAL R-ALGEBROIDS

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İbrahim İlker AKÇA[4]

Introduction

Whitehead introduced crossed modules of groups for the first time in (Whitehead ,1941:409), (Whitehead, 1946: 806). Group crossed modules are equivalent to simplicial groups with Moore complex of length one (Conduche, 1984:155) and similarly for groupoid crossed modules (Mutlu & Porter, 1998: 174). Conduche addressed the idea of a group 2-crossed module and shown in (Conduche, 1984:155) that the category of group 2-crossed modules is equal to the category of simplicial groups with a two-length Moore complex. Also in (Gülsün Akay, 2025: 565) it was given an equivalence relation and groupoid structure on simplicial morphisms and in (Gülsün Akay, 2023) it was obtained crossed module homotopies from simplicial homotopies. Arvasi and Ulualan investigated the relationships between simplicial groups

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with a length of two Moore complex, crossed squares, quadratic modules, and 2-crossed modules in (Arvasi & Ulualan, 2006:1). The definitions of algebra crossed and 2-crossed modules (Arvasi & Porter, 1996 : 426), (Arvasi & Porter, 1998: 455), (Doncel & Grandjean, 1992: 131), (Porter, 1986: 458) are similar to those of the group case, actions by the automorphisms is replaced by the actions by the multipliers, this algebra action is discussed in (Ege Arslan & Hürmetli 2021:72), (Ege Arslan, 2023: 1), (Arvasi & Ege 2003: 478) and its different properties are examined.

Algebra 2-crossed modules and simplicial algebras are closely related, just like in the group case (Conduche, 1984:155), (Mutlu & Porter, 1998: 174), (Mutlu & Porter, 1998: 148). A 2-crossed module can be obtained from a simplicial algebra if it has a Moore complex of length two. Equivalence from category of simplicial algebra with a two-length Moore complex to category algebra crossed module is given in (Porter, 1986: 458), (Arvasi 1997:160), (Grandjean & Vale 1986). Also in (Ege Arslan & ark., 2019: 5293), (Özel, Ege Arslan &Akça, 2024), (Ege Arslan & Kaplan, 2022: 17), (Ege Arslan, 2019:150), a higher-dimensional categorical perspective on 2-crossed modules, fibrations of 2-crossed modules and functorial relations are examined. Moreover in (Akça & Arvasi, 2002: 43), the higher order Peiffer elements in simplicial Lie algebras are examined. The homotopy theory of 2 -crossed modules of commutative algebras studied in (Akça, Emir & Martins, 2016: 99). Then in (Akça, Emir & Martins, 2019: 289), the concept of a 2 -fold homotopy between a pair of 1-fold homotopies connecting 2-crossed module morphisms was defined. As a more broadly, Mitchell in (Mitchell, 1972:1), (Mitchell, 1985: 96.) and Amgott in (Amgott, 1986: 1) specifically studied R-algebroids, where R is a commutative ring. R-algebroids were defined categorically by Mitchell. Later, Mosa introduced crossed modules of R-algebroids as a generalization of crossed modules of associative R-algebras and demonstrated in his thesis (Mosa,1986) that they are equivalent to special double R-algebroids with connections. Additionally, it was mentioned in (Gürmen & Ulualan, 2020: 113) that there was a close relationship between the category of simplicial R-algebroids with the length one Moore complex and the internal categories in the category of R-algebroids. Guin- Waléry and Loday defined crossed squares in (Guin- Waléry & Loday, 1981: 179) as an algebraic model for homotopy 3-type connected spaces. Thus crossed squares model homotopy types in dimensions bigger than 3. Later Ellis defined the commutative algebra version of crossed squares in (Ellis, 1988: 277). In this work we introduce R-algebroid version of crossed square. Then we construct a functor from the category of crossed squares of R-algebroids to the category of simplicial R-algebroids with Moore complex of length two. This chapter produced from Ph. D. Thesis of I. Doğanay Yalğın, (Doğanay Yalğın, 2024).

Preliminaries

Most of the following data, can be found in (Mitchell, 1972:1), (Mitchell, 1978: 867), (Mitchell, 1985: 96), (Amgott, 1986: 1) and (Mosa, 1986).

Let R be a commutative ring. An R-category is a category where composition is R-bilinear and all homsets possess R-module structures. This framework enables the exploration of categorical concepts and constructions within the realm of R-modules, offering a robust foundation for algebraic and categorical inquiries.

An R-algebroid is a small R-category. R- algebroids can be non identity. A set of functions s,t : Mor(U) → Ob(U), the source and target functions, respectively, and an object set Ob(U) = U_0, a morphism set Mor(U), are included with an R-algebroid U. A single object R-algebroid corresponds to an associative R-algebra.

Let U and V be R-algebroids and $U_0 = V_0$, if the family of maps

$$\begin{array}{ccc} V(a,b)\times V(b,c) & \rightarrow & V(a,c) \\ (v,u) & \rightarrow & v^{u} \end{array}$$

satisfies the following conditions

1) $v^{u_1+u_2} = v^{u_1} + v^{u_2}$

2) $(v_1 + v_2)^u = v_1^u + v_2^u$

3) $(v^u)^{u'} = v^{uu'}$

4) $v'v^u = v'v^u$

5) $r \cdot v^u = (r \cdot v)^u = v^{r \cdot u}$

6) $v^{1_{tv}} = v$

for all $a, b, c \in U_0$ and $u, u', u_1, u_2 \in Mor(U), v, v', v_1, v_2 \in Mor(V)$ such that $t(v') = s(v), t(u) = s(u'), t(v) = t(v_1) = t(v_2) = s(u) = s(u_1) = s(u_2)$, $r \in R$, it is called the right action of U on V .

The left action of U on V similarly defined. While U has right and left action on V if the condition

$$({}^{u}v)^{u'} = {}^{u}(v^{u'})$$

is satisfied for all d,a,b,c ∈ U_0, v ∈ V (a,b),u ∈ U(d,a) and u' ∈

U(b,c) then U has an associative action on V .

An R-functor is an R-linear functor between two R-categories, and an R-algebroid morphism is an R-functor between two R-algebroids.

In category Alg(R), all R-algebroids and their morphisms are included.

Let R is an commutative ring U and V be two R-algebroids of the same object set U_0 and V has an associative action on U . For the set $U \rtimes V = \{(u,v): u \in U, v \in V\}$, if the following conditions are satisfied

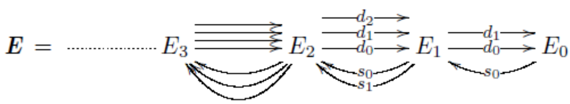
1) $(u,v) + (u',v') = (u+u', v+v')$
2) ${}^{r}(u,v) = ({}^{r}u, {}^{r}v)$
3) $(u,v)(u'',v'') = (uu'' + u^{v''} + {}^{v}u'', vv'')$

$U \rtimes V$ is an R-algebroid, where for all $(u,v) \in U \rtimes V$ and $r \in R$, $s(u,v) = s(u) = s(v)$, $t(u,v) = t(u) = t(v)$, $(u,v),(u',v),(u',v'),(u'',v'') \in U \rtimes V$, $s(u,v) = s(u',v')$, $t(u,v) = t(u',v')$, $t(u,v) = s(u'',v'')$. This R-algebroid is called the semi-direct product R-algebroid of U and V.

A simplicial R-algebroid is a sequence of R-algebroids E $=\{E_0, E_1, \ldots, E_n, \ldots\}$ together with homomorphisms $d_i^n : E_n \to E_{n-1}$ $(0 \le i \le n \ne 0)$ and $s_j^n : E_n \to E_{n+1}$ $(0 \le j \le n)$ for each $(0 \le i \le n \ne 0)$ such that identity on object set, this homomorphisms are required to satisfy the simplicial identities

1) $d_i^{n-1} d_j^n = d_{j-1}^{n-1} d_j^n \quad , \quad 0 \le i < j \le n$
2) $s_i^{n+1} s_j^n = s_{j+1}^{n+1} s_i^n \quad , \quad 0 \le i \le j \le n$
3) $d_i^{n+1} s_j^n = s_{j-1}^{n-1} d_i^n \quad , \quad 0 \le i < j \le n$
4) $d_i^{n+1} s_j^n = Id \quad , \quad i = j \ or \ i = j+1$
5) $d_i^{n+1} s_j^n = s_j^{n-1} d_{i-1}^n \quad , \quad 0 \le j < i-1 \le n$

We denote this simplicial R-algebroid with $\mathbf{E} = (E_n, d_i^n, s_i^n)$.



Let $\mathbf{E} = (E_n, d_i^n, s_i^n)$ and $\mathbf{F} = (F_n, \delta_i^n, \sigma_j^n)$ be R-algebroids. A simplicial map $\mathbf{f} = \{f_n : n \in N\} : \mathbf{E} \to \mathbf{F}$ is a family of homomorphisms $f_n = E_n \to F_n$ satisfying $\delta_i^n f_n = f_{n-1} d_i^n$ and $f_n s_j^{n-1} = \partial_j^{n-1} f_{n-1}$ for all n ∈ N. We have thus defined category of simplicial R-algebroids, which we will denote by **Simp.R-Alg.**

Let **E** be a simplicial R-algebroid. The Moore complex (NE,∂) of **E** is the chain complex defined by $NE_n = \bigcap_{i=0}^{n-1} \ker d_i^n$ with $\partial_n : NE_n \to NE_{n-1}$ induced from ∂_n^n by restriction.

$$\ldots \to NE_2 \xrightarrow{d_2^2} NE_1 \xrightarrow{d_1^1} E_0 = E_0$$

We say that the Moore complex (NE,∂) of E is of length k if $NE_n = 0$ for all $n \geq k+1$. We denote category of simplicial R-algebroids with Moore complex of lenght k by $\text{Simp.R-Alg.}_{\leq k}$.

Crossed Squares of R-algebroids

Guin- Waléry and Loday defined crossed squares in (Guin- Waléry & Loday, 1981: 179) as an algebraic model for homotopy 3-type connected spaces. Thus crossed squares model homotopy types in dimensions bigger than 3. Later Ellis defined the commutative algebra version of crossed squares in (Ellis, 1988: 277). In this section we introduce R-algebroid version of crossed square.

A crossed square is a commutative square of R-algebroids with the same object set M_0

$$\begin{array}{ccc} L & \xrightarrow{\kappa} & M \\ {\scriptstyle \kappa'}\downarrow & & \downarrow{\scriptstyle \omega} \\ N & \xrightarrow[\upsilon]{} & P \end{array}$$

together with associative actions P on L,M,N and a function h : $M \times N \rightarrow L$ identity on M_0. Let M and N act on M,N and L via P. The structure must satisfy the following axioms

CS1) λ and λ' preserve the action of P, and $\lambda,\lambda',\mu,\upsilon$ and $\upsilon\lambda' = \mu\lambda$ are crossed modules.

CS2) $h(m+m_1,n) = h(m,n) + h(m_1+n)$,

$h(m,n+n_1) = h(m,n) + h(m,n_1)$,

CS3) $r \cdot h(m,n) = h(r \cdot m,n) = h(m,r \cdot n), (r \in R)$

CS4) ${}^{p}h(m,n) = h({}^{p}m,n), and\ h(m,n)^{p'} = h(m,n^{p'})$

CS 5) $h(m'm,n) = {}^{m'}h(m,n) = h(m',{}^{m}n)$,

CS 6) $h(m,nn') = h(m,n)^{n'} = h(m^n,n')$,

CS 7) $\lambda(h(m,n)) = m^n$,

CS 8) $\lambda'\left(h(m,n)\right) = {}^{m}n$,

CS 9) $h(\lambda l,n) = l^n$,

CS 10) $h(m,\lambda' l) = {}^{m}l$,

***CS* 11**) $h(m,n)h(m'',n'') = h\left(m^n,{}^{m''}n''\right)$

for all $r \in R, m, m_1, m', m'' \in M, n, n_1, n', n'' \in N, p, p' \in P, l \in L$ with $t(m) = t(m_1) = s(n) = s(n_1)$, $t(p) = s(m)$, $t(n) = s(p'), t(m') = s(m), t(n) = s(n'), t(l) = s(n), t(m) = s(l)$, $t(n) = s(m'')$.

We will denote such a crossed square with $\begin{pmatrix} L & M \\ N & P \end{pmatrix}$.

A morphism $\Phi: \begin{pmatrix} L & M \\ N & P \end{pmatrix} \to \begin{pmatrix} L' & M' \\ N' & P' \end{pmatrix}$ of crossed squares consists of four R-algebroid morphisms $\Phi_L: L \to L', \Phi_M: M \to M', \Phi_N: N \to N'$ and $\Phi_P: P \to P'$ such that: the resulting cube of R-algebroid morphisms is commutative; $\Phi_L(h(m,n)) = h(\Phi_M(m), \Phi_N(n))$ for $m \in M$, $n \in N$; each of the morphisms Φ_L, Φ_M and Φ_N preserve the action of Φ_P. Thus, all Ralgebroid crossed squares and their morphisms form a category denoted by **XSqua**.

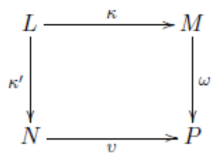
$\boldsymbol{From\ XSqua\ to\ Simp.R - Alg. \leq 2}$

In this section, we will obtain a simplicial R-algebroid $\boldsymbol{E} = (E_n, d_i^n, s_i^n)$ with Moore complex of lenght 2 from a crossed square of R-algebroids.

Proposition 3.1 *Given a crossed square of R-algebroids. We obtain a simplicial R-algebroid* $\mathbf{E} = (E_n, d_i^n, s_i^n)$ *with Moore complex of lenght 2.*

Proof:

Let $\mathbf{K} = (L, M, N, P, \kappa, \kappa', \upsilon, \omega)$ be a crossed square of R-algebroids



Therefore there are associative actions of P on L, M and N. Also M acts on N and L, N acts on M and L.

1) Let $E_0 = P$.

2) We can get M o N with actions of N on M ${}^{n}m = {}^{\upsilon(n)}m$ and $m^{n'} = m^{\upsilon(n')}$. Also it can be get $E_1 = (M \rtimes N) \rtimes P$ with actions of P on $M \rtimes N$

${}^{p}(m,n) = ({}^{p}m, {}^{p}n)$ and $(m,n)^{p} = (m^{p}, n^{p'})$

and we define the morphisms;

- $d_0^1 : E_1 \to E_0$, $d_0^1(m,n,p) = p$
- $d_1^1 : E_1 \to E_0$, $d_1^1(m,n,p) = \omega(m) + v(n) + p$
- $s_0^0 : E_0 \to E_1$, $s_0^0(p) = (\mathbf{0}, \mathbf{0}, p)$

3) We have actions of $M \rtimes N$ on L defined by ${}^{(m,n)}l = h_2(-n, \kappa(l)) = -{}^{n}l$ and $l^{(m',n\prime)} = h_1(\kappa((l), n') = l^{n\prime}$. By means of this actions, $H = L \rtimes (M \rtimes N)$ can be constructed. Also it can be get $E_2 = (L \rtimes (M \rtimes N)) \rtimes ((M \rtimes N) \rtimes P)$ with actions of E_1 on H

${}^{((m,n),p)}(l,(m',n'))$
$=({}^{\omega(m)+v(n)}l + {}^{p}l - h_1(-m,n'), {}^{p}(m',n') + (m,n)(m',n\prime)$

$(l,(m',n'))^{((m'',n''),p)}$
$= l^{\omega(m\prime\prime)+v(n\prime\prime)} + l^{p\prime} - h_2(-n',m''), (m',n')^{p'} + (m',n')(m'',n'')$
and we define the morphisms;

- $d_0^2 : E_2 \to E_1,\quad d_0^2(\mathrm{l},\mathrm{m},\mathrm{n},\mathrm{m}',\mathrm{n}',\mathrm{p}) = (\mathrm{m}',\mathrm{n}',\mathrm{p})$
- $d_1^2 : E_2 \to E_1,\quad d_1^2(\mathrm{l},\mathrm{m},\mathrm{n},\mathrm{m}',\mathrm{n}',\mathrm{p}) = (\mathrm{m}+\mathrm{m}',\mathrm{n}+\mathrm{n}',\mathrm{p})$
- $d_2^2 : E_2 \to E_1,\quad d_2^2(\mathrm{l},\mathrm{m},\mathrm{n},\mathrm{m}',\mathrm{n}',\mathrm{p}) = (-\kappa(\mathrm{l})+\mathrm{m},\kappa'(\mathrm{l})+\mathrm{n},\omega(\mathrm{m}')+\mathrm{v}(\mathrm{n}')+\mathrm{p})$
- $s_0^1 : E_1 \to E_2,\quad s_0^1(m,n,p) = (0,0,0,m',n',p)$
- $s_1^1 : E_1 \to E_2,\quad s_1^1(m',n',p) = (0,m',n',0,0,p)$

4) We have actions of $H = L \rtimes (M \rtimes N)$ on L defined by ${}^{(l,m,n)}l' = (ll' - {}^{n}l')$ and $l'^{(l'',m\prime,n\prime)} = l'l'' + l'^{n\prime}$. By means of this actions $J = L \rtimes (L \rtimes (M \rtimes N))$ can be costructed. Also it can be get

$E_3 = (L \rtimes (L \rtimes (M \rtimes N))) \rtimes (L \rtimes (M \rtimes N))L \rtimes ((M \rtimes N) \rtimes P)$

with actions of E_2 on J

$$
\begin{aligned}
&{}^{(l,m,n,m',n',p)}(l',l'',m'',n'') \\
&= {}^{\omega(m)+v(n)}l' + {}^{\omega(m')+v(n')}l'' + \\
&{}^{\omega(m')+v(n')}l' + {}^{p}l' - h_1(-m,n'') - h_2(-n,\kappa(l'')) - ll'' - h_1(-\kappa(l),n''), \\
&{}^{\omega(m')+v(n')}l'' + {}^{p}l'' - h_1(-m',n'') + h_2\big(-n,\kappa(l'')\big) + h_1(-\kappa(l),n'') + ll', \\
&mm'' + m^{n''} + {}^{n}m'' + m'm'' + m'^{n''} + {}^{n'}m'' + {}^{p}m'', \\
&nn'', n'n'', {}^{p}n'')
\end{aligned}
$$

$$
\begin{aligned}
&(l',l'',m'',n'')^{(k,w,v,w',v',q)} \\
&= (l'^{\omega(m)+v(n)} + l''^{\omega(m)+v(n)} + l'^{\omega(m')+v(n')} \\
&+l'^{q} - h_2(-n'',w) - h_2\big(-n'',\kappa(k)\big) - h_1(-\kappa(l''),v) - l''k, \\
&l''^{(\omega(w')+v(v'))} + l''^{q} - h_2(-n'',w') + h_2(-n'',\kappa'^{(k)}) + h_1\big(-\kappa'^{(l'')},w\big) + l''k, m''w + m''^{v} \\
&+{}^{n''}w + m''w' + m''^{v'} + {}^{n''}w' + m''^{q}, n''v + n''v' + n''^{q}).
\end{aligned}
$$

Let the morphisms $d_0^3, d_1^3, d_2^3, d_3^3 : E_3 \to E_2$ be defined so as to map element $(l,l',m,n,l'',m',n',m'',n'',p)$ of E_3 to the elements (l'',m',n',m'',n'',p), $(l'+l'',m+m',n+n',m'',n'',p)$, $\big(l+l',m,n,m'+m'',n'+n'',p\big)$, $(l,-\kappa(l')+m,\kappa(l')+n,-\kappa(l'')+m',\kappa(l'')+n',\omega(m'')+v(n'')+p)$ of E_2 , respectively, and let the morphisms $s_0^2, s_1^2, s_2^2 : E_2 \to E_3$ be defined so as to map the element (l,m,n,m',n',p) of E_2 to the elements $(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},l,m,n,m',n',p)$, $(\mathbf{0},l,m,n,\mathbf{0},\mathbf{0},\mathbf{0},m',n',p)$, $(l,\mathbf{0},m,n,\mathbf{0},\mathbf{0},\mathbf{0},m',n',p)$ of E_3 , respectively. Thus, we obtain

$$
\begin{aligned}
&ker d_0^3 = \{(l,l',m,n,\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) \mid l,l' \in L, m \in M, n \in N\} \\
&ker d_1^3 = \{(l,-l'',-m',-n',l'',m',n',\mathbf{0},\mathbf{0},\mathbf{0}) \mid l,l'' \in L, m' \in M, n' \in N\} \\
&ker d_2^3 = \{(l,-l'',-m',-n',l'',m',n',\mathbf{0},\mathbf{0},\mathbf{0}) \mid l,l'' \in L, m' \in M, n' \in N\} \\
&ker d_0^3 \cap ker d_1^3 \cap ker d_2^3 = \{(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})\} = NE_3
\end{aligned}
$$

Consequently, these definitions give rise to simplisel R-algebroids with Moore complex of lenght 2.

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QUARTIC TRIGONOMETRIC TENSION B-SPLINE COLLOCATION METHOD FOR FITZHUGH-NAGUMO EQUATION

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KÜBRA KAYMAK[6]

Introduction

The Fitzhugh-Nagumo (FHN) equation is a reduced version of the Hodgkin-Huxley model. They is a simplified neuron model, while retaining fundamental features, used to describe the dynamics of excitable systems. This structure allows for analytical and numerical analysis. Neural networks are widely used in many fields, including biophysics, medicine, and computational neuroscience. The FHN equation was solved using a spectral and colocation-based approach with a special type of basis function related to Chebyshev polynomials of generalized Gegenbauer polynomials (Abd-Elhameed, Alqubori, & Atta, 2025:2). A spectral approach based on two-dimensional shifted Legendre polynomials was used for partial

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differential equations of type FHN (Uma et al.,2025:2). The modified FHN neuron model under the influence of an external electric field was addressed using a discrete matching approach (Zhang et al.,2023:3). Numerical results were analyzed using a two-step effective hybrid block method (Rufai et al., 2023:1)

B-spline functions are commonly used approximation functions in the numerical solution of differential equations. These functions can be defined in various forms, such as polynomial, exponential, trigonometric, and trigonometric tension. In this study, quartic trigonometric tension B-spline functions are used. The tension B-splines are defined by a tension parameter that varies within specific ranges. Trigonometric tension B-spline functions have also been applied to various differential equations; Burgers-Huxley equation (Alina & Zarebnia, 2019:3) Burger's equation (Yigit, Hepson & Allahviranloo, 2024:2), RLW equation (Iqbal, Akram & Alsharif, 2024:2).

In this study, the spatial integration of the FHN equation, which has a partial differential equation structure, was applied using the quartic trigonometric tension B-spline collocation method. The FHN equation was solved numerically using the Crank–Nicolson method, and various test problems were applied to evaluate the accuracy of the method. FHN Eq. is

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - u(1-u)(u-p) = 0, x \in [a,b], t \in [0,T). \quad (1)$$

Here, p represents an arbitrarily chosen constant. The initial condition (IC) and boundary conditions (BCs) of the problem are as follows:

$$u(x,0) = u_0 \quad (2)$$

with

$$\boldsymbol{u(a,t) = f_0, \quad u(b,t) = f_1,}$$

$$\boldsymbol{\frac{\partial u(a,t)}{\partial x} = 0, \ \frac{\partial u(b,t)}{\partial x} = 0,} \tag{3}$$

$$\boldsymbol{\frac{\partial^2 u(a,t)}{\partial x^2} = 0, \ \frac{\partial^2 u(b,t)}{\partial x^2} = 0.}$$

Quartic Trigonometric Tension B-Spline Collocation Method

This section introduces the B-spline function, based on quartic trigonometric tension curves, to be used in the colocation scheme. Then, the spatial and temporal discretization procedures, along with the linearization step, are examined in detail to obtain the fully discretized scheme.

First consider the interval $[a, b]$ as uniformly divided knots such that, $h = x_{m+1} - x_m, m = 0,1, \dots, N-1$ with the points $a = x_0 < x_1 < \cdots < x_N = b$. Furthermore, the fictitious knots outside the domain $x_{-4}, x_{-3}, x_{-2}, x_{-1}$ and $x_{N+1}, x_{N+2}, x_{N+3}, x_{N+4}$ are included to form the b-spline base on the domain $[x_0, x_N]$. Therefore, this tension B-spline curve is given by

$$
QTT = \frac{r}{2\tau^2}\begin{cases}
(\tau^2 q_{m-2}^2 + 2\omega_{m-2} - 2), & [x_{m-2}, x_{m-1}] \\
-\tau^2(3h^2 + 6hq_{m-2} + 2q_{m-2}^2) - 2K_1(\tau^2 q_{m-1}^2 - 2) + 6\omega_{m-1} + 2\omega_m - 4, & [x_{m-1}, x_m] \\
\tau^2(13h^2 + 10hq_{m-2} + 2q_{m-2}^2) + 2K_1\tau^2(11h^2 + 10hq_{m-2}) + 4K_1\tau^2 q_{m-2}^2 - 8K_1 + 6\omega_{m+1} - 4, & [x_m, x_{m+1}] \\
-\tau^2(23h^2 + 14hq_{m-2} + 2q_{m-2}^2) - 2K_1(\tau^2 q_{m+2}^2 - 2) + 2\omega_{m+1} + 6\omega_{m+2} - 4, & [x_{m+1}, x_{m+2}] \\
\tau^2 q_{m+3}^2 + 2\omega_{m+3} - 2, & [x_{m+2}, x_{m+3}] \\
0 & \text{otherwise}
\end{cases} \quad m = 0,1,\dots,N+1 \qquad (4)
$$

where $r = \frac{1}{2h^2(1-K_1)}$, $\omega_{m+j} = \cos\left(\tau\left(x_{m+j} - x\right)\right)$, $q_{m+j} = x_{m+j} - x$, $K_1 = \cos(\tau h)$, $K_2 = \sin(\tau h)$, $\tau \leq \sqrt{\lambda}$, $\lambda = \frac{\pi}{h}$ $(\lambda \in R)$ is tension parameter. The set QTT_0, $QTT_1, \dots,$ QTT_{N+1} constitutes a basis for the space of functions defined on the interval $[a, b]$.

Table 1 ***Values of $QTT(x)$ and its derivatives at knots***

	$QTT(x_m)$	$QTT'(x_m)$	$QTT''(x_m)$	$QTT'''(x_m)$
x_{m-3}	0	0	0	0
x_{m-2}	$r\left(\frac{h^2\tau^2+2K_1-2}{2\tau^2}\right)$	$r\left(\frac{h\tau-K_2}{\tau}\right)$	$r(1-K_1)$	$r\tau K_2$
x_{m-1}	$r\left(\frac{h^2\tau^2-2K_1(h^2\tau^2+1)+2}{2\tau^2}\right)$	$r\left(\frac{h\tau-3K_2+2K_1h\tau}{\tau}\right)$	$r(K_1-1)$	$-3r\tau K_2$
x_m	$r\left(\frac{h^2\tau^2-2K_1(h^2\tau^2+1)+2}{2\tau^2}\right)$	$r\left(\frac{3K_2-2K_h\tau-h\tau}{\tau}\right)$	$r(K_1-1)$	$3r\tau K_2$
x_{m+1}	$r\left(\frac{h^2\tau^2+2K_1-2}{2\tau^2}\right)$	$r\left(\frac{h\tau-K_2}{\tau}\right)$	$r(1-K_1)$	$-r\tau K_2$
x_{m+2}	0	0	0	0

Building on the unified spline approach of Wang and Fang (2008:2), Alinia and Zarebnia (2018:3) employed trigonometric tension B-spline basis functions to develop a collocation method for problems in the calculus of variations. An approximation $U(x,t)$ of the analytical solution $u(x,t)$ can then be expressed, as in

$$U(x,t)=\sum_{m=-2}^{N+1}\delta_m(t)QTT_m(x), \tag{5}$$

as a linear combination of quartic trigonometric tension B-spline basis functions, where δ_m denotes time-dependent parameters. The approximate solution $U(x,t)$ and its derivative at the knot points x_m are expressed in terms of the time-dependent parameters δ_m using Eq. (5) as follows:

$$U(x_m,t)=a_1\delta_{m-2}(t)+a_2\delta_{m-1}(t)+a_2\delta_m(t)+a_1\delta_{m+1}(t)$$

$$U'(x_m,t)=b_1\delta_{m-2}(t)+b_2\delta_{m-1}(t)+b_2\delta_m(t)+b_1\delta_{m+1}(t) \quad (6)$$

$$U''(x_m,t)=c_1\delta_{m-2}(t)-c_1\delta_{m-1}(t)-c_1\delta_m(t)+c_1\delta_{m+1}(t)$$

where

$$a_1 = r\left(\frac{h^2\tau^2 + 2K_1 - 2}{2\tau^2}\right),$$

$$a_2 = r\left(\frac{h^2\tau^2 - 2(h^2\tau^2 + 1)2K_1 + 2}{2\tau^2}\right),$$

$$b_1 = r\left(\frac{h\tau - K_2}{\tau}\right), \tag{7}$$

$$b_2 = -r\left(\frac{h\tau - 3K_2 + 2K_1 h\tau}{\tau}\right),$$

$$c_1 = r(1 - K_1).$$

The temporal discretization of Eq. (1) is carried out using the Crank–Nicolson method. The application of this method to Eq. (1) is given below:

$$\frac{U^{n+1} - U^n}{\Delta t} - \frac{U_{xx}^{n+1} + U_{xx}^n}{2} + \frac{(U^3)^{n+1} + (U^3)^n}{2}$$
$$-(p+1)\frac{(U^2)^{n+1} + (U^2)^n}{2} + p\frac{U^{n+1} + U^n}{2} = 0 \tag{8}$$

where and U^{n+1} are defined as $U(x, t_n + \Delta t)$. According to the Taylor expansion, the nonlinear term, when linearized, is given as follows:

$$(U^2)^{(n+1)} = 2U^n\, U^{n+1} - (U^n)^2 \tag{9}$$

and

$$(U^3)^{n+1} = 3(U^n)^2 U^{n+1} - 2(U^n)^3. \tag{10}$$

Hence, Eq. (8) takes the following form:

$$\frac{U^{n+1}-U^{n}}{\Delta t}-\frac{U_{xx}^{n+1}+U_{xx}^{n}}{2}+\frac{3(U^{n})^{2}U^{n+1}-(U^{n})^{3}}{2}$$
$$-(p+1)U^{n}U^{n+1}+p\frac{U^{n+1}+U^{n}}{2}=0. \qquad (11)$$

Therefore, Eq. (11) is reduced to its simplified form:

$$\eta_1\delta_{m-2}^{n+1}+\eta_2\delta_{m-1}^{n+1}+\eta_2\delta_{m}^{n+1}+\eta_1\delta_{m+1}^{n+1}$$
$$=\eta_3\delta_{m-2}^{n}+\eta_4\delta_{m-1}^{n}+\eta_4\delta_{m}^{n}+\eta_3\delta_{m+1}^{n} \qquad (12)$$

where

$$\begin{aligned}
\eta_1 &= \left(1+3\frac{\Delta t}{2}L^2-(p+1)\Delta tL+p\frac{\Delta t}{2}\right)a_1-\frac{\Delta t}{2}c_1,\\
\eta_2 &= \left(1+3\frac{\Delta t}{2}L^2-(p+1)\Delta tL+p\frac{\Delta t}{2}\right)a_2+\frac{\Delta t}{2}c_1,\\
\eta_3 &= \left(1+\frac{\Delta t}{2}L-p\frac{\Delta t}{2}\right)a_1+\frac{\Delta t}{2}c_1,\\
\eta_4 &= \left(1+\frac{\Delta t}{2}L-p\frac{\Delta t}{2}\right)a_2-\frac{\Delta t}{2}c_1,
\end{aligned} \qquad (13)$$

and the coefficients $\boldsymbol{L_m}$ is defined as:

$$L_m=a_1\delta_{m-2}^{n}(t)+a_2\delta_{m-1}^{n}(t)+a_2\delta_{m}^{n}(t)+a_1\delta_{m+1}^{n}(t). \qquad (14)$$

We now obtain a system with $(N+4)$ unknowns and $(N+1)$ equations. For the system to be solvable, the numbers of unknowns and equations must be equal. Therefore, three unknowns $(\delta_{-2}^{n+1},\delta_{-1}^{n+1},\delta_{N+1}^{n+1})$ are eliminated using the BCs, and $(N+1)\times(N+1)$ a linear system is derived. The solution of this system yields the approximate values at the knots. Then, the quantities δ_m^{n+1} are computed using the initial data δ_m^0.

Numerical Results

The accuracy of the proposed scheme is assessed by evaluating the maximum error norm

$$L_\infty = |u(x,t) - U(x,t)|_\infty = \max_m |u(x_m,t) - U(x_m,t)| \quad (15)$$

where $u(x,t)$ denotes the exact solution and $U(x,t)$ represents the numerical solution at time t.

Test-1

To evaluate the accuracy of the proposed method, the analytical solution of the FHN equation is given as:

$$u(x,t) = -\frac{1}{2} + \frac{1}{2}\tanh\left(0.3536x - \frac{3}{4}t\right) \quad (16)$$

with IC (Pathak et al., 2024:7):

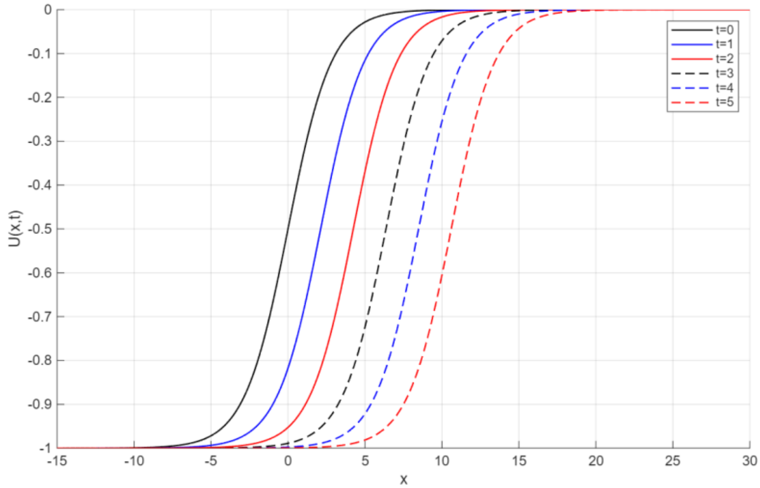
$$u(x,t) = -\frac{1}{2} + \frac{1}{2}\tanh(0.3536x). \quad (17)$$

Numerical solutions were obtained for different values of $\Delta t = 0.01, 0.05$; $h = 0.1, 0.2, 0.5$; η and $p = -1$ over the range $[-15,30]$. The results of these calculations are shown in Table-2.

Table 2 $\boldsymbol{Error\ norms\ for\ Test1,\ 15 \leq x \leq 30\ and\ t = 5}$

h	$\boldsymbol{\Delta t}$	$\boldsymbol{QTT^4}$ ($\boldsymbol{\tau = \sqrt{\lambda}}$)	$\boldsymbol{QTT^4}$ ($\boldsymbol{\tau = \sqrt{\lambda}/10}$)	$\boldsymbol{QTT^4}$ ($\boldsymbol{\tau = \sqrt{\lambda}/5}$)
0.1	*0.01*	1.3882×10^{-4}	1.3926×10^{-4}	1.3925×10^{-4}
	0.05	7.3179×10^{-4}	7.3216×10^{-4}	7.3215×10^{-4}
0.2	*0.01*	1.3565×10^{-4}	1.3921×10^{-4}	1.3910×10^{-4}
	0.05	7.2868×10^{-4}	7.3179×10^{-4}	7.3170×10^{-4}
0.5	*0.01*	8.5479×10^{-5}	1.3776×10^{-4}	1.3611×10^{-4}
	0.05	6.8272×10^{-4}	7.3179×10^{-4}	7.3039×10^{-4}

Figure 1 Numerical Simulations for Test-1



Test-2

In the following problem, we consider another analytical solution of the FHN equation.

$$u(x,t) = \frac{1}{1 + e^{\frac{-s}{\sqrt{2}}}} \tag{18}$$

where $s = x + ct, c = \sqrt{2}\left(\frac{1}{2} - \alpha\right)$. IC (Inan et al., 2021:9)

$$u(x,0) = \frac{1}{1 + e^{\frac{-x}{\sqrt{2}}}} \tag{19}$$

BCs

$$u(1,t) = \frac{1}{1 + e^{\frac{-ct}{\sqrt{2}}}} \tag{20}$$

and

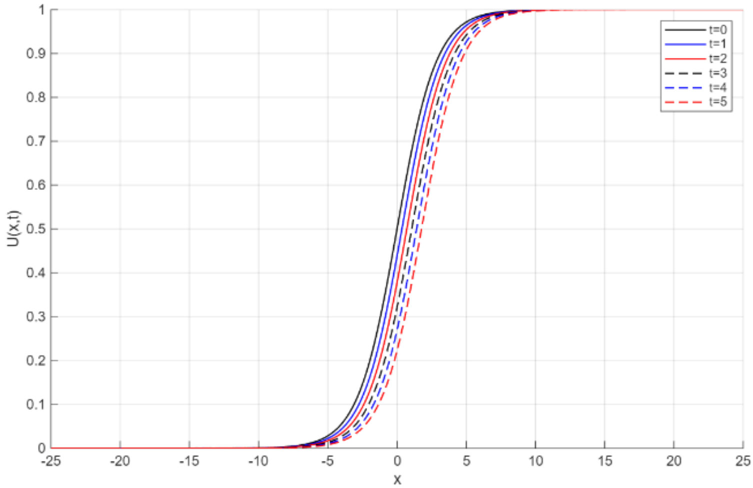
$$u(1,t) = \frac{1}{1 + e^{\frac{-(1+ct)}{\sqrt{2}}}} \tag{21}$$

Numerical solutions were obtained for different values of $\Delta t = 0.01, 0.05; h = 0.1, 0.2, 0.5;$ η and $p = 0.75$ over the range $[-25,25]$. The results of these calculations are shown in Table-3.

Table 3 $\boldsymbol{Error\ norms\ for\ Test\ 2,\ -25 \leq x \leq 25\ and\ t = 5}$

h	$\boldsymbol{\Delta t}$	$\boldsymbol{QTT^4}$ $(\boldsymbol{\tau} = \sqrt{\boldsymbol{\lambda}})$	$\boldsymbol{QTT^4}$ $(\boldsymbol{\tau} = \sqrt{\boldsymbol{\lambda}}/\mathbf{10})$	$\boldsymbol{QTT^4}$ $(\boldsymbol{\tau} = \sqrt{\boldsymbol{\lambda}}/\mathbf{5})$
0.1	*0.01*	6.7624×10^{-7}	1.7142×10^{-7}	1.6456×10^{-7}
	0.05	3.9246×10^{-6}	4.1579×10^{-6}	4.1508×10^{-6}
0.2	*0.01*	5.7831×10^{-6}	3.1174×10^{-7}	2.2412×10^{-7}
	0.05	4.5790×10^{-6}	4.2436×10^{-6}	4.1861×10^{-6}
0.5	*0.01*	8.8190×10^{-5}	1.1320×10^{-5}	8.6726×10^{-6}
	0.05	8.6441×10^{-5}	1.2516×10^{-5}	9.6966×10^{-6}

Figure 2 Numerical Simulations for Test-2



Test-3

In this problem, the analytical solution of the FHN equation is expressed as follows:

$$u(x,t) = \frac{1}{2} + \frac{1}{2} tanh[k(x - ct)] \quad (22)$$

where $k = \frac{1}{2\sqrt{2}}, c = \frac{2\alpha - 1}{\sqrt{2}}$. IC (Inan et al., 2020:11):

$$u(x,0) = \frac{1}{2} + \frac{1}{2} tanh\left(\frac{x}{2\sqrt{2}}\right), \quad (23)$$

BCs:

$$u(-1,t) = \frac{1}{2} + \frac{1}{2} tanh[k(-1 - ct)] \quad (24)$$

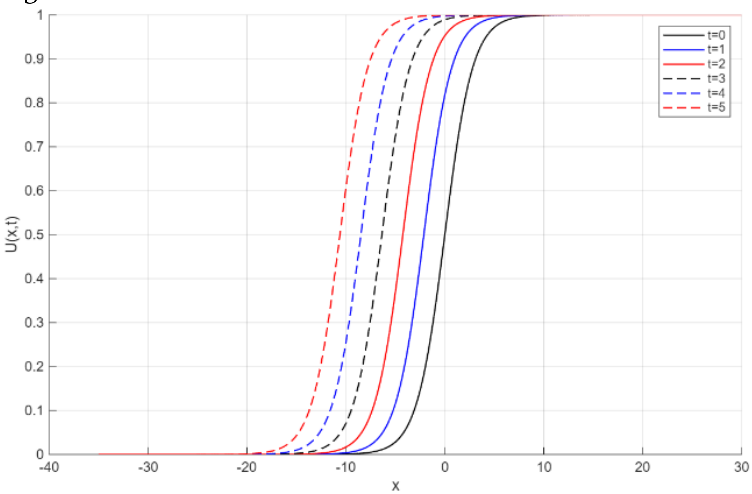
$$u(1,t) = \frac{1}{2} + \frac{1}{2} tanh[k(1 - ct)] \quad (25)$$

Numerical solutions were obtained for different values of $\Delta t = 0.01, 0.05; h = 0.1, 0.2, 0.5; \eta$ and $p = -1$ over the range $[-35,30]$. The results of these calculations are shown in Table-4.

Table 4 $\boldsymbol{Error\ norms\ for\ Test3, -35 \leq x \leq 30\ and\ t = 5}$

$\boldsymbol{h}$	$\boldsymbol{\Delta t}$	$\boldsymbol{QTT^4}$ $(\boldsymbol{\tau = \sqrt{\lambda}})$	$\boldsymbol{QTT^4}$ $(\boldsymbol{\tau = \sqrt{\lambda}/10})$	$\boldsymbol{QTT^4}$ $(\boldsymbol{\tau = \sqrt{\lambda}/5})$)
0.1	*0.01*	2.4614×10^{-5}	2.4978×10^{-5}	2.4967×10^{-5}
	0.05	6.2248×10^{-4}	6.2284×10^{-4}	6.2283×10^{-4}
0.2	*0.01*	2.2062×10^{-5}	2.4965×10^{-5}	2.4878×10^{-5}
	0.05	6.1988×10^{-4}	6.2283×10^{-4}	6.2274×10^{-4}
0.5	*0.01*	8.3999×10^{-5}	2.6620×10^{-5}	2.5451×10^{-5}
	0.05	5.7273×10^{-4}	6.2178×10^{-4}	6.2038×10^{-4}

Figure 2: Numerical Simulations for Test-3



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