SOME GENERALIZATIONS OF ERNST NUMBERS

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ERNST NUMBERS and THEIR CONNECTION TO NON-HOMOGENEOUS LINEAR RECURRENCES with EXPONENTIAL FORCING

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Introduction

Jacobsthal numbers are named after the German mathematician Ernst Jacobsthal. The classical definition of these numbers is that each term is obtained by adding the previous term and twice the previous term:

$$J_n = J_{n-1} + 2J_{n-2}, \text{ initial conditions } J_0 = 0, J_1 = 1.$$

The closed Binet-type formula for Jacobsthal numbers is

$$J_n = \frac{2^n - (-1)^n}{3}.$$

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Numerous works have investigated the field of Jacobsthal numbers. For example: (Horadam, 1996), (Daşdemir, 2014), (Al-Kateeb, 2019), (Asci & Gurel, 2014), (Çimen & İpek, 2017), (Tasci, 2017), (Soykan & Taşdemir, 2022), (Kuloğlu & Özkan, 2023), (Köken & Bozkurt, 2008), (Irge & Soykan, 2024), (Soykan & Taşdemir, 2024). These works provide evidence for the existence of Jacobsthal numbers and the numerous applications they have in the field of algebraic and combinatorial structures aside from the pure number theory.

Following works on Jacobsthal numbers, numerous sequences have been introduced in literature from various viewpoints and these numbers. One of these sequences is Ernst sequence. The expression Ernst numbers is used in the theory of linear recurrence relations and associated with an external constant forcing term. These sequences can be considered as extensions of the classical homogeneous recurrences of higher order with the structure of an external constant. Such sequences have been the object of study in the literature for what is termed as the generalized Ernst numbers.

For instance, in (Soykan, 2022) is documented an advanced version of the Ernst sequence and the Binet formulas associated with it, the generating functions, the matrices of the sequence in various representations and the structural identities of the sequence which provide algebraic relations of the sequence. These studies show that the Ernst numbers constitute a link between number sequences derived from purely homogeneous recurrences and number sequences generated by recurrences of the non-homogeneous type with external forcing, and that these structures are fundamental in the theory of discrete dynamical systems. In (Çolak, Bilgin & Soykan, 2024) the sequence of Gaussian Ernst numbers is given and fundamental algebraic properties are studied.

In (Costa, Catarino & Carvalho, 2025), the Tricomplex Ernst-type sequence is investigated, extending classical Ernst sequences into the tricomplex number system. Fundamental properties are established, including Binet formulas, generating functions, matrix representations.

The goal of this paper is to analyze a new class of integer sequences that is named Generalized Ernst numbers. These numbers come from a particular non-homogeneous linear recurrence relation which contains exponential terms. The Ernst sequence is a classic sequence which these numbers generalize and are directly related to recurrences of the Jacobsthal type. The paper finds and derives the sequence's generating function from the sequence and then uses that generating function to find Binet type closed formulas and an asymptotic form which describes the growth of the sequence. The paper also contains the study of some of the structural relationships that Generalized Ernst numbers hold with some well-known sequences that can be proved using generating function identities and convolution relations. The paper also closes some algebraic relations where it contains some foundational identities such as the Cassini type, Catalan, and d'Ocagne relations which show that the sequence is well defined algebraically. The paper also creates the Hankel matrix associated with sequence and discusses the matrix's determinant properties from the context of linear independence. To complete the paper, the authors provide numerical and graphical evidence in order to validate the theoretical results and illustrate how the results delineate the sequence's behavior. The work is a further advancement of the Ernst and Jacobsthal numbers by providing an extensive investigation of non-homogeneous recurrences with exponential terms.

In this work, the authors derive and analyze Simson-type and related identities, extending the classical theory of real Ernst numbers into the complex domain. This extension not only broadens the analytical scope of Ernst sequences but also demonstrates their potential connections with complex recurrence relations and matrix-based representations.

In the literature, the connection between Ernst sequences and Jacobsthal sequences mainly arises from the fact that the homogeneous part of the Ernst recurrence is of the Jacobsthal type.

Therefore, in solving Ernst sequences, the homogeneous solution of the Jacobsthal sequence is frequently referenced as a fundamental component.

Non-homogeneous linear recurrence relations, in classical literature, refer to sequences that include a forcing term, which is often expressed in constant, polynomial, or exponential form. When the forcing term is exponential (for instance, a^n), there is a potential overlap with the roots of the characteristic polynomial. In such cases, the particular solution must be chosen carefully, often taking the form of $n.a^n, n^2.a^n$, depending on the multiplicity of the root. Analytical tools such as matrix methods, generating function techniques, root analysis, and asymptotic investigations are commonly employed in studying these equations.

Several studies—particularly those under titles like *"Nonhomogeneous Linear Recurrence Relations"*—focus on analyzing how dominant terms emerge and how error terms can be bounded within such recurrence structures (Pituk, 2002), (Li & Mehrotra, 2008), (Singh, Meena & Singh, 2025), (Vrajitoru & Knight, 2014), (Yang & Han, 2021), (Andrade & Pethe, 1992).

In this study, the generalized Ernst numbers are introduced as a new class of sequences defined by a linear recurrence relation that includes a nonlinear structure and an exponential term**.** The main objective is to situate this sequence within the framework of classical number theory**,** to examine its fundamental structures such as the

Cassini, Catalan and Hankel identities, and to explore its connections with the Jacobsthal and Ernst numbers.

Generalized Form of the Sequence and Techniques for Term Evaluation

At the outset, the definition of the Ernst number sequence, which forms the foundation of this study, will be recalled. In (Soykan, 2022), Ernst numbers were defined as follows.

$$E_n = E_{n-1} + 2E_{n-2} + 1, \text{ with } E_0 = 0, E_1 = 1, n \geq 2. \tag{2.1}$$

Now, the numbers will be defined in the form below by changing the added term:

$$\mathcal{E}_0 = 0, \mathcal{E}_1 = 1 \text{ and for } n \geq 2,$$

$$\mathcal{E}_n = \mathcal{E}_{n-1} + 2\mathcal{E}_{n-2} + 2^n. \tag{2.2}$$

This type of sequence is in the form of a nonhomogeneous second-order difference equation. Such sequences are referred to as "non-homogeneous linear recurrences with exponential terms" in classical difference equations boks (Li & Mehrotra, 2008), (Singh, Meena & Singh, 2025). Solving such equations usually involves two steps: a) Solve the homogeneous part, b) Find the particular solution. Then write the general solution.

Binet Formula

The Binet formula is a closed-form expression that facilitates the analysis of a sequence's growth rate. It offers significant advantages in theoretical investigations, limit evaluations, and combinatorial problems by providing insight into the behavior of the sequence. This approach enables direct computation of terms without relying on recursive methods, thereby simplifying asymptotic analysis.

The Binet formula can be directly applied to sequences defined by linear homogeneous recurrence relations. Its primary function is to compute any term of the sequence without relying on previous terms, allowing for the direct evaluation of the th term. In the case of non-homogeneous recurrence relations, the process differs slightly: the general solution is obtained by combining the solution to the homogeneous part (where a Binet-like formula applies) with a particular solution corresponding to the non-homogeneous component.

The following step-by-step calculations produce the closed-form (Binet-style) expression for the sequence. Accordingly, the general solution for the newly defined sequence is determined by obtaining both the homogeneous and particular solutions as follows:

Solve the homogeneous part

The homogeneous equation is:

$$\mathcal{E}_n^{(h)} = \mathcal{E}_{n-1}^{(h)} + 2\mathcal{E}_{n-2}^{(h)}.$$

Its characteristic equation is $r^2 - r - 2 = 0$.

Finding the roots:

$$r = \frac{1 \pm \sqrt{9}}{2} \Longrightarrow r_1 = 2, r_2 = -1.$$

Thus, the homogeneous solution is

$$\mathcal{E}_n^{(h)} = A2^n + B(-1)^n \qquad (2.3)$$

Find a particular solution

If α is a root of multiplicity m in the characteristic equation, then the particular solution is assumed to be of the following form:

$$\mathcal{E}_n^{(p)} = C.n^m.\alpha^n$$

In our case, $\alpha = 2$ is a simple root (i.e., of multiplicity $m = 1$); therefore, the particular solution takes the standard form associated with single roots.

$$\mathcal{E}_n^{(p)} = C.n.2^n$$

The non-homogeneous term is 2^n. Since 2 is already a root of the characteristic equation, a particular solution of the form is proposed:

$$\mathcal{E}_n^{(p)} = C.n.2^n$$

Substituting into the recurrance:

$$Cn2^n = C(n-1)2^{n-1} + 2C(n-2)2^{n-2} + 2^n$$

dividing by 2^{n-2}

$$4Cn = 2C(n-1) + 2C(n-2) + 4$$

$$4Cn = 4Cn - 6C + 4$$

Solving for C:

$$0 = -6C + 4 \Longrightarrow C = \frac{2}{3}.$$

Hence, the particular solution is:

$$\mathcal{E}_n^{(p)} = \frac{2}{3}n.2^n. \tag{2.4}$$

General solution

$$\mathcal{E}_n = \mathcal{E}_n^{(h)} + \mathcal{E}_n^{(p)} = A.2^n + B.(-1)^n + \frac{2}{3}n.2^n \tag{2.5}$$

Apply initial conditions

- For $n = 0$: $\mathcal{E}_0 = 0 = A + B \Longrightarrow B = -A$.
- For $n = 1$: $\mathcal{E}_1 = 1 = 2A - B + \frac{4}{3}$.

If $B = -A$ is substituted into the equation obtained from the second initial condition, the resulting expression becomes

$$3A + \frac{4}{3} = 1 \Longrightarrow A = -\frac{1}{9}, B = \frac{1}{9}.$$

Explicit formula

$$\mathcal{E}_n = -\frac{1}{9}.2^n + \frac{1}{9}.(-1)^n + \frac{2}{3}n.2^n$$

Simplified with common denominator

$$\mathcal{E}_n = \frac{(6n-1).2^n + (-1)^n}{9} \qquad (2.6)$$

This formula provides the value of the sequence for any $n \geq 0$.

Verification for small n

- $n = 0$: $\mathcal{E}_0 = 0$,
- $n = 1$: $\mathcal{E}_1 = 1$,
- $n = 2$: $\mathcal{E}_2 = 5$.

On the other hands, using the recurrence

$$\mathcal{E}_2 = \mathcal{E}_1 + 2\mathcal{E}_0 + 2^2 = 1 + 0 + 4 = 5.$$

The first 10 terms corresponding to $n = 0,1,\dots,9$ are listed in the Table below, based on the explicit formula.

Table *First 10 terms of (2.2)*

n	$\mathcal{E}_n$
0	0
1	1
2	5
3	15
4	41
5	103
6	249
7	583
8	1337
9	3015

These results fully match both the direct recurrence calculation and the explicit formula.

Lemma

The $\mathcal{E}_n$ sequence can be expressed as a third-order non-homogen linear equation as follows:

$$\mathcal{E}_0 = 0, \mathcal{E}_1 = 1, \mathcal{E}_2 = 5 \text{ and for } n \geq 2,$$
$$\mathcal{E}_{n+1} = 3\mathcal{E}_{n-1} + 2\mathcal{E}_{n-2} + 3.2^n \qquad (2.7)$$

Proof

The recurrence relation is shifted one step backward.

$$\mathcal{E}_{n-1} = \mathcal{E}_{n-2} + 2\mathcal{E}_{n-3} + 2^{n-1}.$$

Then, substituting this expression into (2.2) yields the following:

$$n \geq 2, \qquad \mathcal{E}_n = \mathcal{E}_{n-1} + 2\mathcal{E}_{n-2} + 2^n$$
$$= \mathcal{E}_{n-2} + 2\mathcal{E}_{n-3} + 2^{n-1} + 2\mathcal{E}_{n-2} + 2^n$$
$$= 3\mathcal{E}_{n-2} + 2\mathcal{E}_{n-3} + 3.2^{n-1}.$$

Shifting this final equation one step forward gives the following

$$\mathcal{E}_{n+1} = 3\mathcal{E}_{n-1} + 2\mathcal{E}_{n-2} + 3.2^n .$$

Lemma

The $\mathcal{E}_n$ sequence can be expressed as a third-order homogen linear equation as follows:

$$\mathcal{E}_0 = 0, \mathcal{E}_1 = 1, \mathcal{E}_2 = 5 \text{ and for } n \geq 2,$$

$$\mathcal{E}_{n+1} = 3\mathcal{E}_n - 4\mathcal{E}_{n-2} \quad (2.8)$$

Proof

The proof is presented in a similar way to the previous lemma.

Behavior of the Sequence

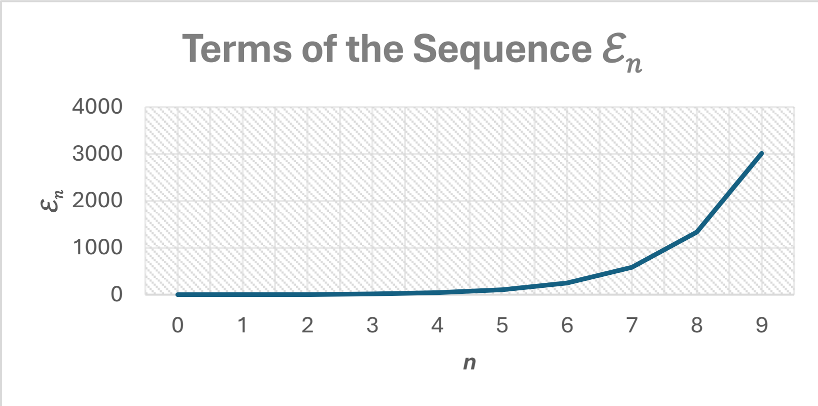
Now, the general growth behavior of the sequence can also be illustrated. Since the dominant term in the explicit formula (2.6) is $\frac{(6n-1).2^n}{9}$, for lange n the sequence grows approximately like

$$\mathcal{E}_n \sim \frac{2}{3} n.2^n$$

This shows that the sequence grows exponentially with a linear factor n. The $(-1)^n$ term oscillates but becomes negligible for large n.

So asymptotically, $\mathcal{E}_n$ behaves like $O(n.2^n)$.

Figure *The graph illustrating the terms of the sequence*



The exponential growth pattern of the sequence $\mathcal{E}_n$ can be clearly seen in its graph, starting from 0 and rapidly increasing as n grows.

The program code used to compute the terms of the sequence and plot its graph is provided in the attachment.

Table *A Code Fragment Illustrating the Behavior of the Sequence*

```
# --- Definitions ---
# Define the sequence ℰ(n) using the closed form:
ℰ := n -> ((6*n - 1)*2^n + (-1)^n)/9;
# Number of terms to compute (change N as desired)
N := 9;    # computes n = 0..9 (first 10 terms)
# Compute and display the sequence values
seq_values := [ seq(ℰ(n), n=0..N ) ];
print("Sequence terms ℰ(0).. ℰ(N):", seq_values);
# Prepare points (n, ℰ(n)) for plotting
```

```
pts := [ seq( [n, evalf(ℰ(n))], n=0..N ) ];   # evalf to ensure
numeric plotting
# --- Plot (requires the plots package) ---
with(plots):
p := listplot(   pts,   style = pointline,         # points connected
by lines
   symbol = solidcircle,      # point symbol
   symbolsize = 12,           # size of points
   linewidth = 2,             # line thickness
   axes = boxed,
   labels = ["n", " ℰ_n"],      # axes labels
   title = "Terms of the Sequence \\(\\mathscr{ ℰ}_n\\)",  #
English title font = ["Helvetica", 12],    tickmarks = [0..N,
default] # x-ticks at integers 0..N);
display(p);
```

Obtaining the asymptotic formula

Using

$$\mathcal{E}_n = \frac{1}{9}((-1)^n - 2^n + 6n2^n) = 2^n\left(\frac{6n-1}{9}\right) + \frac{(-1)^n}{9}.$$

The following asymptotic expressions are directly obtained: Leading term:

$$\mathcal{E}_n \sim \frac{2}{3}n2^n \ \ (n \to \infty).$$

More detailed explicit formulation:

$$\mathcal{E}_n = \left(\frac{2}{3}n - \frac{1}{9}\right)2^n + O(1).$$

Asymptotic growth class: Θ notation (Big-Theta):

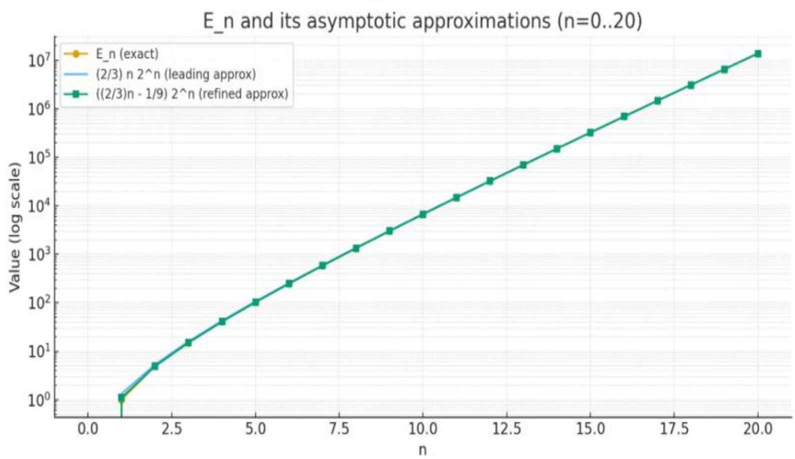
$$\mathcal{E}_n = \Theta(n2^n).$$

So,

$$\lim_{n\to\infty}\frac{\mathcal{E}_n}{n2^n} = \lim_{n\to\infty}\frac{1}{9}\left(\frac{6n - 1 + \frac{(-1)^n}{2^n}}{n}\right) = \lim_{n\to\infty}\frac{1}{9}\left(6 - \frac{1}{n} + \frac{(-1)^n}{n2^n}\right)$$

$$= \frac{2}{3}.$$

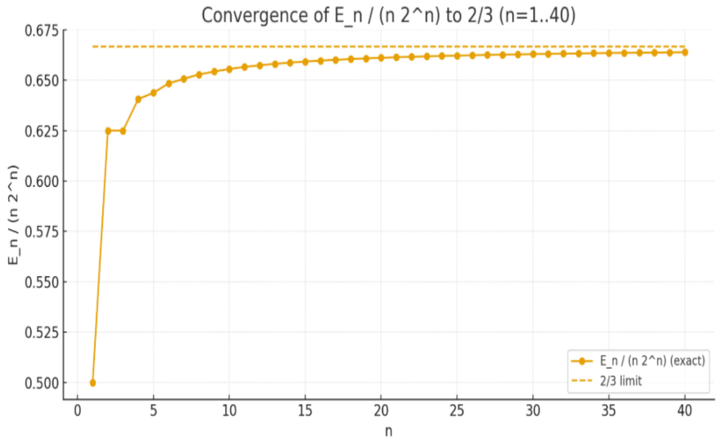
Figure *Comparative Asymptotic Approximation.*



In the upper graph, the function $\mathcal{E}_n$ is plotted alongside its asymptotic approximations $(2/3)n2^n$ and $\left((2/3)n - \frac{1}{9}\right)2^n$, using a logarithmic scale on the y-axis.

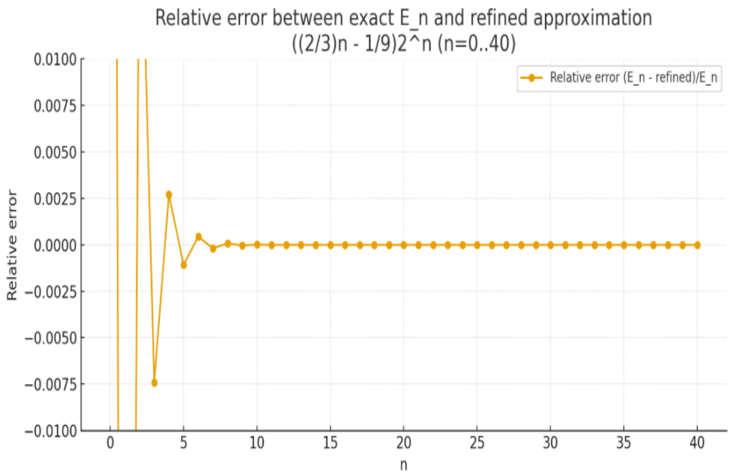
The following graph depicts the behavior of the sequence $\mathcal{E}_n/n2^n$ along the real axis, clearly demonstrating its convergence toward the asymptotic limit of $2/3$.

***Figure** Asymptotic Approximation of* $\mathcal{E}_n/n2^n$.



In the following graph, the relative error $((\mathcal{E}_n - refined)/\mathcal{E}_n)$ is plotted to compare the actual value with its refined approximation. The vertical axis is constrained to the range $[-0.01, 0.01]$, thereby enhancing the visibility of subtle deviations between the plotted curves.

***Figure** Graph Illustrating the Relative Error*



The first 12 rows are shown below. The last column of the table clearly confirms that the value of $\mathcal{E}_n/n2^n$ approaches $2/3$ as n increases. For instance, when $n = 10$, the value is approximately 0.6556, and for $n = 11$, it remains around 0.6556, thereby verifying that the limit is indeed $2/3$.

***Table** Asymptotic Control Table of the $\mathcal{E}_n$ Sequence*

n	$\mathcal{E}_n$	Roughly Value $\left(\left(\frac{2}{3}n-\frac{1}{9}\right)2^n\right)$	Difference $(\mathcal{E}_n - App.)$	Relative error	$\mathcal{E}_n/n2^n$
1	1	1	-0.1111	-0.1111	0.5000
2	5	4	+0.1111	+0.0222	0.6250
3	15	15	-0.1111	-0.0074	0.6250
4	41	40	+0.1111	+0.0027	0.6406
5	103	103	-0.1111	-0.0011	0.6438
8	1337	1336	+0.1111	+0.000083	0.6528
10	6713	6712	+0.1111	+0.000016	0.6556
12	32313	32312	+0.1111	+0.000003	0.6574
15	324039	324039	-0.1111	-0.0000003	0.6593
20	13864505	13864504	+0.1111	+0.000000008	0.6611
30	21355531833	21355531832	+0.1111	$+5.10^{-12}$	0.6630
40	29198142115385	2919814211584	+0.1111	$+3.9.10^{-15}$	0.6639

The subsequent section is devoted to the formal derivation of the generating function associated with the newly introduced sequence.

Finding the generating function

Ordinary generating functions (OGFs) serve as a robust analytical framework for transforming discrete mathematical problems into algebraic formulations. Once a sequence is encoded within an OGF, one can employ calculus-inspired operations—such as differentiation, integration, and algebraic manipulation—to derive meaningful insights, solve recurrence relations, and obtain closed-form expressions. At this stage, the generating function corresponding to the given sequence will be derived.

Theorem

$(\mathcal{E}_n)$ be a sequence defined by (2.2). Then, the ordinary generating function $G(x)$ is given by

$$G(x) = \frac{x(1+2x)}{(1-2x)^2(1+x)} \tag{2.9}$$

Proof

The OGF of the sequence can be derived by applying the relation

$$G(x) = \sum_{n\geq 0} \mathcal{E}_n x^n$$

as follows: For $(\mathcal{E}_n)$ the generator function will be calculated in detail.

1) Recurrence Relation for the Sequence $(n \geq 2)$: $\mathcal{E}_n - \mathcal{E}_{n-1} - 2\mathcal{E}_{n-2} = 2^n, \quad \mathcal{E}_0 = 0, \mathcal{E}_1 = 1.$

2) The recurrence relation is multiplied on both sides by x^n, followed by summation from $n = 2$ to infinity:

$$\sum_{n\geq 2} \mathcal{E}_n x^n - \sum_{n\geq 2} \mathcal{E}_{n-1} x^n - 2\sum_{n\geq 2} \mathcal{E}_{n-2} x^n = \sum_{n\geq 2} 2^n x^n.$$

Each summation is expressed in terms of $G(x)$: (using $\mathcal{E}_0 = 0, \mathcal{E}_1 = 1$)

$$\sum_{n\geq 2} \mathcal{E}_n x^n = G(x) - \mathcal{E}_0 - \mathcal{E}_1 x = G(x) - 0 - x = G(x) - x.$$

$$\sum_{n\geq 2} \mathcal{E}_{n-1} x^n = x\sum_{n\geq 2} \mathcal{E}_{n-1} x^{n-1} = x\sum_{m\geq 1} \mathcal{E}_m x^m = x\big(G(x) - \mathcal{E}_0\big)$$

$$= xG(x).$$

$$\sum_{n\geq 2} \mathcal{E}_{n-2} x^n = x^2\sum_{n\geq 2} \mathcal{E}_{n-2} x^{n-2} = x^2\sum_{k\geq 0} \mathcal{E}_k x^k = x^2 G(x).$$

The right-hand side:

$$\sum_{n\geq 2} 2^n x^n = \sum_{n\geq 2} (2x)^n = \frac{(2x)^2}{1-2x} = \frac{4x^2}{1-2x}, \left(for\ |x| < \frac{1}{2}\right).$$

3) Combining all the components and simplifying the resulting expression

Upon substituting the summations on the left-hand side,

$$G(x) - x - xG(x) - 2x^2G(x) = \frac{4x^2}{1-2x}.$$

Simplification,

$$G(x)(1 - x - 2x^2) - x = \frac{4x^2}{1-2x}.$$

Then,

$$G(x)(1 - x - 2x^2) = x + \frac{4x^2}{1-2x}.$$

4) Express the right-hand side as a single rational expression and reduce it to its simplest form:

$$x + \frac{4x^2}{1-2x} = \frac{x(1-2x)+4x^2}{1-2x} = \frac{x-2x^2+4x^2}{1-2x} = \frac{x+2x^2}{1-2x}$$

$$= \frac{x(1+2x)}{1-2x}.$$

Furthermore, factoring the left-hand side expression $1 - x - 2x^2$ and placing it in the denominator yields:

$$G(x) = \frac{\frac{x(1+2x)}{1-2x}}{1-x-2x^2} = \frac{x(1+2x)}{(1-2x)(1-x-2x^2)}$$

$$= \frac{x(1+2x)}{(1-2x)^2(1+x)}.$$

Here, this is the desired ordinary generating function (OGF), valid in the domain $|x| < \frac{1}{2}$.

Remark

Decomposing this ordinary generating function (OGF) into partial fractions directly yields the closed-form expression of the type $(\alpha n + \beta)2^n + \gamma(-1)^n$, that is, a linear combination of the terms $n2^n, 2^n$ and $(-1)^n$.

Indeed, the transition from the ordinary generating function (OGF) to the closed-form expression via partial fraction decomposition is carried out as follows:

Given that the denominator of OGF is $(1+x)(1-2x)^2$, its partial fraction decomposition takes the following form:

$$G(x) = \frac{K}{1+x} + \frac{L}{1-2x} + \frac{M}{(1-2x)^2}.$$

To determine the coefficients, one may expand the expression and evaluate it at strategically chosen values, such as $x = -1, x = \frac{1}{2}$ and $x = 0$.

By substituting $x = \frac{1}{2}$, only the term involving M remains, yielding $M = \frac{2}{3}$.

Substituting $x = -1$ isolates the term with K, giving $K = \frac{1}{9}$.

Finally, evaluating at $x = 0$ leads to the equation $K + L + M = 0$, which implies

$L = -\frac{7}{9}$.

Thus,

$$G(x) = \frac{1/9}{1+x} - \frac{7/9}{1-2x} + \frac{2/3}{(1-2x)^2}.$$

If each term is expanded into its power series:

1) $$\frac{1}{1+x} = \sum_{n\geq 0} (-1)^n x^n,$$

2) $$\frac{1}{1-2x} = \sum_{n\geq 0} 2^n x^n \qquad \left(for\ |x| < \frac{1}{2}\right),$$

3) $$\frac{1}{(1-2x)^2} = \sum_{n\geq 0} (n+1)2^n x^n \qquad \left(for\ |x| < \frac{1}{2}\right).$$

Thus, by collecting the coefficients from each series expansion, the following expression is obtained:

$$\mathcal{E}_n = \frac{1}{9}(-1)^n - \frac{7}{9}2^n + \frac{2}{3}(n+1)2^n = \frac{(-1)^n}{9} + \frac{(6n-1)2^n}{9}.$$

So, the following identity is obtained, which is exactly the same as the previous closed-form expression.

$$\mathcal{E}_n = \frac{(6n-1)2^n + (-1)^n}{9}.$$

Note

As a result, when analyzing the partial fraction decomposition of the ordinary generating function (OGF) leading to the closed-form expression, it is observed that the term of the form $n2^n$ arises due to the presence of $(1-2x)^2$ in the denominator.

So, the recurrence provides a recursive structure for the sequence, while the OGF serves as a powerful analytical tool for studying its properties and deriving closed-form expressions.

Summary of Steps

- The recurrence has both homogeneous and non-homogeneous parts.
- Solving the homogeneous equation $r^2 - r - 2 = 0$ gives roots $r = 2$ and $r = -1$.
- The particular solution for the 2^n term introduces an additional $n2^n$ component.
- Combining these gives the Binet formula:

$$\mathcal{E}_n = \frac{(6n-1).2^n + (-1)^n}{9}.$$

- The Maple function *E := n-> ((6*n-1)*2^n + (-1)^n)/9;* implements this directly.
- `listplot` visualizes the discrete growth pattern derived from the formula.

The Maple program command that gives the Binet closed form for new type Ernst sequence is as follows:

***Table** Code Snippet for Generating Sequence Terms via Binet's Formula*

```
# === Ernst sequence (Binet closed form) ===
# Recurrence: E[n] = E[n-1] + 2*E[n-2] + 2^n
# Initial values: E[0]=0, E[1]=1
# Derived closed form (Binet formula):
E := n -> ((6*n - 1)*2^n + (-1)^n)/9;
# --- Example evaluations ---
E(0);  # should be 0
E(1);  # should be 1
E(2);  # next term
E(9);  # term at n = 9
# --- Generate a sequence (first 10 terms) ---
seq( E(n), n=0..9 );
# --- Optional: exact vs decimal ---
seq( evalf(E(n)), n=0..9 );
# --- Optional: plot ---
with(plots):
listplot( [seq([n, E(n)], n=0..15)],
style=pointline, symbol=solidcircle, color=blue,
thickness=2, title="Binet formula of the Ernst Sequence",
labels=["n", "E(n)"] );
```

Fundamental Relations of the Sequence

The relationship between the sequence we defined and similar sequences in the literature is given below.

The two sequences to be related will now be recalled. The generalized Ernst sequence was defined as follows in (Çolak, Bilgin & Soykan, 2024):

$E_n = E_{n-1} + 2E_{n-2} + 1$, $E_0 = 0, E_1 = 1$. Binet formula of (E_n) is $\mathrm{E}_n = \frac{2^{n+1}}{3} - \frac{(-1)^n}{6} - \frac{1}{2}$.

Additionally, another related sequence, the Jacobsthal sequence (J_n), is defined in (Horadam, 1996) as follows:

$J_n = J_{n-1} + 2J_{n-2}, J_0 = 0,\ J_1 = 1,$

and its closed-form expression is given by $J_n = \frac{2^n-(-1)^n}{3}$.

Clear and useful connections are now to be derived between $\mathcal{E}_n = \mathcal{E}_{n-1} + 2\mathcal{E}_{n-2} + 2^n,\quad \mathcal{E}_0 = 0, \mathcal{E}_1 = 1,$ the classical Ernst sequence, and the Jacobsthal sequence.

Binet formula of $(\mathcal{E}_n)$ is given in (2.6) as $\mathcal{E}_n = \frac{(6n-1).2^n+(-1)^n}{9}$.

Lemma

The following equalities illustrate the connection between $(\mathcal{E}_n)$ and (J_n).

i) $\mathcal{E}_n = \frac{2}{3}n2^n - \frac{1}{3}J_n,$

ii) $\boldsymbol{\varepsilon_n} = \frac{1}{3}((2n+1)2^n - 2\mathrm{E}_n - 1),$

iii) $\mathcal{E}_n - \mathrm{E}_n = \frac{(6n-7)2^{n+1}+5(-1)^n+9}{18}$.

Proof i)

By expressing the Jacobsthal identity as $(-1)^n = 2^n - 3J_n$ and substituting it into the closed-form expression of $\mathcal{E}_n$, the terms can be rearranged to yield the equality stated above.

This equality establishes a direct algebraic relationship between $(\mathcal{E}_n)$ and the Jacobsthal sequence, allowing one to be computed in closed form when the other is known.

ii) By isolating $(-1)^n$ in both sequences and equating the resulting expressions, the desired identity is obtained.

iii) Using Binet formulas of (E_n) and $(\mathcal{E}_n)$, the desired equality can be readily obtained.

The Cassini-type identity corresponding to the sequence introduced in (2.6) is given below.

Lemma (Cassini-type (Simson) identity)

For $(\mathcal{E}_n)$, the following identity is valid.

$$\mathcal{E}_{n+1}\mathcal{E}_{n-1} - \mathcal{E}_n^2 = -\frac{(6n+1)}{9}2^{n-1}(-1)^n - \frac{2^{2n+2}}{9} \qquad (3.1)$$

Proof:

If the two terms are multiplied and various algebraic operations are applied, then

$$\mathcal{E}_{n+1}\mathcal{E}_{n-1} = \frac{(6n-7)(6n+5)2^{2n} - (6n-7)2^{n-1}(-1)^n}{81}$$

$$+\frac{-(6n+5)2^{n+1}(-1)^n + 1}{81}$$

$$\mathcal{E}_n^2 = \frac{(6n-1)^2 4^n + 2(6n-1)2^n(-1)^n + 1}{81}.$$

So,

$$\mathcal{E}_{n+1}\mathcal{E}_{n-1} - \mathcal{E}_n^2 = -\frac{(6n+1)}{9}2^{n-1}(-1)^n - \frac{2^{2n+2}}{9}.$$

The terms required to compute the Cassini identity for the sequence have been calculated, and the first five terms are presented in table. Analysis of the table reveals that the computed values are consistent with the Cassini identity formula.

Table Control Table of Cassini identity formula

n	$\mathcal{E}_{n-1}$	$\mathcal{E}_n$	$\mathcal{E}_{n+1}$	$W_n = \mathcal{E}_{n+1}\mathcal{E}_{n-1} - \mathcal{E}_n^2$
1	0	1	5	-1
2	1	5	15	-10
3	5	15	41	-20
4	15	41	103	-136
5	41	103	249	-400

Subsequently, a Catalan-type identity corresponding to the defined sequence will be established.

Lemma (Catalan type identity)

For $(\mathcal{E}_n)$, the following identity is valid.

$$\mathcal{E}_n^2 - \mathcal{E}_{n-r}\mathcal{E}_{n+r} = -\frac{(6n+6r-1)2^{n+r}(-1)^{n+r}}{81} - \frac{(6n-6r-1)2^{n-r}(-1)^{n+r}}{81} + \frac{(6n-1)2^{n+1}(-1)^n}{81} + \frac{36r^2 2^{2n}}{81}. \quad (3.2)$$

Proof

Each necessary term is computed separately and subsequently assembled to establish the target identity via algebraic operations.

$$\mathcal{E}_n^2 = \frac{(6n-1)^2 4^n + 2(6n-1)2^n(-1)^n + 1}{81},$$

$$\mathcal{E}_{n-r}\mathcal{E}_{n+r} = \frac{(6n-6r-1)(6n+6r-1)2^{2n}}{81} + \frac{(6n-6r-1)2^{n-r}(-1)^{n+r}}{81} + \frac{(6n+6r-1)2^{n+r}(-1)^{n+r}}{81} + \frac{1}{81}.$$

Then,

$$\mathcal{E}_n^2 - \mathcal{E}_{n-r}\mathcal{E}_{n+r} = -\frac{(6n+6r-1)2^{n+r}(-1)^{n+r}}{81}$$

$$-\frac{(6n-6r-1)2^{n-r}(-1)^{n+r}}{81}$$

$$+\frac{(6n-1)2^{n+1}(-1)^n}{81}+\frac{36r^2 2^{2n}}{81}.$$

The verification based on a specific selection of n and r is carried out as follows:

Control

For $n = 3, r = 1$, then $\mathcal{E}_2 = 5, \mathcal{E}_3 = 15, \mathcal{E}_4 = 41.$

$$\mathcal{E}_3^2 - \mathcal{E}_2\mathcal{E}_4 = 15^2 - 5.41 = 20.$$

$$-\frac{(6.3+6.1-1)2^4(-1)^4}{81}-\frac{(6.3-6.1-1)2^2(-1)^4}{81}$$

$$+\frac{(6.3-1)2^{3+1}(-1)^3}{81}+\frac{361^2 2^{2.3}}{81}$$

$$=\frac{2304}{81}-\frac{272}{81}-\frac{44}{81}-\frac{368}{81}=\frac{1620}{81}=20.$$

Remark ($r = 1$ in Catalan type identity)

For $(\mathcal{E}_n)$, take $r = 1$ in Catalan type identity then the following identity is valid.

$$\mathcal{E}_n^2 - \mathcal{E}_{n-1}\mathcal{E}_{n+1} = \frac{(6n-1)2^{n+1}(-1)^n}{81}-\frac{(6n+5)2^{n+1}(-1)^{n+1}}{81}$$

$$-\frac{(6n-7)2^{n-1}(-1)^{n+1}}{81}+\frac{36.\,2^{2n}}{81}.$$

Control

For $n=5, r=1$, then $\mathcal{E}_4 = 41, \mathcal{E}_5 = 103, \mathcal{E}_6 = 249.$

$\mathcal{E}_5^2 - \mathcal{E}_4\mathcal{E}_6 = 103^2 - 249.41 = 400.$

On the other hand,

$$\frac{(6.5-1)2^6(-1)^5}{81} - \frac{(6.5+5)2^6(-1)^6}{81} - \frac{(6.5-7)2^4(-1)^6}{81}$$

$$+\frac{36.2^{10}}{81}$$

$$= \frac{36864-1856-2240-368}{81} = 400.$$

Now for $(\mathcal{E}_n)$, d'Ocagne-type identity will be presented.

Lemma (d'Ocagne type identity)

$$\mathcal{E}_m\mathcal{E}_{n+1} - \mathcal{E}_{m+1}\mathcal{E}_n = \frac{4(m-n)2^{m+n+1}}{9} - \frac{(2m+1)2^m(-1)^n}{9}$$
$$+\frac{(2n+1)2^n(-1)^m}{9} \qquad (3.3)$$

Proof

Using

$$\mathcal{E}_m = \frac{(6m-1).2^m + (-1)^m}{9}, \mathcal{E}_n = \frac{(6n-1).2^n + (-1)^n}{9}.$$

$$\mathcal{E}_{n+1} = \frac{(6n+5).2^{n+1} + (-1)^{n+1}}{9}, \mathcal{E}_{m+1}$$
$$= \frac{(6m+5).2^{m+1} + (-1)^{m+1}}{9}.$$

The following equalities can be written.

$\mathcal{E}_m\mathcal{E}_{n+1}$

$$=\frac{[(6m-1).2^{m}+(-1)^{m}][(6n+5).2^{n+1}+(-1)^{n+1}]}{81},$$

$$\mathcal{E}_{m+1}\mathcal{E}_{n}$$

$$=\frac{[(6m+5).2^{m+1}+(-1)^{m+1}][(6n-1).2^{n}+(-1)^{n}]}{81}.$$

Then,

$$\mathcal{E}_{m}\mathcal{E}_{n+1}-\mathcal{E}_{m+1}\mathcal{E}_{n}$$

$$=\frac{[(6m-1).2^{m}+(-1)^{m}][(6n+5).2^{n+1}+(-1)^{n+1}]}{81}$$

$$-\frac{[(6m+5).2^{m+1}+(-1)^{m+1}][(6n-1).2^{n}+(-1)^{n}]}{81}$$

$$\mathcal{E}_{m}\mathcal{E}_{n+1}-\mathcal{E}_{m+1}\mathcal{E}_{n}$$

$$=\frac{4(m-n).2^{m+n+1}}{9}$$

$$-\frac{(2m+1)2^{m}(-1)^{n}+(2n+1)2^{n}(-1)^{m}}{9}.$$

This is the desired equality.

Control

For $m=5, n=4$, then $\mathcal{E}_4=41, \mathcal{E}_5=103, \mathcal{E}_6=249.$

$$\mathcal{E}_5\mathcal{E}_5-\mathcal{E}_6\mathcal{E}_4=400.$$

On the other hand,

$$\frac{4(5-4).2^{9+1}-(2.5+1)2^{5}(-1)^{4}+(2.4+1)2^{4}(-1)^{5}}{9}$$

$$=\frac{4096-352-144}{9}=400.$$

Matrix Formation Based on the Proposed Sequence

On a Hankel-Type Matrix and Its Determinant Properties

By using the defined sequence $\mathcal{E}_n$, a Hankel-type matrix, which is one of the special matrices found in the literature, is defined as follows:

$$H_k(n) = \begin{pmatrix} \mathcal{E}_n & \mathcal{E}_{n+1} & \mathcal{E}_{n+2} & \cdots & \mathcal{E}_{n+k-1} \\ \mathcal{E}_{n+1} & \mathcal{E}_{n+2} & \mathcal{E}_{n+3} & \cdots & \mathcal{E}_{n+k} \\ \mathcal{E}_{n+2} & \mathcal{E}_{n+3} & \mathcal{E}_{n+4} & \cdots & \mathcal{E}_{n+k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{E}_{n+k-1} & \mathcal{E}_{n+k} & \mathcal{E}_{n+k+1} & \cdots & \mathcal{E}_{n+2k-2} \end{pmatrix}$$

The matrix is composed of the terms of the sequence, organized symmetrically.

For example, when $n = 0$ and $k = 4$, since the sequence terms satisfy

$\mathcal{E}_0 = 0, \mathcal{E}_1 = 1, \mathcal{E}_2 = 5, \mathcal{E}_3 = 15, \mathcal{E}_4 = 41, \mathcal{E}_5 = 103,$

$\mathcal{E}_6 = 249$

it follows that

$$H_4(0) = \begin{pmatrix} 0 & 1 & 5 & 15 \\ 1 & 5 & 15 & 41 \\ 5 & 15 & 41 & 103 \\ 15 & 41 & 103 & 249 \end{pmatrix}.$$

Determinant Zero Condition and Its Relation to Recurrence Order

If $\det(H_k(0)) = 0$ this implies that:

- The sequence $(\mathcal{E}_n)$ can be expressed by a linear difference equation of degree k or less. In other words, the subsequent terms of the sequence can be written as a linear combination of at most k preceding terms.

For example, in the Fibonacci sequence, $\det(H_3(0)) = 0$ since the sequence is defined by a second-order recurrence,

$F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1.$

For the sequence $\mathcal{E}_n = \mathcal{E}_{n-1} + 2\mathcal{E}_{n-2} + 2^n$, since its homogeneous part is of second order, it is expected that for sufficiently large n, the Hankel determinants with $k > 2$ will be zero.

$H_1(0) = (0) \Rightarrow \det(H_1(0)) = \Delta_1 = 0.$

$H_2(0) = \begin{pmatrix} 0 & 1 \\ 1 & 5 \end{pmatrix} \Rightarrow \det(H_2(0)) = \Delta_2 = -1.$

$$H_3(0) = \begin{pmatrix} 0 & 1 & 5 \\ 1 & 5 & 15 \\ 5 & 15 & 41 \end{pmatrix} \Rightarrow \det(H_3(0)) = \Delta_3 = -16.$$

$$H_4(0) = \begin{pmatrix} 0 & 1 & 5 & 15 \\ 1 & 5 & 15 & 41 \\ 5 & 15 & 41 & 103 \\ 15 & 41 & 103 & 249 \end{pmatrix} \Rightarrow \det(H_4(0)) = \Delta_4 = 0.$$

In this case, the fact that $\det(H_4(0)) = 0$ indicates that the sequence can be fully described by a recurrence relation of order 3 or lower — which is consistent with the fact that our sequence is already defined by a second-order difference equation.

Determinant Zero Condition and Its Relation to Linearly Independent

If $\det(H_k(0)) \neq 0$, then the terms of the sequence up to that order are linearly independent. This indicates that the sequence genuinely possesses a structure of order k, and cannot be described by a recurrence relation of lower order.

On a Circulant-Type Matrix and Its Determinant Properties

Circulant matrices are matrices in which each row is obtained from the previous one by a cyclic shift of its entries to the right (or to the left) by one position. More formally, given a sequence of elements $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_{n-1})$, the first row of the matrix is formed by this sequence, and each subsequent row is generated by a cyclic rotation of the preceding row. Then,

$$C_n = circ(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_{n-1})$$

$$= \begin{pmatrix} \mathcal{E}_0 & \mathcal{E}_1 & \mathcal{E}_2 & \cdots & \mathcal{E}_{n-1} \\ \mathcal{E}_{n-1} & \mathcal{E}_0 & \mathcal{E}_1 & \cdots & \mathcal{E}_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 & \cdots & \mathcal{E}_0 \end{pmatrix}$$

For $n = 2$,

$$C_2 = \begin{pmatrix} \mathcal{E}_0 & \mathcal{E}_1 \\ \mathcal{E}_1 & \mathcal{E}_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow |C_2| = -1.$$

The inverse of a matrix C_2 is denoted by ${C_2}^{-1}$,

$${C_2}^{-1} = \frac{1}{|C_2|} \begin{pmatrix} \mathcal{E}_0 & -\mathcal{E}_1 \\ -\mathcal{E}_1 & \mathcal{E}_0 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For $n = 3$,

$$C_3 = \begin{pmatrix} 0 & 1 & 5 \\ 5 & 0 & 1 \\ 1 & 5 & 0 \end{pmatrix} \Rightarrow |C_3| = 126.$$

The inverse of a matrix C_3 is denoted by ${C_3}^{-1}$,

$${C_3}^{-1} = \frac{1}{|C_3|} \begin{pmatrix} -5 & 25 & 1 \\ 1 & -5 & 25 \\ 25 & 1 & -5 \end{pmatrix} = \frac{1}{126} \begin{pmatrix} -5 & 25 & 1 \\ 1 & -5 & 25 \\ 25 & 1 & -5 \end{pmatrix}.$$

For $n = 4$,

$$C_4 = \begin{pmatrix} 0 & 1 & 5 & 15 \\ 15 & 0 & 1 & 5 \\ 5 & 15 & 0 & 1 \\ 1 & 5 & 15 & 0 \end{pmatrix} \Rightarrow |C_4| = -51051.$$

The inverse of a matrix C_4 is denoted by ${C_4}^{-1}$,

$${C_4}^{-1} = \frac{1}{51051}\begin{pmatrix} -1130 & 3385 & 25 & 151 \\ 151 & -1130 & 3385 & 25 \\ 25 & 151 & -1130 & 3385 \\ 3385 & 25 & 151 & -1130 \end{pmatrix}.$$

Now, with the help of this matrix, we will proceed to the polynomial representation.

$$C_n = \text{circ}(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_{n-1}), \qquad P(x) = \sum_{i=0}^{n-1} \mathcal{E}_i x^i.$$

For $n = 1$, $P_1(x) = \mathcal{E}_0 = 0.$

For $n = 2$, $P_2(x) = \mathcal{E}_0 + \mathcal{E}_1 x = 0 + x = x.$

For $n = 3$, $P_3(x) = \mathcal{E}_0 + \mathcal{E}_1 x + \mathcal{E}_2 x^2$

$= 0 + x + 5x^2 = 5x^2 + x.$

For $n = 4$, $P_4(x) = \mathcal{E}_0 + \mathcal{E}_1 x + \mathcal{E}_2 x^2 + \mathcal{E}_3 x^3$

$= 15x^3 + 5x^2 + x.$

For $n = 5$, $P_5(x) = \mathcal{E}_0 + \mathcal{E}_1 x + \mathcal{E}_2 x^2 + \mathcal{E}_3 x^3 + \mathcal{E}_4 x^4$

$= 41x^4 + 15x^3 + 5x^2 + x.$

Remark

The eigenvalues of the matrices (C_n) is calculated with

$\lambda_j = P_n(\omega^j)$ for $j = 0, \dots, n-1$ and $\omega = e^{\frac{2\pi i}{n}}$,

$$|C_n| = \prod_{j=0}^{n-1} P(\omega^j).$$

- For $n = 2$, $P_2(x) = x$.

  $j = 0: \lambda_0 = P(1) = 1$ and $j = 1: \lambda_1 = P(-1) = -1$.

- For $n = 3$, $P_3(x) = 5x^2 + x$

using $\omega = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ and $\omega^2 = e^{\frac{4\pi i}{3}} = cos\left(\frac{4\pi}{3}\right) + isin\left(\frac{4\pi}{3}\right) \Rightarrow \omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$.

$j = 0: \lambda_0 = P(1) = 6$ and $j = 1: \lambda_1 = P(\omega) = \omega + 5\omega^2$

$= -3 - 2\sqrt{3}i$,

$j = 2: \lambda_2 = P(\omega^2) = \omega^2 + 5\omega = -3 + 2\sqrt{3}i$.

- For $n = 4$, $P_4(x) = 15x^3 + 5x^2 + x$.

$j = 0: \lambda_0 = P(1) = 21$, $j = 1: \lambda_1 = P(i) = -5 - 14i$,

$j = 2: \lambda_2 = P(-1) = -11$, $j = 3: \lambda_3 = P(-i) = -5 + 14i$.

- For $n = 5$, $P_5(x) = 41x^4 + 15x^3 + 5x^2 + x$.

$j = 0: \lambda_0 = P(1) = 62$, $j = 1: \lambda_1 \cong -3.201626 - i\ 43.920113$,

$j = 2: \lambda_2 \cong -27.798374 - i\ 14.000845$,

$j = 3: \lambda_3 = -27.798374 + i\ 14.000845$,

$j = 4: \lambda_4 = -3.201626 + i\ 43.920113$.

Remark

Note that the trace of (C_n) equals the sum of its eigenvalues.

$$tr(C_n) = \sum_{i=0}^{n-1} \lambda_i.$$

Eigenvalues

$$\lambda_j = \sum_{i=0}^{n-1} \mathcal{E}_i\, \omega^{ij}, \omega = e^{\frac{2\pi i}{n}}.$$

Then,

$$tr(C_n) = \sum_{j=0}^{n-1}\sum_{i=0}^{n-1} \mathcal{E}_i \omega^{ij} = \sum_{i=0}^{n-1} \mathcal{E}_i \sum_{j=0}^{n-1} \omega^{ij}.$$

The sum inside:

$$\sum_{j=0}^{n-1} \omega^{ij} = \begin{cases} n, & j \equiv 0 \ (mod\ n) \\ 0, & otherwise \end{cases}$$

Here, since $0 \le i \le n-1$ and, apart from $i = 0$, we have $\omega^i \neq 1$, so inner sum is equal to n only only for $i = 0$, and it is equal 0 for all other values of i.

$$tr(C_n) = \mathcal{E}_0 . n = 0. n = 0.$$

Thus, the trace of C_n is zero for all, since all elements on the diagonal are $\mathcal{E}_0 = 0$.

Conclusion

In this study, a new sequence called the Generalized Ernst numbers has been systematically introduced and examined as a generalized form of non-homogeneous linear recurrence relations involving exponential terms.

The analysis established structural and algebraic connections between the Generalized Ernst sequence and classical sequences such as Jacobsthal and Ernst numbers, presenting these relationships through explicit closed-form expressions. Furthermore, analogues of classical identities—including Cassini, Catalan, and d'Ocagne type relations—have been developed for the Generalized Ernst numbers, demonstrating that these traditional formulations extend naturally to the newly defined sequence.

In addition, Hankel matrix analyses were performed to explore the determinantal properties of the sequence, offering deeper insight into its algebraic structure. Theoretical findings were further validated through numerical examples, graphical representations, and asymptotic evaluations, which confirmed the consistency and predictive accuracy of the derived results.

Overall, the existing studies on Jacobsthal and Ernst numbers served as both a conceptual foundation and a comparative framework. Within this context, the Generalized Ernst numbers provide a novel contribution to the theory of recurrence relations by extending classical results to a broader class of non-homogeneous, exponentially driven systems. Also, this study can be extended to newly obtained sequences with the aid of (Soykan &Numbers, 2020) and (Soykan, 2020), allowing for examination from different perspectives.

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MULATU–ERNST NUMBERS: CONNECTIONS BETWEEN TWO RECURSIVELY DEFINED SEQUENCES

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Introduction

Recurrence relations are fundamental tools in discrete mathematics and number theory. The study of integer sequences arising from linear recurrences has yielded numerous classical examples such as Fibonacci, Lucas, and Jacobsthal sequences (Koshy, 2001), (Sloane, 2007), (Horadam, 1984).

In this work, we focus on two sequences defined by the recurrence

$$X_n = X_{n-1} + 2X_{n-2} + 1, \quad n \geq 2, \tag{1.1}$$

with same initial conditions.

The first sequence, denoted by J_n, is defined in (Horadam, 1984) by

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$$J_0 = 0, J_1 = 1,\ J_n = J_{n-1} + 2J_{n-2},\ (n \geq 2) \tag{1.2}$$

The first few terms of these sequences,

$J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 3, J_4 = 5, J_5 = 11, J_6 = 21, J_7 = 43,$

$J_8 = 85, J_9 = 171, \ldots$

The second sequence, denoted by $\tilde{E}_n$, is given in (Soykan, 2022) by

$$\tilde{E}_n = \tilde{E}_{n-1} + 2\tilde{E}_{n-2} + 1, \quad n \geq 2, \quad \tilde{E}_0 = 0, \tilde{E}_1 = 1 \tag{1.3}$$

This sequence was named the Lichtenberg sequence by Hinz in 2017, represented by (A000975) in (Hinz, 2017).

The first few terms of these sequences,

$\tilde{E}_0 = 0, \tilde{E}_1 = 1, \tilde{E}_2 = 2, \tilde{E}_3 = 5, \tilde{E}_4 = 10, \tilde{E}_5 = 21, \tilde{E}_6 = 42,$

$\tilde{E}_7 = 85, \tilde{E}_8 = 170, \tilde{E}_9 = 341, \ldots$

For quite some time, the Ernst sequence has caught the attention of researchers, thanks to its distinctive structure and its important place in discrete mathematics. Although the homogeneous form has been studied in detail, the non-homogeneous versions haven't been explored with the same level of care. These variations are significant, though—they expand the range of possible solutions and offer fresh insights into how related combinatorial and algebraic systems work.

At this point, the essential properties of the Jacobsthal sequence that are required for this study will be recalled.

For Jacobsthal sequence Homogeneous recurrence relation is defined in (Horadam, 1984) by

$J_n - J_{n-1} - 2J_{n-2} = 0$

The characteristic equation is $r^2 - r - 2 = 0$ which has the roots $r_1 = 2, r_2 = -1$.

General solution

$J_n = A.2^n + B.(-1)^n.$

Determination of constants

From the initial conditions

For $n = 0$: $0 = A + B \Rightarrow B = -A$

For $n = 1$: $1 = 2A - B \Rightarrow 2A - (-A) = 3A \Rightarrow A = \frac{1}{3}, B = -\frac{1}{3}.$

Closed-form (Binet formula)

$$J_n = \frac{1}{3}(2^n - (-1)^n) \quad (1.4)$$

In numerical sequences, recurrence relations usually require starting with the first term and progressively computing subsequent terms in order to obtain the desired element. However, the availability of closed-form expressions allows for the quick calculation of any given term. These equations are particularly useful when the relationships between sequences need to be expressed because they yield very effective results.

The closed form of $Ể_n$ from (Soykan, 2022) is now recalled.

This linear recurrence relation contains a non-homogeneous term. For a detailed comparison with the computational steps of the other sequence, the general solution given in (Soykan, 2022) will be examined.

Homogeneous part

First, consider the homogeneous recurrence and disregard the non-homogeneous term.

$Ể_n^h = Ể_{n-1}^h + 2Ể_{n-2}^h$

The characteristic equation is:

$r^2 - r - 2 = 0,\ r_1 = 2, r_2 = -1$

Hence, the homogeneous solution is:

$\text{Ể}_n^h = A.2^n + B.(-1)^n$

Particular solution

Since the non-homogeneous term is a constant, a constant particular solution $\text{Ể}_n^p = C$ is attempted.

$\text{Ể}_n^p = \text{Ể}_{n-1}^p + 2\text{Ể}_{n-2}^p + 1.$

Substituting C: $C = C + 2C + 1 \Rightarrow -2C = 1 \Rightarrow C = -\frac{1}{2}.$

So the particular solution is $\text{Ể}_n^p = -\frac{1}{2}$.

General solution

$\text{Ể}_n = \text{Ể}_n^h + \text{Ể}_n^p = A.2^n + B.(-1)^n - \frac{1}{2}$

Determine constants using initial conditions

For $n = 0$: $\text{Ể}_0 = 0 = A.2^0 + B.(-1)^0 - \frac{1}{2} = A + B - \frac{1}{2}$

$\Rightarrow A + B = \frac{1}{2}.$

For $n = 1$: $\text{Ể}_1 = 1 = A.2^1 + B.(-1)^1 - \frac{1}{2} = 2A - B - \frac{1}{2}$

$\Rightarrow 2A - B = \frac{3}{2}.$

Solve the system: $A + B = \frac{1}{2},\ \ 2A - B = \frac{3}{2}.$

Then, $B = -\frac{1}{6}, A = \frac{2}{3}.$

Closed form

$$\tilde{E}_n = \frac{2}{3}.2^n - \frac{1}{6}.(-1)^n - \frac{1}{2} \qquad (1.5)$$

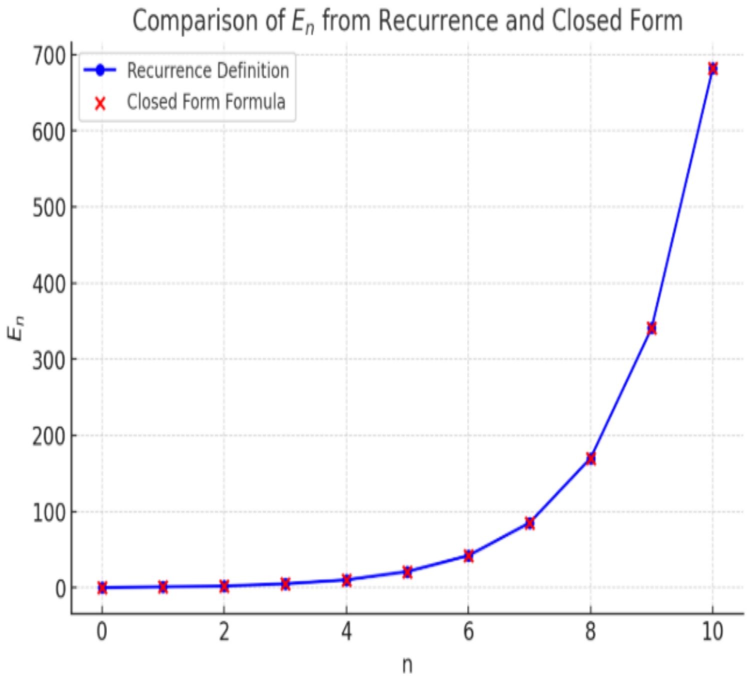
The next table presents the first six terms of the sequence, computed both from the recurrence relation and from the closed-form expression, showing that the two approaches yield identical results.

Table *First Few Terms of* $\tilde{E}_n$

n	Recurrence Definition of $\tilde{E}_n$	Closed Form of $\tilde{E}_n$
0	$\tilde{E}_0 = 0$	$\frac{2}{3}2^0 - \frac{1}{6}(-1)^0 - \frac{1}{2} = 0$
1	$\tilde{E}_1 = 1$	$\frac{2}{3}2^1 - \frac{1}{6}(-1)^1 - \frac{1}{2} = 1$
2	$\tilde{E}_2 = \tilde{E}_1 + 2\tilde{E}_0 + 1 = 2$	$\frac{2}{3}2^2 - \frac{1}{6}(-1)^2 - \frac{1}{2} = 2$
3	$\tilde{E}_3 = \tilde{E}_2 + 2\tilde{E}_1 + 1 = 5$	$\frac{2}{3}2^3 - \frac{1}{6}(-1)^3 - \frac{1}{2} = 5$
4	$\tilde{E}_4 = \tilde{E}_3 + 2\tilde{E}_2 + 1 = 10$	$\frac{2}{3}2^4 - \frac{1}{6}(-1)^4 - \frac{1}{2} = 10$
5	$\tilde{E}_5 = \tilde{E}_4 + 2\tilde{E}_3 + 1 = 21$	$\frac{2}{3}2^5 - \frac{1}{6}(-1)^5 - \frac{1}{2} = 21$

Now, in the following figure, the graphs obtained from the recurrence relation of the sequence and its closed-form expression will be compared.

Figure *First 10 Terms of* $\tilde{E}_n$



In the graph, the blue curve corresponds to the recurrence relation, while the red points represent the closed-form formula; the two coincide perfectly.

Now we will mention the Mulatu sequences, the Mulatu series, one of the preferred series for combining elements in late years. In recent times, there has been a growing interest in the Mulatu sequences. For studies on Mulatu sequences, (Lemma, Lambright & Atena, 2016), (Lemma, 2019), (Prabowo, 2020), (Lemma & Lambright, 2021), (Lemma & et al., 2021), (Erduvan, 2023), (Erduvan & Duman, 2023), (Lemma, Mohammed & Lambright, 2024), (Adédji, Adjakidjè & Togbé, 2024), (Costa, da Costa Mesquita & Catarino, 2025), (Adédji, Bachabi & Togbé, 2025), (da Costa Mesquita & et al., 2025), (Derso & Admasu, 2025), (Adédji, 2025) can be examined.

Lemma et al. (2016) presented noteworthy properties and novel findings concerning the mathematical structure of the Mulatu number sequences in (Lemma, Lambright & Atena, 2016). In his 2023 study, Erduvan demonstrated that Mulatu numbers can be generated through the concatenation of two Lucas numbers, thereby revealing an alternative construction method for the sequence in (Erduvan, 2023). In their 2024 study, Adédji et all. established that Mulatu numbers admit a formulation as the product of three generalized Lucas numbers in (Adédji, Adjakidjè & Togbé, 2024). Costa et al. (2025) investigate the shared structural features of the Fibonacci and Mulatu sequences, deriving novel relations and identities from these commonalities in (Costa, da Costa Mesquita & Catarino, 2025). Adédji et al. (2025) examined Thabit and Williams numbers in relation to Fibonacci and Mulatu numbers, focusing on their representation through additive and subtractive connections in (Adédji, Bachabi & Togbé, 2025).

In this paper, we look at two integer sequences defined by the same recurrence relation but starting from different initial values. We call them $\mathcal{M}Ẻ_n$ and $Ẻ_n$, and show that they share strong structural ties with the Jacobsthal sequence. Closed-form formulas are presented, linear connections between the sequences are highlighted, and their sums and differences are studied. We back up the results with numerical examples and bring them to life through visual illustrations.

General Solution of a Non-Homogeneous Extension of the Ernst Sequence

This section derives the general solution for a nonhomogeneous extension of the Ernst sequence. First, the homogeneous iteration is examined, and then the nonhomogeneous term is introduced using standard methods for specific solutions. The computational steps are methodically compared with those presented in previous studies (Soykan, 2022) to highlight both similarities and the unique features of the generalized solution. This detailed analysis clarifies the mathematical framework and lays the groundwork for further applications in related fields.

To preserve the initial conditions of the Mulatu sequence, a generalization of the Ernst sequence is defined as follows.

$$\mathcal{M}\hat{E}_n = \mathcal{M}\hat{E}_{n-1} + 2\mathcal{M}\hat{E}_{n-2} + 1, \qquad n \geq 2,$$
$$\mathcal{M}\hat{E}_0 = 4, \mathcal{M}\hat{E}_1 = 1 \tag{2.1}$$

$\mathcal{M}\hat{E}_0 = 4, \mathcal{M}\hat{E}_1 = 1, \mathcal{M}\hat{E}_2 = 10, \mathcal{M}\hat{E}_3 = 13,$

$\mathcal{M}\hat{E}_4 = 34, \mathcal{M}\hat{E}_5 = 61, \mathcal{M}\hat{E}_6 = 130, \mathcal{M}\hat{E}_7 = 253,$

$\mathcal{M}\hat{E}_8 = 514, \mathcal{M}\hat{E}_9 = 1021 \ldots$

This is a linear non-homogeneous recurrence relation. The solution will be carried out step by step.

Homogeneous Solution

When the non-homogeneous term is ignored, the corresponding homogeneous recurrence can be expressed through its characteristic equation.

Hence, the homogeneous solution is

$\mathcal{M}\hat{E}_n^h = \mathcal{M}\hat{E}_{n-1}^h + 2\mathcal{M}\hat{E}_{n-2}^h$

with characteristic equation $r^2 - r - 2 = 0$.

The roots are $r_1 = 2, r_2 = -1$. Hence, the homogeneous solution is

$\mathcal{M}\hat{E}_n^h = A.2^n + B.(-1)^n$.

Particular Solution

Since the non-homogeneous term is constant, a constant particular solution $\mathcal{M}\hat{\tilde{E}}_n^p = C$ is considered.

Substituting this into the recurrence $\mathcal{M}\hat{\tilde{E}}_n = \mathcal{M}\hat{\tilde{E}}_{n-1} + 2\mathcal{M}\hat{\tilde{E}}_{n-2} + 1$ yields $C = C + 2C + 1 \Rightarrow C = -\frac{1}{2}$, which simplifies to. Thus, the particular solution is $\mathcal{M}\hat{\tilde{E}}_n^p = -\frac{1}{2}$.

General Solution

The general solution is

$$\mathcal{M}\hat{\tilde{E}}_n = \mathcal{M}\hat{\tilde{E}}_n^h + \mathcal{M}\hat{\tilde{E}}_n^p = A.2^n + B.(-1)^n - \frac{1}{2}.$$

Determination of Constants

Using the initial conditions:

$$n = 0: \mathcal{M}\hat{\tilde{E}}_0 = 4 = A + B - \frac{1}{2} \Rightarrow A + B = \frac{9}{2},$$

$$n = 1: \mathcal{M}\hat{\tilde{E}}_1 = 1 = 2A - B - \frac{1}{2} \Rightarrow 2A - B = \frac{3}{2}.$$

Solving this system $A + B = \frac{9}{2}, 2A - B = \frac{3}{2}$, then

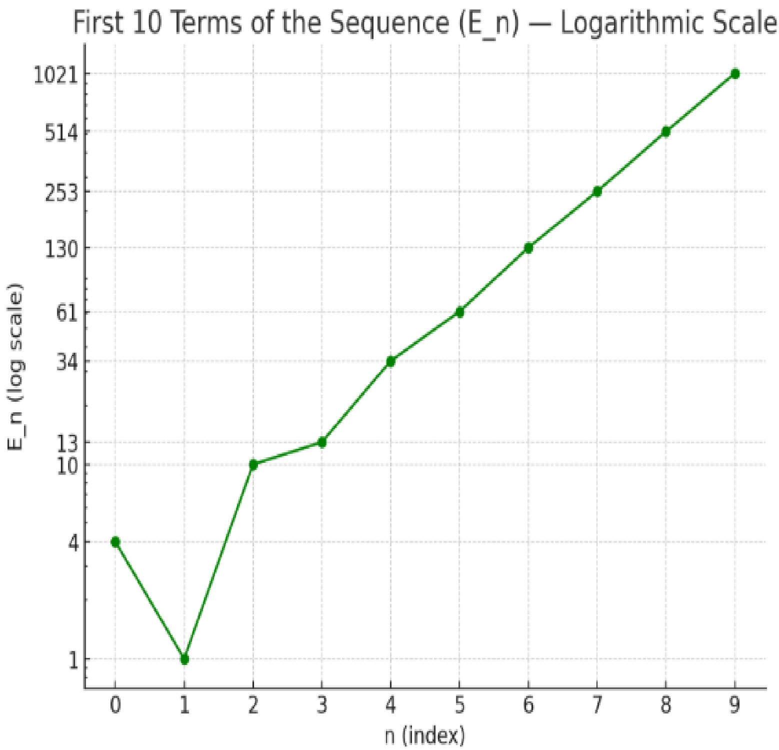
$$A = 2, B = \frac{5}{2}.$$

Closed-Form Expression

Finally, the closed-form expression of the sequence is

$$\mathcal{M}\hat{\tilde{E}}_n = 2.2^n + \frac{5}{2}.(-1)^n - \frac{1}{2} \qquad (2.2)$$

Figure *The graph of the sequence $\mathcal{M}\tilde{\hat{E}}_n$ drawn according to its closed form*



Both the rapid growth (particularly due to the 2^n factor) and the minor oscillations arising from the $(-1)^n$ term are clearly observable.

The Relationships Between the Sequences

We want to examine these two sequences, compare their growth, and determine how they relate to the Jacobsthal numbers. The recurrence relation $\tilde{\hat{E}}_n = \tilde{\hat{E}}_{n-1} + 2\tilde{\hat{E}}_{n-2} + 1$, initial conditions $\tilde{\hat{E}}_0 = 0, \tilde{\hat{E}}_1 = 1$ and the closed-form $\tilde{\hat{E}}_n = \frac{2}{3}2^n - \frac{1}{6}(-1)^n - \frac{1}{2}$ with the recurrence relation

$\mathcal{M}\text{Ể}_n = \mathcal{M}\text{Ể}_{n-1} + 2\mathcal{M}\text{Ể}_{n-2} + 1,$ initial conditions $\mathcal{M}\text{Ể}_0 = 4, \mathcal{M}\text{Ể}_1 = 1$ and the closed-form $\mathcal{M}\text{Ể}_n = 2.2^n + \frac{5}{2}.(-1)^n - \frac{1}{2}$.

The relationship between these two sequences can be described as follows:

The difference sequence of $\mathcal{M}\text{Ể}_n$ and Ể_n

Using closed form

$$\text{Ể}_n = \frac{2}{3}2^n - \frac{1}{6}(-1)^n - \frac{1}{2}, \qquad \mathcal{M}\text{Ể}_n = 2.2^n + \frac{5}{2}.(-1)^n - \frac{1}{2},$$

$$D_n := \text{Ể}_n - \mathcal{M}\text{Ể}_n$$

is calculated as follows:

$$D_n = \left(\frac{2}{3}2^n - \frac{1}{6}(-1)^n - \frac{1}{2}\right) - \left(2.2^n + \frac{5}{2}.(-1)^n - \frac{1}{2}\right)$$

$$= \frac{2}{3}2^n - \frac{1}{6}(-1)^n - \frac{1}{2} - 2.2^n - \frac{5}{2}.(-1)^n + \frac{1}{2}$$

$$= -\frac{4}{3}2^n - \frac{8}{3}(-1)^n.$$

Then

$$D_n = \text{Ể}_n - \mathcal{M}\text{Ể}_n = -\frac{4}{3}2^n - \frac{8}{3}(-1)^n \qquad (3.1)$$

So,

$$\mathcal{M}\text{Ể}_n - \text{Ể}_n = \frac{4}{3}2^n + \frac{8}{3}(-1)^n.$$

The relationship between $\mathcal{M}\text{Ể}_n$ and Ể_n,

$$\mathcal{M}\text{Ể}_n - \text{Ể}_n = \frac{4}{3}2^n + \frac{8}{3}(-1)^n.$$

Equivalently, the relation can be expressed as

$$\mathcal{M}\text{Ẻ}_n = \text{Ẻ}_n + \frac{4}{3}2^n + \frac{8}{3}(-1)^n \qquad (3.2)$$

(As a brief check: for $n = 2$ one obtains $\mathcal{M}\text{Ẻ}_2 = 10, \text{Ẻ}_2 = 2$ and thus their difference is 8.

On the other hand, substituting into the closed-form expression yields, which is consistent $\frac{4}{3}.4 + \frac{8}{3}.1 = 8$)

Table *Values of* $\mathcal{M}\text{Ẻ}_n, \text{Ẻ}_n$ *and difference sequence*

n	$\mathcal{M}\text{Ẻ}_n$	Ẻ_n	$\mathcal{M}\text{Ẻ}_n - \text{Ẻ}_n$
0	4	0	4
1	1	1	0
2	10	2	8
3	13	5	8
4	34	10	24
5	61	21	40
6	130	42	88
7	253	85	168
8	514	170	344
9	1021	341	680

In Maple, the definition of such recursive sequences and the plotting of their graphs can be carried out using the following commands:

Table *Program Command for Sequence Terms*

```
with(plots):
# Closed-form definitions of the sequences
MEn := n -> 2^(n+1) + (5*(-1)^n - 1)/2:
En  := n -> (2/3)*2^n - (1/6)*(-1)^n - 1/2:
# Prepare the points (for n=0..9)
pointsME := [seq([n, MEn(n)], n=0..9)]:
pointsE  := [seq([n, En(n)],  n=0..9)]:
# Generate a graph
plot([pointsME, pointsE],
style=pointline,
symbol=[circle, square],
color=[blue, red],
scaling=logarithmic,
labels=["n", "Value"],
legend=["ME_n", "E_n"]);
```

The relationship between $Ể_n$ and J_n,

It is known that the Jacobsthal sequence is defined by

$$J_0 = 0, J_1 = 1, \qquad J_n = J_{n-1} + 2J_{n-2}, \qquad (n \geq 2)$$

and its closed form is

$$J_n = \frac{2^n - (-1)^n}{3}.$$

First, the expressions 2^n and $(-1)^n$ should be formulated with J_n and J_{n+1}.

From the closed form it is obtained:

$3J_n = 2^n - (-1)^n$,

$3J_{n+1} = 2^{n+1} - (-1)^{n+1} = 2.2^n + (-1)^n$.

Adding these two equations gives:

$3J_n + 3J_{n+1} = 3.2^n \Rightarrow 2^n = J_n + J_{n+1}$.

Substituting this back into the first equation:

$(-1)^n = 2^n - 3J_n = (J_n + J_{n+1}) - 3J_n = J_{n+1} - 2J_n$

Thus,

$$2^n = J_n + J_{n+1}, \quad (-1)^n = J_{n+1} - 2J_n. \qquad (3.3)$$

In this case, $\tilde{\hat{E}}_{\boldsymbol{n}}$ is expressed in terms of J_n:

Using

$$\tilde{\hat{E}}_n = \frac{2}{3}2^n - \frac{1}{6}(-1)^n - \frac{1}{2},$$

Substitute $2^n = J_n + J_{n+1}$ and $(-1)^n = J_{n+1} - 2J_n$,

$$\tilde{\hat{E}}_n = \frac{2}{3}(J_n + J_{n+1}) - \frac{1}{6}(J_{n+1} - 2J_n) - \frac{1}{2}$$

$$= \left(\frac{2}{3} + \frac{2}{6}\right)J_n + \left(\frac{2}{3} - \frac{1}{6}\right)J_{n+1} - \frac{1}{2}$$

$$= J_n + \frac{1}{2}J_{n+1} - \frac{1}{2}.$$

So,

$$\tilde{\hat{E}}_n = J_n + \frac{1}{2}J_{n+1} - \frac{1}{2}. \qquad (3.4)$$

On the other hand using,

$\tilde{\hat{E}}_n = \frac{2}{3}2^n - \frac{1}{6}(-1)^n - \frac{1}{2}$ and $2^n = 3J_n + (-1)^n$

then,

$$Ể_n = \frac{2}{3}(3J_n + (-1)^n) - \frac{1}{6}(-1)^n - \frac{1}{2}$$

$$= 2J_n + \frac{2(-1)^n}{3} - \frac{(-1)^n}{6} - \frac{1}{2} = 2J_n + \frac{(-1)^n - 1}{2}.$$

So,

$$Ể_n = 2J_n + \frac{(-1)^n - 1}{2}. \qquad (3.5)$$

Table *Values of* $J_n, Ể_n$ *and* $\left(2J_n + \frac{(-1)^n-1}{2}\right)$

n	J_n	$Ể_n$	$2J_n + \frac{(-1)^n - 1}{2}$
0	0	0	$2.0 + \frac{1-1}{2} = 0$
1	1	1	$2.1 + \frac{-1-1}{2} = 1$
2	1	2	$2.1 + \frac{1-1}{2} = 2$
3	3	5	$2.3 + \frac{-1-1}{2} = 5$
4	5	10	$2.5 + \frac{1-1}{2} = 10$
5	11	21	$2.11 + \frac{-1-1}{2} = 21$
6	21	42	$2.21 + \frac{1-1}{2} = 42$
7	43	85	$2.43 + \frac{-1-1}{2} = 85$
8	85	170	$2.85 + \frac{1-1}{2} = 170$
9	171	341	$2.171 + \frac{-1-1}{2} = 341$
10	341	682	$2.341 + \frac{1-1}{2} = 682$

The relationship between $\mathcal{M}\text{Ể}_n$ and J_n

Using

$$\mathcal{M}\text{Ể}_n = 2.2^n + \frac{5}{2}.(-1)^n - \frac{1}{2} ,$$

Then,

$2^{n+1} = 2(J_n + J_{n+1})$ and $(-1)^n = J_{n+1} - 2J_n$.

$$\mathcal{M}\text{Ể}_n = 2(J_n + J_{n+1}) + \frac{1}{2}(5(J_{n+1} - 2J_n) - 1)$$

$$= \frac{1}{2}(4J_n + 4J_{n+1} + 5J_{n+1} - 10J_n - 1)$$

$$= \frac{1}{2}(-6J_n + 9J_{n+1} - 1)$$

$$= -3J_n + \frac{9}{2}J_{n+1} - \frac{1}{2}.$$

Thus,

$$\boxed{\mathcal{M}\text{Ể}_n = -3J_n + \frac{9}{2}J_{n+1} - \frac{1}{2}. \qquad (3.6)}$$

Table *Values of* $J_n, \mathcal{M}\tilde{\hat{E}}_n$ *and* $\left(-3J_n + \frac{9}{2}J_{n+1} - \frac{1}{2}\right)$

n	J_n	$\mathcal{M}\tilde{\hat{E}}_n$	$-3J_n + \frac{9}{2}J_{n+1} - \frac{1}{2}$
0	0	4	$0 + \frac{9}{2} - \frac{1}{2} = 4$
1	1	1	$-3 + \frac{9}{2} - \frac{1}{2} = 1$
2	1	10	$-3 + \frac{27}{2} - \frac{1}{2} = 10$
3	3	13	$-9 + \frac{45}{2} - \frac{1}{2} = 13$
4	5	34	$-15 + \frac{99}{2} - \frac{1}{2} = 34$
5	11	61	$-33 + \frac{189}{2} - \frac{1}{2} = 61$
6	21	130	$-63 + \frac{387}{2} - \frac{1}{2} = 130$
7	43	253	$-129 + \frac{765}{2} - \frac{1}{2} = 253$
8	85	514	$-255 + \frac{1539}{2} - \frac{1}{2} = 514$
9	171	1021	$-513 + \frac{3069}{2} - \frac{1}{2} = 1021$
10	341	2050	$-1023 + \frac{6147}{2} - \frac{1}{2} = 2050$

Also,

$$\mathcal{M}\tilde{\hat{E}}_n - J_n = \frac{5}{3}.2^n + \frac{17}{6}.(-1)^n - \frac{1}{2} \tag{3.7}$$

Table *Values of* $J_n, \mathcal{M}Ể_n$ *and difference sequence*

n	J_n	$\mathcal{M}Ể_n$	$\mathcal{M}Ể_n - J_n$
0	0	4	4
1	1	1	0
2	1	10	9
3	3	13	10
4	5	34	29
5	11	61	50
6	21	130	109
7	43	253	210
8	85	514	429
9	171	1021	850
10	341	2050	1709

Consequently,

$$Ể_n \quad = J_n + \frac{1}{2}J_{n+1} - \frac{1}{2},$$

$$\mathcal{M}Ể_n = -3J_n + \frac{9}{2}J_{n+1} - \frac{1}{2}.$$

Equivalently, the second equation can be expressed as

$$2\mathcal{M}Ể_n = 9J_{n+1} - 6J_n - 1.$$

These are the explicit forms of $Ể_n$ and $\mathcal{M}Ể_n$ expressed entirely in terms of the Jacobsthal sequence J_n.

Representation of Jacobsthal sequences using $\mathcal{M}Ể_n$

The closed-form solution of the sequence can also be expressed using the Jacobsthal numbers J_n, defined by the recurrence

$$J_n = J_{n-1} + 2J_{n-2}, \qquad J_0 = 0, \qquad J_1 = 1.$$

Using Binet-type formula for the Jacobsthal sequence and from (3.3), it is written

$$2^n = J_n + J_{n+1}, \qquad (-1)^n = J_{n+1} - 2J_n.$$

By appropriate algebraic manipulations, the closed-form expression of $\mathcal{M}\hat{\tilde{E}}_n$ can be rewritten in terms of J_n as:

$$\mathcal{M}\hat{\tilde{E}}_n = 2.2^n + \frac{5}{2}.(-1)^n - \frac{1}{2}$$

and the Jacobsthal sequence satisfies

$$3J_n = 2^n - (-1)^n$$

Our goal is to express J_n directly in terms of $\mathcal{M}\hat{\tilde{E}}_n$ and $\mathcal{M}\hat{\tilde{E}}_{n-1}$. Then, for the purpose of expressing 2^n and $(-1)^n$ in terms of the numbers $\mathcal{M}\hat{\tilde{E}}_n$ and $\mathcal{M}\hat{\tilde{E}}_{n-1}$, the recurrence relation is formulated for n and $n-1$:

$$\begin{cases} \mathcal{M}\hat{\tilde{E}}_n = 2.2^n + \frac{5}{2}(-1)^n - \frac{1}{2}, \\ \mathcal{M}\hat{\tilde{E}}_{n-1} = 2.2^{n-1} + \frac{5}{2}(-1)^{n-1} - \frac{1}{2}. \end{cases}$$

By solving this linear system for 2^n and $(-1)^n$, so

$$(-1)^n = \frac{2\mathcal{M}\hat{\tilde{E}}_n - 4\mathcal{M}\hat{\tilde{E}}_{n-1} - 1}{15}, \qquad 2^n = \frac{\mathcal{M}\hat{\tilde{E}}_n + \mathcal{M}\hat{\tilde{E}}_{n-1} + 1}{3}.$$

Thus, substitute into the Jacobsthal formula

Using $3J_n = 2^n - (-1)^n$, then

$$J_n = \frac{1}{3}\left(\frac{\mathcal{M}\hat{\tilde{E}}_n + \mathcal{M}\hat{\tilde{E}}_{n-1} + 1}{3} - \frac{2\mathcal{M}\hat{\tilde{E}}_n - 4\mathcal{M}\hat{\tilde{E}}_{n-1} - 1}{15}\right)$$

Simplifying this expression yields

$$\boxed{J_n = \frac{\mathcal{M}\hat{\tilde{E}}_n + 3\mathcal{M}\hat{\tilde{E}}_{n-1} + 2}{15}, \qquad n \geq 1} \qquad (3.8)$$

Verification

Fort he given initial conditions $\mathcal{M}\hat{\tilde{E}}_0 = 4$ and $\mathcal{M}\hat{\tilde{E}}_1 = 1$:

$$J_1 = \frac{1 + 3.4 + 2}{15} = \frac{15}{15} = 1,$$

which matches the Jacobsthal sequence definition.

$$J_2 = \frac{10 + 3.1 + 2}{15} = \frac{15}{15} = 1, J_3 = \frac{13 + 3.10 + 2}{15} = \frac{45}{15} = 3,$$

$$J_4 = \frac{34 + 3.13 + 2}{15} = \frac{75}{15} = 5,$$

$$J_5 = \frac{61 + 3.34 + 2}{15} = \frac{165}{15} = 11,$$

$$J_6 = \frac{130 + 3.61 + 2}{15} = \frac{315}{15} = 21, \ldots$$

This representation highlights the structural relationship between the sequence $\mathcal{M}\text{Ễ}_n$ and the Jacobsthal numbers.

Such an expression not only connects $\mathcal{M}\text{Ễ}_n$ with a well-studied classical sequence but also facilitates further algebraic and combinatorial interpretations.

The relationship between D_n and J_n

In equation (3.1), the closed form of the sequence D_n was found as

$$D_n = -\frac{4}{3}2^n - \frac{8}{3}(-1)^n.$$

From closed form:

$$J_n = \frac{2^n - (-1)^n}{3} \Rightarrow 2^n = 3J_n + (-1)^n.$$

If D_n is substituted,

$$D_n = -\frac{4}{3}(3J_n + (-1)^n) - \frac{8}{3}(-1)^n$$

$$D_n = -4J_n - \frac{4}{3}(-1)^n - \frac{8}{3}(-1)^n$$

$$D_n = -4J_n - 4(-1)^n.$$

is obtained. So,

$$D_n = -4(J_n + (-1)^n) \qquad (3.9)$$

The sum of sequence Ể_n, $\mathcal{M}\text{Ể}_n$

For $T_n := \text{Ể}_n + \mathcal{M}\text{Ể}_n$, using definition of sequences

$$T_n = \left(\frac{2}{3}2^n - \frac{1}{6}(-1)^n - \frac{1}{2}\right) + \left(2.2^n + \frac{5}{2}(-1)^n - \frac{1}{2}\right).$$

Then,

$$T_n = \frac{8}{3}2^n + \frac{7}{3}(-1)^n - 1 \qquad (3.10)$$

The relationship between T_n and J_n

From closed form of J_n

$$2^n = 3J_n + (-1)^n.$$

Then,

$$T_n = \frac{8}{3}(3J_n + (-1)^n) + \frac{7}{3}(-1)^n - 1$$

$$= 8J_n + 5(-1)^n - 1.$$

So,

$$T_n = 8J_n + 5(-1)^n - 1. \qquad (3.11)$$

(3.4)-(3.6), (3.9) and (3.11) demonstrate that all four sequences ($\mathcal{M}\text{Ể}_n, \text{Ể}_n, D_n, T_n$) can be expressed as linear combinations of Jacobsthal numbers and alternating terms.

All sequences can be expressed in terms of the Jacobsthal (J_n) and the parity term $(-1)^{\mathrm{n}}$ as follows:

Table *Table of some important formulas*

$*\quad \mathcal{M}\text{Ể}_n = 6J_n + \frac{9(-1)^n - 1}{2},$
$*\quad \text{Ể}_n = 2J_n + \frac{(-1)^n - 1}{2},$
$*\quad D_n = -4J_n - 4(-1)^n,$
$*\quad T_n = 8J_n + 5(-1)^n - 1.$

The sum of sequence $\boldsymbol{J_n}$, $\boldsymbol{\mathcal{M}\text{Ể}_n}$

We define a new sequence

$S_n = J_n + \mathcal{M}\text{Ể}_n.$

For closed form of S_n using $J_n = \frac{1}{3}(2^n - (-1)^n)$, $\mathcal{M}\text{Ể}_n = 2.2^n + \frac{5}{2}(-1)^n - \frac{1}{2}$,

then for $n \geq 1$,

$$S_n = \frac{7}{3}2^n + \frac{13}{6}(-1)^n - \frac{1}{2} \tag{3.12}$$

Table *Comparison of the sum of recurrences and the closed formula of the sum*

n	J_n	$\mathcal{M}\text{Ể}_n$	$J_n + \mathcal{M}\text{Ể}_n$	$\frac{7}{3}2^n + \frac{13}{6}(-1)^n - \frac{1}{2}$
1	1	1	2	2
2	1	10	11	11
3	3	13	16	16
4	5	34	39	39
5	11	61	72	72
6	21	130	151	151
7	43	253	296	296
8	85	514	599	599
9	171	1021	1192	1192
10	341	2050	2391	2391

These expressions are consistent both in terms of recurrence formulas and closed form.

***Table** Values of the terms of six sequences*

n	J_n	$\mathcal{M}\hat{\tilde{E}}_n$	$\hat{\tilde{E}}_n$	D_n	T_n	S_n
0	0	4	0	-4	4	
1	1	1	1	0	2	2
2	1	10	2	-8	12	11
3	3	13	5	-8	18	16
4	5	34	10	-24	44	39
5	11	61	21	-40	82	72
6	21	130	42	-88	172	151
7	43	253	85	-168	338	296
8	85	514	170	-344	684	599
9	171	1021	341	-680	1362	1192
10	341	2050	682	-1368	2732	2391

Important Relations Derived from the Sequence

In this section, some identities related to the sequence that are widely accepted in the literature will be presented.

First, a Cassini-type identity will be derived for the $\mathcal{M}\hat{\tilde{E}}_n$ sequence using the previously obtained closed form.

Theorem (Cassini-type identity)

For $\mathcal{M}\hat{\tilde{E}}_n$,

$$\mathcal{M}\hat{\tilde{E}}_{n+1}\mathcal{M}\hat{\tilde{E}}_{n-1} - \mathcal{M}\hat{\tilde{E}}_n^2 = \frac{10(-1)^n - 45(-2)^n - 2^n}{2} \qquad (4.1)$$

Proof

From

$$\mathcal{M}\hat{\tilde{E}}_n = 2.2^n + \frac{5}{2}(-1)^n - \frac{1}{2},$$

an expression for the determinant-like quantity was sought under the condition

$K_n \coloneqq \mathcal{M}\hat{\text{Ể}}_{n+1}\mathcal{M}\hat{\text{Ể}}_{n-1} - \mathcal{M}\hat{\text{Ể}}_n^2.$

Each term was expressed in closed form and subsequently expanded,

$$\mathcal{M}\hat{\text{Ể}}_n = 2.2^n + \frac{5}{2}(-1)^n - \frac{1}{2}.$$

$$\mathcal{M}\hat{\text{Ể}}_{n+1} = 2.2^{n+1} + \frac{5}{2}(-1)^{n+1} - \frac{1}{2},$$

$$\mathcal{M}\hat{\text{Ể}}_{n-1} = 2.2^{n-1} + \frac{5}{2}(-1)^{n-1} - \frac{1}{2},$$

Compute $K_n = \mathcal{M}\hat{\text{Ể}}_{n+1}\mathcal{M}\hat{\text{Ể}}_{n-1} - \mathcal{M}\hat{\text{Ể}}_n^2$ and an algebraic simplification was carried out by grouping terms in powers of 2^n and $(-1)^n$, through which the closed form was eventually derived.

First we will express it with simpler symbols. Let

$$U = 2^n, \qquad V = (-1)^n, \qquad c = \frac{5}{2}.$$

Then

$$\mathcal{M}\hat{\text{Ể}}_n = 2U + cV - \frac{1}{2}.$$

Because $2^{n+1} = 2U, 2^{n-1} = U/2$, and $(-1)^{n\pm1} = -V$,

$$\mathcal{M}\hat{\text{Ể}}_{n+1} = 4U - cV - \frac{1}{2}, \qquad \mathcal{M}\hat{\text{Ể}}_{n-1} = U - cV - \frac{1}{2}.$$

We need

$$K_n = \left(\mathcal{M}\hat{\text{Ể}}_{n+1}\right)\left(\mathcal{M}\hat{\text{Ể}}_{n-1}\right) - \left(\mathcal{M}\hat{\text{Ể}}_n\right)^2$$

If the expression $\mathcal{M}\hat{\text{Ể}}_{n+1}\mathcal{M}\hat{\text{Ể}}_{n-1}$ is written explicitly,

$$\mathcal{M}\hat{\text{Ể}}_{n+1}\mathcal{M}\hat{\text{Ể}}_{n-1} = \left(4U - cV - \frac{1}{2}\right)\left(U - cV - \frac{1}{2}\right)$$

$$= 4U^2 - 5cUV - \frac{5}{2}U + c^2V^2 + cV + \frac{1}{4}.$$

If a similar operation is made in $\mathcal{M}\hat{\text{Ẽ}}_n^2$,

$$\mathcal{M}\hat{\text{Ẽ}}_n^2 = \left(2U + cV - \frac{1}{2}\right)^2$$

$$= 4U^2 + 4cUV - 2U + c^2V^2 - cV + \frac{1}{4}.$$

If the difference of these two terms is taken,

$$K_n = \left(4U^2 - 5cUV - \frac{5}{2}U + c^2Y^2 + cY + \frac{1}{4}\right)$$

$$-\left(4U^2 + 4cUY - 2U + c^2Y^2 - cY + \frac{1}{4}\right)$$

$$= (-9cUY) - \frac{1}{2}U + 2cY.$$

When the equivalents of c, U, Y are put back in place, the following expression is obtained:

$$K_n = -9\left(\frac{5}{2}\right)2^n(-1)^n - \frac{1}{2}2^n + 2\left(\frac{5}{2}\right)(-1)^n$$

$$= -\frac{45}{2}(-2)^n - \frac{1}{2}2^n + 5(-1)^n$$

$$= \frac{10(-1)^n - 45(-2)^n - 2^n}{2}$$

Using $(-2)^n = (-1)^n 2^n$, then

$$K_n = \mathcal{M}\hat{\text{Ẽ}}_{n+1}\mathcal{M}\hat{\text{Ẽ}}_{n-1} - \mathcal{M}\hat{\text{Ẽ}}_n^2 = \frac{10(-1)^n - 45(-2)^n - 2^n}{2}.$$

__Table__ Comparing recurrence values with formula (4.1)

n	$\mathcal{M}\hat{E}_{n-1}$	$\mathcal{M}\hat{E}_{n}$	$\mathcal{M}\hat{E}_{n+1}$	$\mathcal{M}\hat{E}_{n+1}\mathcal{M}\hat{E}_{n-1} - \mathcal{M}\hat{E}_{n}^2$	**From (4.1)**
1	4	1	10	$4.10 - 1 = 39$	39
2	1	10	13	$13.1 - 100 = -87$	-87
3	10	13	34	$340 - 169 = 171$	171
4	13	34	61	$13.61 - 34.34 = -363$	-363
5	34	61	130	$34.130 - 61.61 = 699$	699
6	61	130	253	$61.253 - 130.130 = -1467$	-1467
7	130	253	514	$130.514 - 253.253 = 2811$	2811
8	253	514	1021	$253.1021 - 514.514 = -5883$	-5883
9	514	1021	2050	$514.2050 - 1021.1021 = 11259$	11259

Now, a Catalan-type identity will be presented for $\mathcal{M}\hat{E}_n$. In fact, the Cassini identity is a special case of the Catalan identity. In the following theorem, if $r = 1$, the Cassini identity multiplied by -1 is obtained.

Theorem (Catalan-type identity)

For $\mathcal{M}\hat{E}_n$, let r is a fixed positive integer, then

$$\mathcal{M}\hat{E}_n^2 - \mathcal{M}\hat{E}_{n-r}\mathcal{M}\hat{E}_{n+r} = 5(-1)^{n+r}[(-2^{n+r} - 2^{n-r} + 2^{-1})] + 2^n(2^r + 2^{-r} - 2) + 5(-1)^n \frac{2^{n+2} - 1}{2} \quad (4.2)$$

We will derive a closed formula for the quantity

$$K_{n,r} := \mathcal{M}\hat{E}_n^2 - \mathcal{M}\hat{E}_{n-r}\mathcal{M}\hat{E}_{n+r}.$$

Let's recall the closed form (2.2) we already have for $\mathcal{M}\hat{E}_n$;

$$\mathcal{M}\hat{E}_n = 2.2^n + \frac{5}{2}(-1)^n - \frac{1}{2}.$$

We need to make some notation choices to perform operations faster:

$U \coloneqq 2^n, \qquad V \coloneqq (-1)^n, \qquad u \coloneqq 2^r, \qquad v \coloneqq (-1)^r,$

$y \coloneqq 2,\ z \coloneqq \frac{5}{2},\ a \coloneqq -\frac{1}{2}.$

Then

$\mathcal{M}\text{Ễ}_n := yU + zV + a,$

$\mathcal{M}\text{Ễ}_{n+r} := y(Uu) + z(Vv) + a,$

$\mathcal{M}\text{Ễ}_{n-r} := y\left(\frac{U}{u}\right) + z(Vv) + a.$

So,

$\mathcal{M}\text{Ễ}_n^2 = (yU)^2 + (zV)^2 + a^2 + 2(yU)(zV) + 2(yU)a + 2(zV)a$

$= y^2U^2 + z^2V^2 + a^2 + 2yzUV + 2yaU + 2zaV.$

If we compute $\mathcal{M}\text{Ễ}_{n-r}\mathcal{M}\text{Ễ}_{n+r}$:

$\mathcal{M}\text{Ễ}_{n-r}\mathcal{M}\text{Ễ}_{n+r} = \left(y\frac{U}{u} + zvV + a\right)(yuU + zvV + a)$

$= (yuU)\left(y\frac{U}{u}\right) + (zvV)\left(y\frac{U}{u}\right) + a\left(y\frac{U}{u}\right)$

$+(yuU)(zvV) + (zvV)(zvV) + a(zvV) + (yuU)a + (zvV)a$

$+a^2$

$= y^2U^2 + yz\left(\frac{U}{u}vV + uUvV\right) + ya\left(\frac{U}{u} + uU\right) + z^2v^2V^2$

$+za(vV + vV) + a^2.$

If these expressions are written in $C_{n,r}$,

$C_{n,r} = \mathcal{M}\text{Ễ}_n^2 - \mathcal{M}\text{Ễ}_{n-r}\mathcal{M}\text{Ễ}_{n+r}$

$= (yU + zV + a)^2 - \left(y\frac{U}{u} + zvV + a\right)(yuU + zvV + a).$

Here, from $V=(-1)^n, v=(-1)^r$ then $V^2=1$ and $v^2=1$. If various simplifications are made here, simplifying the same y^2U^2 and a^2 terms and substituting $V^2=1,\ v^2=1$ (by adding the coefficients of the three fundamental types of constants, U^2 (cancelled), UV, U, V, and U^2), we get:

$$C_{n,r}=(2UVyz)-\left(\left(\frac{U}{u}vV+uUvV\right)yz\right)+(2aUy)$$

$$-\left(\left(\frac{U}{u}+uU\right)ya\right)+(z^2)-(z^2)+2aVz-2Vvza.$$

Now substitute the numeric parameters $y=2, z=\frac{5}{2}, a=-\frac{1}{2}$ and simplify each group.

Then,

$$C_{n,r}=-\frac{1}{4u}(20u^2UVv-4u^2U-40uUV+8uU-10uVv$$

$$+10uV+20UVv-4U)$$

This expression is algebraically equivalent to the generalized form from the step before—just combined with $4u$ in the common denominator for simplicity and split into factors.

If the expressions $U=2^n, V=(-1)^n, u=2^r, v=(-1)^r$,we use for abbreviation are written in their places and then symbolic variables should be replaced with their definitions.

$$\mathcal{M}\tilde{\hat{E}}_n^2-\mathcal{M}\tilde{\hat{E}}_{n-r}\mathcal{M}\tilde{\hat{E}}_{n+r}=-\frac{1}{4.2^r}(-4.2^n-4.2^{2r}.2^n+8.2^r.2^n$$

$$-40.2^n.2^r.(-1)^n-10.2^r.(-1)^r.(-1)^n+$$

$$10.2^r.(-1)^n+20.2^n.(-1)^r(-1)^n-20.2^n.2^{2r}.(-1)^n.(-1)^r)$$

This can be simplified into different but equivalent looking closed forms. One convenient way to write it – grouping by factors 2^n, $(-1)^n$ and powers of $(-1)^{n+r}$ – is:

$$\mathcal{M}Ể_n^2 - \mathcal{M}Ể_{n-r}\mathcal{M}Ể_{n+r} = 5(-1)^{n+r}[-2^n(2^r + 2^{-r}) + 2^{-1}]$$

$$+5(-1)^n[2.2^n - 2^{-1}] + 2^n(2^r + 2^{-r} - 2)$$

So, we get the Catalan-type identity for our sequences. Here, (4.2) includes a linear combination of terms involving $2^j, (-1)^j$ and powers for $j = n, k$.

Remark

If we take $r = 1$, we get Cassini-type identity.

$$\mathcal{M}Ể_n^2 - \mathcal{M}Ể_{n-1}\mathcal{M}Ể_{n+1} = -\left(\mathcal{M}Ể_{n+1}\mathcal{M}Ể_{n-1} - \mathcal{M}Ể_n^2\right)$$

$$= -\left(\frac{10(-1)^n - 45(-2)^n - 2^n}{2}\right).$$

If we take $n = 7, r = 2$. Using the Table above,

$$\mathcal{M}Ể_7 = 253, \mathcal{M}Ể_5 = 61, \mathcal{M}Ể_9 = 1021.$$

Left side: $\mathcal{M}Ể_7^2 - \mathcal{M}Ể_5\mathcal{M}Ể_9 = 253^2 - 61.1021$

$$= 64009 - 62281 = 1728.$$

In the table below, the results obtained from recurrences for some n and r values are compared with the results obtained from equality 4.2.

***Table** Comparing recurrence values with formula (4.2)*

n	r	$\mathcal{M}Ễ_{n-r}$	$\mathcal{M}Ễ_n$	$\mathcal{M}Ễ_{n+r}$	$\mathcal{M}Ễ_n^2 - \mathcal{M}Ễ_{n+r}\mathcal{M}Ễ_{n-r}$	From (4.2)
3	1	$\mathcal{M}Ễ_2 = 10$	13	$\mathcal{M}Ễ_4 = 34$	-171	-171
3	2	$\mathcal{M}Ễ_1 = 1$	13	$\mathcal{M}Ễ_5 = 61$	108	108
3	3	$\mathcal{M}Ễ_0 = 4$	13	$\mathcal{M}Ễ_6 = 130$	-351	-351
4	1	$\mathcal{M}Ễ_3 = 13$	34	$\mathcal{M}Ễ_5 = 61$	363	363
4	2	$\mathcal{M}Ễ_2 = 10$	34	$\mathcal{M}Ễ_6 = 130$	-144	-144
4	3	$\mathcal{M}Ễ_1 = 1$	34	$\mathcal{M}Ễ_7 = 253$	903	903
5	1	$\mathcal{M}Ễ_4 = 34$	61	$\mathcal{M}Ễ_6 = 130$	-699	-699
5	3	$\mathcal{M}Ễ_2 = 10$	61	$\mathcal{M}Ễ_8 = 514$	-1419	-1419
5	4	$\mathcal{M}Ễ_1 = 1$	61	$\mathcal{M}Ễ_9 = 1021$	2700	2700

Now, we compute the Cassini-type identity for $\mathcal{M}Ễ_n$ using the matrix method.

Theorem

The following equality is satisfied for $\mathcal{M}Ễ_n$ and for all $n \geq 1$:

$$\mathcal{M}Ễ_{n+1}\mathcal{M}Ễ_{n-1} - \mathcal{M}Ễ_n^2 = \det([g_{n+1}, g_n]) \tag{4.3}$$

Proof

We remember

$$\mathcal{M}Ễ_n = \mathcal{M}Ễ_{n-1} + 2\mathcal{M}Ễ_{n-2} + 1, \qquad n \geq 2,$$

$$\mathcal{M}Ễ_0 = 4, \mathcal{M}Ễ_1 = 1,$$

$$K_n = \mathcal{M}Ễ_{n+1}\mathcal{M}Ễ_{n-1} - \mathcal{M}Ễ_n^2.$$

First we must define a state vector and its accompanying matrix:

$$g_n := \begin{bmatrix} \mathcal{M}Ễ_n \\ \mathcal{M}Ễ_{n-1} \end{bmatrix}$$

Here the recurrence equation can be written using the sum of the matrix and the vector:

$$g_{n+1} = Bg_n + e$$

where

$$B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \qquad e = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Clearly,

$$g_{n+1} = \begin{bmatrix} \mathcal{M}\text{Ễ}_{n+1} \\ \mathcal{M}\text{Ễ}_{n} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{M}\text{Ễ}_{n} \\ \mathcal{M}\text{Ễ}_{n-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{M}\text{Ễ}_{n} + 2\mathcal{M}\text{Ễ}_{n-1} + 1 \\ \mathcal{M}\text{Ễ}_{n} \end{bmatrix}$$

holds for g_{n+1}.

Now, in order to represent the Cassini-type term with a determinant, we will make the following definition. Let's define a 2×2 matrix using column matrices g_{n+1} and g_n and denote this matrix by G_n:

$$G_n := [g_{n+1} \quad g_n] = \begin{bmatrix} \mathcal{M}\text{Ễ}_{n+1} & \mathcal{M}\text{Ễ}_{n} \\ \mathcal{M}\text{Ễ}_{n} & \mathcal{M}\text{Ễ}_{n-1} \end{bmatrix}.$$

Then by definition:

$$K_n = |G_n|.$$

If the equality provided by g_{n+1} is written in place of G_n,

$$g_{n+1} = Bg_n + e.$$

So

$$G_n = [g_{n+1}, g_n] = [Bg_n + e, g_n] = [Bg_n, g_n] + [e, g_n].$$

Now expand the determinant using linearity of the first column:

$$\det([Bg_n + e, g_n]) = \det([Bg_n, g_n]) + \det[(e, g_n)] \qquad (4.4)$$

We can calculate the second determinant as follows:

$$\det([e, g_n]) = \det \begin{bmatrix} 1 & \mathcal{M}\text{Ễ}_{n} \\ 0 & \mathcal{M}\text{Ễ}_{n-1} \end{bmatrix} = 1.\mathcal{M}\text{Ễ}_{n-1} - 0.\mathcal{M}\text{Ễ}_{n}$$

$= \mathcal{M}\hat{\text{Ể}}_{n-1}$.

So

$$\det[(e, g_n)] = \mathcal{M}\text{Ể}_{n-1} \qquad (4.5)$$

This gives a non-homogeneous term. The method of calculating the first determinant using the following equation, which is valid for invertible matrices B, will be used.

$\det(BX, Y) = \det(B)\det(X, B^{-1}Y)$.

So,

$$\det([Bg_n, g_n]) = \det(B) . \det([g_n, B^{-1}g_n]). \qquad (4.6)$$

From $\det(B) = -2$, we can write

$\det([Bg_n, g_n]) = -2. \det([g_n, B^{-1}g_n])$.

To find $B^{-1}g_n$, we must first find the inverse of matrix B. Using

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then

$$B^{-1} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Thus

$$B^{-1}g_n = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \mathcal{M}\text{Ể}_n \\ \mathcal{M}\text{Ể}_{n-1} \end{bmatrix} = \begin{bmatrix} \mathcal{M}\text{Ể}_{n-1} \\ \frac{1}{2}\mathcal{M}\text{Ể}_n - \frac{1}{2}\mathcal{M}\text{Ể}_{n-1} \end{bmatrix}.$$

So,

$$\det \begin{bmatrix} \mathcal{M}\text{Ể}_n & \mathcal{M}\text{Ể}_{n-1} \\ \mathcal{M}\text{Ể}_{n-1} & \frac{1}{2}\mathcal{M}\text{Ể}_n - \frac{1}{2}\mathcal{M}\text{Ể}_{n-1} \end{bmatrix}$$

$$= \mathcal{M}\hat{E}_n \left(\frac{1}{2}\mathcal{M}\hat{E}_n - \frac{1}{2}\mathcal{M}\hat{E}_{n-1}\right) - \mathcal{M}\hat{E}^2_{n-1}$$

$$= \frac{1}{2}\mathcal{M}\hat{E}^2_n - \frac{1}{2}\mathcal{M}\hat{E}_n\mathcal{M}\hat{E}_{n-1} - \mathcal{M}\hat{E}^2_{n-1}.$$

Using (4.6),

$$\det([Bg_n, g_n]) = -2.\left(\frac{1}{2}\mathcal{M}\hat{E}^2_n - \frac{1}{2}\mathcal{M}\hat{E}_n\mathcal{M}\hat{E}_{n-1} - \mathcal{M}\hat{E}^2_{n-1}\right)$$

$$= -\mathcal{M}\hat{E}^2_n + \mathcal{M}\hat{E}_n\mathcal{M}\hat{E}_{n-1} + 2\mathcal{M}\hat{E}^2_{n-1}.$$

If this expression is written in place of (4.4),

$$K_n = \det([Bg_n, g_n]) + \det([e, g_n])$$

$$= \left(-\mathcal{M}\hat{E}^2_n + \mathcal{M}\hat{E}_n\mathcal{M}\hat{E}_{n-1} + 2\mathcal{M}\hat{E}^2_{n-1}\right) + \mathcal{M}\hat{E}_{n-1}$$

So,

$$K_n = -\mathcal{M}\hat{E}^2_n + \mathcal{M}\hat{E}_n\mathcal{M}\hat{E}_{n-1} + 2\mathcal{M}\hat{E}^2_{n-1} + \mathcal{M}\hat{E}_{n-1} \qquad (4.7)$$

Here, the last three terms here actually give $\mathcal{M}\hat{E}_{n+1}\mathcal{M}\hat{E}_{n-1}$. To see this, using

$$\mathcal{M}\hat{E}_n = \mathcal{M}\hat{E}_{n-1} + 2\mathcal{M}\hat{E}_{n-2} + 1, \mathcal{M}\hat{E}_{n+1} = \mathcal{M}\hat{E}_n + 2\mathcal{M}\hat{E}_{n-1} + 1$$

and

$$\mathcal{M}\hat{E}_n = \mathcal{M}\hat{E}_{n+1} - 2\mathcal{M}\hat{E}_{n-1} - 1$$

then

$$\mathcal{M}\hat{E}_n\mathcal{M}\hat{E}_{n-1} = \left(\mathcal{M}\hat{E}_{n+1} - 2\mathcal{M}\hat{E}_{n-1} - 1\right)\mathcal{M}\hat{E}_{n-1}$$

$$= \mathcal{M}\hat{E}_{n+1}\mathcal{M}\hat{E}_{n-1} - 2\mathcal{M}\hat{E}_{n-1}\mathcal{M}\hat{E}_{n-1} - \mathcal{M}\hat{E}_{n-1}.$$

From,

$$\mathcal{M}\hat{E}_n\mathcal{M}\hat{E}_{n-1} = \mathcal{M}\hat{E}_{n+1}\mathcal{M}\hat{E}_{n-1} - 2\mathcal{M}\hat{E}_{n-1}\mathcal{M}\hat{E}_{n-1} - \mathcal{M}\hat{E}_{n-1}$$

so,

$$\mathcal{M}\text{Ễ}_{n+1}\mathcal{M}\text{Ễ}_{n-1} = \mathcal{M}\text{Ễ}_{n}\mathcal{M}\text{Ễ}_{n-1} + 2\mathcal{M}\text{Ễ}_{n-1}\mathcal{M}\text{Ễ}_{n-1} + \mathcal{M}\text{Ễ}_{n-1}.$$

Thus, the desired (4.3) are shown.

Remark

If the closed form of $\mathcal{M}\text{Ễ}_n$ is used in (4.7), the Cassini type identity is clearly obtained.

Indeed, using

$$\mathcal{M}\text{Ễ}_n = 2.2^n + \frac{5}{2}(-1)^n - \frac{1}{2},$$

$$K_n = -\mathcal{M}\text{Ễ}_n^2 + \mathcal{M}\text{Ễ}_n\mathcal{M}\text{Ễ}_{n-1} + 2\mathcal{M}\text{Ễ}_{n-1}^2 + \mathcal{M}\text{Ễ}_{n-1}$$

$$= -\left(2.2^n + \frac{5}{2}(-1)^n - \frac{1}{2}\right)^2$$

$$+\left(2.2^n + \frac{5}{2}(-1)^n - \frac{1}{2}\right)\left(2.2^{n-1} + \frac{5}{2}(-1)^{n-1} - \frac{1}{2}\right)$$

$$+2\left(2.2^{n-1} + \frac{5}{2}(-1)^{n-1} - \frac{1}{2}\right)^2 + \left(2.2^{n-1} + \frac{5}{2}(-1)^{n-1} - \frac{1}{2}\right)$$

$$= \frac{-2^{n+1} + 20(-1)^n - 90(-2)^n}{4}.$$

This is equivalent to the equality given in (4.1).

It will now be shown that the sequence $\mathcal{M}\text{Ễ}_n$ satisfies some formulas.

Lemma

The sequence $\mathcal{M}\text{Ễ}_n$ satisfies the following equalities for $m \geq n \geq 0$:

$$\mathcal{M}\text{Ễ}_{m-n} = 2.2^{m-n} + \frac{5}{2}(-1)^{m-n} - \frac{1}{2}, \qquad (4.8)$$

$$\frac{\mathcal{M}\hat{\text{Ẽ}}_m\mathcal{M}\hat{\text{Ẽ}}_{n+1} - \mathcal{M}\hat{\text{Ẽ}}_{m+1}\mathcal{M}\hat{\text{Ẽ}}_n}{(-2)^n} = 15(-1)^{m-n} - 15.\,2^{m-n}$$
$$+\frac{5}{2^{n+1}}\left(1-(-1)^{m-n}\right) + 2^{m-n}(-1)^n - (-1)^n - \frac{25}{2^{n+1}}(-1)^m \qquad (4.9)$$

Proof

Using

$$\mathcal{M}\hat{\text{Ẽ}}_n = \mathcal{M}\hat{\text{Ẽ}}_{n-1} + 2\mathcal{M}\hat{\text{Ẽ}}_{n-2} + 1, \qquad \mathcal{M}\hat{\text{Ẽ}}_0 = 4, \qquad \mathcal{M}\hat{\text{Ẽ}}_1 = 1$$

and closed form of sequences:

$$\mathcal{M}\hat{\text{Ẽ}}_n = 2.\,2^n + \frac{5}{2}(-1)^n - \frac{1}{2}, \;\; \mathcal{M}\hat{\text{Ẽ}}_{n+1}$$
$$= 2.\,2^{n+1} + \frac{5}{2}(-1)^{n+1} - \frac{1}{2}$$

Then,

$$\mathcal{M}\hat{\text{Ẽ}}_{m-n} = 2.\,2^{m-n} + \frac{5}{2}(-1)^{m-n} - \frac{1}{2}.$$

On the other hand,

$$\mathcal{M}\hat{\text{Ẽ}}_m\mathcal{M}\hat{\text{Ẽ}}_{n+1} - \mathcal{M}\hat{\text{Ẽ}}_{m+1}\mathcal{M}\hat{\text{Ẽ}}_n = \left(2.\,2^m + \frac{5}{2}(-1)^m - \frac{1}{2}\right)$$
$$*\left(2.\,2^{n+1} + \frac{5}{2}(-1)^{n+1} - \frac{1}{2}\right)$$
$$-\left(2.\,2^{m+1} + \frac{5}{2}(-1)^{m+1} - \frac{1}{2}\right) * \left(2.\,2^n + \frac{5}{2}(-1)^n - \frac{1}{2}\right)$$

$$\mathcal{M}\hat{\text{Ẽ}}_m\mathcal{M}\hat{\text{Ẽ}}_{n+1} = 2^{m+3+n} + \frac{5}{2}\left(2^{m+1}(-1)^{n+1} + 2^{n+2}(-1)^m\right)$$
$$-\frac{1}{2}\left(2^{m+1} + 2^{n+2}\right) + \frac{25}{4}(-1)^{m+n+1}$$

$$-\frac{5}{4}[(-1)^m + (-1)^{n+1}] + \frac{1}{4}.$$

$$\mathcal{M}\hat{\tilde{E}}_{m+1}\mathcal{M}\hat{\tilde{E}}_n = 2^{m+3+n} + 5.2^{m+1}(-1)^n - 2^{m+1} - 5.2^n(-1)^m$$

$$-2^n - \frac{25}{4}(-1)^{m+n+1} + \frac{5}{4}(-1)^m - \frac{5}{4}(-1)^n + \frac{1}{4}$$

$$\mathcal{M}\hat{\tilde{E}}_m\mathcal{M}\hat{\tilde{E}}_{n+1} - \mathcal{M}\hat{\tilde{E}}_{m+1}\mathcal{M}\hat{\tilde{E}}_n = 15(-1)^m 2^n - (-1)^n 2^m 15$$

$$+\frac{5}{2}((-1)^n - (-1)^m) + 2^m - 2^n - \frac{25}{2}(-1)^{m+n}$$

So,

$$\frac{\mathcal{M}\hat{\tilde{E}}_m\mathcal{M}\hat{\tilde{E}}_{n+1} - \mathcal{M}\hat{\tilde{E}}_{m+1}\mathcal{M}\hat{\tilde{E}}_n}{(-2)^n} = 15(-1)^{m-n} - 15.2^{m-n}$$

$$+\frac{5}{2^{n+1}}(1 - (-1)^{m-n}) + 2^{m-n}(-1)^n - (-1)^n - \frac{25}{2^{n+1}}(-1)^m.$$

Now, we find the generating function of the sequence $\mathcal{M}\hat{\tilde{E}}_n$ in next theorems.

Theorem

For $\mathcal{M}\hat{\tilde{E}}_n$, the generating function is defined as follows.

$$N(x) = \sum_{n=0}^{\infty} \mathcal{M}\hat{\tilde{E}}_n x^n = \frac{4 - 7x + 3x^2}{(1-x)(1-x-2x^2)}. \qquad (4.10)$$

Proof

Using

$$\mathcal{M}\hat{\tilde{E}}_n = \mathcal{M}\hat{\tilde{E}}_{n-1} + 2\mathcal{M}\hat{\tilde{E}}_{n-2} + 1, \qquad n \geq 2,$$

$$\mathcal{M}\hat{\tilde{E}}_0 = 4, \mathcal{M}\hat{\tilde{E}}_1 = 1$$

$$B(x) = \sum_{n=0}^{\infty} T_n x^n.$$

We aim to find the generating function of the sequence:

$$N(x) = \sum_{n=0}^{\infty} \mathcal{M}\text{Ể}_n x^n.$$

If we multiply both sides of the recurrence by x^n and if we get sum for $n \geq 2$:

$$\sum_{n=2}^{\infty} \mathcal{M}\text{Ể}_n x^n = \sum_{n=2}^{\infty} \mathcal{M}\text{Ể}_{n-1} x^n + 2\sum_{n=2}^{\infty} \mathcal{M}\text{Ể}_{n-2} x^n + \sum_{n=2}^{\infty} x^n$$

Here, taking into account that the following equalities hold

$$\sum_{n=2}^{\infty} \mathcal{M}\text{Ể}_{n-1} x^n = x\sum_{n=1}^{\infty} \mathcal{M}\text{Ể}_n x^n = x\left(N(x) - \mathcal{M}\text{Ể}_0\right)$$

$$\sum_{n=2}^{\infty} \mathcal{M}\text{Ể}_{n-2} x^n = x^2\sum_{n=0}^{\infty} \mathcal{M}\text{Ể}_n x^n = x^2 N(x)$$

$$\sum_{n=2}^{\infty} x^n = \frac{1}{1-x} - 1 - x = \frac{x^2}{1-x}, \qquad |x| < 1.$$

So the recurrence is written next form:

$$N(x) - \mathcal{M}\text{Ể}_0 - \mathcal{M}\text{Ể}_1 x = x\left(N(x) - \mathcal{M}\text{Ể}_0\right) + 2x^2 N(x) + \frac{x^2}{1-x}.$$

Using $\mathcal{M}\text{Ể}_0 = 4, \mathcal{M}\text{Ể}_1 = 1$, we get

$$N(x) - 4 - x = xN(x) - 4x + 2x^2 N(x) + \frac{x^2}{1-x}.$$

Then,

$$N(x) - xN(x) - 2x^2N(x) - 4 + 4x - x = \frac{x^2}{1-x}$$

$$N(x) - xN(x) - 2x^2N(x) - 4 - x = -4x + \frac{x^2}{1-x}.$$

We add $4 + x$ to both sides,

$$N(x) - xN(x) - 2x^2N(x) = 4 + x - 4x + \frac{x^2}{1-x}.$$

So we get

$$N(x)(1 - x - 2x^2) = \frac{x^2}{1-x} + 4 - 3x.$$

If we solve according to $N(x)$:

$$N(x) = \frac{\frac{x^2}{1-x} + 4 - 3x}{1 - x - 2x^2} = \frac{4 - 7x + 4x^2}{(1-x)(1-x-2x^2)}.$$

Then the generating function of $\mathcal{M}\text{Ễ}_n$,

$$N(x) = \frac{4 - 7x + 4x^2}{(1-x)(1-2x)(1+x)}.$$

In the following theorem, the Binet formula will be obtained using the generator function we found above.

Theorem

The Binet formula of $\mathcal{M}\text{Ễ}_n$ is obtained using partial fractions for $\mathcal{M}\text{Ễ}_n$ with the following generating function

$$N(x) = \frac{4 - 7x + 4x^2}{(1-x)(1-2x)(1+x)}.$$

For $\mathcal{M}\text{Ễ}_n$, using generating function, and next equality

$$\frac{4-7x+3x^2}{(1-x)(1-2x)(1+x)} = \frac{A}{1-x} + \frac{B}{1-2x} + \frac{C}{x+1}.$$

Then,

$$A = -\frac{1}{2}, B = 2, C = \frac{5}{2}.$$

So, we can write

$$N(x) = -\frac{1}{2}\frac{1}{1-x} + 2\frac{1}{1-2x} + \frac{5}{2}\frac{1}{x+1}.$$

In the final step, by utilizing expand as power series

$$\frac{1}{1-rx} = \sum_{n=0}^{\infty} r^n x^n,$$

we get

$$N(x) = \sum_{n=0}^{\infty} \left(-\frac{1}{2}1^n + 22^n + \frac{5}{2}(-1)^n\right) x^n$$

Then,

$$\mathcal{M}\text{Ể}_n = 2.2^n + \frac{5}{2}(-1)^n - \frac{1}{2}.$$

Conclusion

Mulatu sequences are among the sequences that have recently been studied in combination with other sequences. By adapting the initial condition of the sequence to the mulatu structure and evaluating it together with the sequence known in the literature as the Ernst or Lichtenberg sequence, a relationship was established with the Jacobsthal sequence, Ernst sequence. Subsequently, various equations were obtained that included the relationships between the different sequences obtained. We showed that the sequences $\mathcal{M}\text{Ể}_n$ and Ể_n, although defined by identical recurrence relations with different initial conditions, have a deep connection with the Jacobsthal sequence. Their closed forms, sums, and differences are reduced to simple expressions in terms of J_n and alternating signs. This study, which seeks answers to the usual steps in the literature, is a guiding work for researchers who want to apply Mulatu sequences to other sequences. For example: (Soykan, 2023), (Kalca & Soykan, 2025), (Demirci & Soykan, 2025), (Dogan & Soykan, 2025), (Soykan, 2025). This work highlights how small changes in initial conditions can lead to structured and predictable relationships between integer sequences. To transfer the study we prepared to different fields, the studies (Çolak, Bilgin & Soykan, 2024) and (Costa, Catarino & Carvalho, 2025) related to the defined series can be used.

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ON A PARAMETRIC GENERALIZED OF ERNST NUMBERS: k-ERNST SEQUENCES

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Introduction

Studies in the field of number theory date back to very early times. The history of work with specific number sequences extends back to the definition of the Fibonacci sequence. This field, which has developed rapidly and never lost its popularity due to its applicability to many areas and the approach of solving emerging problems using sequences, has gained a significant advantage with the introduction of a method for finding the largest terms of number sequences. This notation, called Binet form, provides an explicit formula for each term in the sequence, allowing the method to perform detailed comparative analysis for various parameter values. Several algebraic identities, difference-product relations, and some equations can be established analytically using Binet form.

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We want to discuss a number sequence known in the literature as a generalization of the Jacobstal sequence: the Ernst sequence (Soykan, 2022), which can also be obtained using the Lichtenberg recurrence relation (A000975) in (Hinz, 2017), (Sloane). Ernst numbers represent an important numerical sequence that emerges within a broad class of sequences defined by linear recurrence relations and closely connected to combinatorial structures.

One method for generalizing number sequences is parameter-based generalization. For some generalizations made in this sense, sources (Tasci & Kilic, 2004), (Falcon & Plaza, 2007), (Yılmaz & Bozkurt, 2009), (Jhala, Sisodiya, & Rathore, 2013), (Campos, Catarino, Aires, Vasco, & Borges, 2014), (Falcón, 2014), (Nilsrakoo & Nilsrakoo, 2025) can be examined. For the parameter-dependent generalization in the literature regarding Jacobsthal numbers, which forms the basis of our paper, the study in (Nilsrakoo & Nilsrakoo, 2025), which is described below, was taken as a basis.

$J_{k,n} = (k-1)J_{k,n-1} + kJ_{k,n-2}, \qquad n \geq 2,$ with $J_{k,0} = 0$ and $J_{k,1} = 1$.

This study examines a generalized form of the classical Ernst sequence, referred to as the k-Ernst sequence. In this study, a new numerical sequence called the k-Ernst sequence is introduced as a generalized extension of the classical Ernst numbers. Unlike the original form, the proposed sequence incorporates parameter-dependent coefficients and a constant term in its recursive structure, thereby defining a linear yet non-homogeneous difference relation. This new formulation extends the original definition by adding a constant term and a control parameter k, thereby enriching the algebraic and analytic properties of the sequence. This identities unites classical and generalized models under a unified framework by allowing for a broader class of behavior governed by the k parameter. The main objective of this work is to derive a Binet-type closed-form expression for the sequence, examine its limit and summation properties, and investigate the structural transitions that occur as the k parameter approaches positive and negative unity. So, the characteristic equation of the k-Ernst sequence has been derived, and a Binet-type closed-form expression has been obtained via its characteristic roots.

The findings confirm that the k-Ernst sequence is not merely a parametric generalization of Ernst numbers, but also provides a new perspective within the theory of inhomogeneous recurrence relations. Therefore, the proposed sequence offers a rich algebraic framework with potential applications in number theory, combinatorial analysis, and discrete dynamical systems.

Now, we recall of definition of Jacobsthal and Ernst (Lichtenberg) sequences.

J_n is defined in (Horadam A. , 1984) by

$$J_0 = 0, J_1 = 1,\ J_n = J_{n-1} + 2J_{n-2},\ (n \geq 2) \quad (1.1)$$

The first few terms of these sequences,

$J_0 = 0, J_1 = 1, J_2 = 1, J_3 = 3, J_4 = 5, J_5 = 11, J_6 = 21, J_7 = 43,$

$J_8 = 85, J_9 = 171, \dots$

E_n, is defined in (Soykan, 2022), (Hinz, 2017) by

$$E_0 = 0, E_1 = 1,\ E_n = E_{n-1} + 2E_{n-2} + 1, \quad n \geq 2, \quad (1.2)$$

The first few terms of these sequences,

$E_0 = 0, E_1 = 1, E_2 = 2, E_3 = 5, E_4 = 10, E_5 = 21, E_6 = 42,$

$E_7 = 85, E_8 = 170, E_9 = 341, \dots$

k- Ernst Parametric Sequence

Definition

The *k- Ernst parametric sequence* is defined by the following recurrence relation:

$$E_{k,n} = (k-1)E_{k,n-1} + kE_{k,n-2} + 1, \qquad n \geq 2, \qquad (2.1)$$

with initial conditions

$E_{k,0} = 0, \qquad E_{k,1} = 1.$

The following Remark describes the relationship between the k-Ernst parametric sequence and classical Ernst sequences.

Remark

The k- Ernst parametric sequence represents a generalization of the classical Ernst numbers. When the parameter k is set to $k = 2$, the reccurence relation reduces to

$E_{2,n} = E_{2,n-1} + 2E_{2,n-2} + 1,$

which is exactly the defining relation of the classical Ernst sequence introduced in (Soykan, 2022).

Hence, the k-Ernst numbers can be viewed as an extension of the standard Ernst numbers, where the parameter k controls the growth rate and structural behavior of the sequence.

Table *Some values of* $E_{k,n}$

k	$E_{k,2} = k$	$E_{k,3} = k^2 + 1$	$E_{k,4} = k^3 + k$	$E_{k,5} = k^4 + k^2 + 1$	$E_{k,6} = k^5 + k^3 + k$
-5	-5	26	-130	651	-3255
-4	-4	17	-68	273	-1092
-3	-3	10	-30	91	-273
-2	-2	5	-10	21	-42
2	2	5	10	21	42
3	3	10	30	91	273
4	4	17	68	273	1092
5	5	26	130	651	3255

Theorem

Closed-form of the k- Ernst parametric sequence is defined as follows:

$$E_{k,n} = \frac{k}{(k-1)(k+1)}k^n - \frac{1}{2(k+1)}(-1)^n - \frac{1}{2(k-1)} \qquad (2.2)$$

Proof

First, we need to find the characteristic equation and roots of the homogeneous part. The homogeneous part of the recurrence relation

$$E_{k,n} - (k-1)E_{k,n-1} - kE_{k,n-2} = 0$$

has the characteristic equation

$$r^2 - (k-1)r - k = 0.$$

Then

$\Delta = (k-1)^2 + 4k = (k+1)^2$, which yields the distinct roots

$$r_1 = k, \qquad r_2 = -1.$$

Including the non-homogeneous constant term $+1$, the general solution of the reccurence is

$$E_{k,n} = Ak^n + B(-1)^n + C.$$

The particular solution is constant, and substituting into the recurrence gives

$$C - (k-1)C - kC = 1 \Rightarrow C = \frac{1}{2(1-k)}.$$

Then,

$$E_{k,n} = Ak^n + B(-1)^n + \frac{1}{2(1-k)}$$

Using the initial conditions $E_{k,0} = 0$ and $E_{k,1} = 1$, the coefficients A and B are found as

$$A = \frac{k}{(k-1)(k+1)}, \qquad B = -\frac{1}{2(k+1)}.$$

Thus, complete closed-form expression is

$$E_{k,n} = \frac{k}{(k-1)(k+1)}k^n - \frac{1}{2(k+1)}(-1)^n - \frac{1}{2(k-1)}.$$

Remark

Here, for $k = 2$, the reccurence becomes

$$E_{2,n} = E_{2,n-1} + 2E_{2,n-2} + 1, \quad E_{2,0} = 0, \quad E_{2,1} = 1.$$

Substituting $k = 2$ into the the closed form gives

$$E_{2,n} = \frac{2}{3}2^n - \frac{1}{6}(-1)^n - \frac{1}{2},$$

which yields the sequence

$0, 1, 2, 5, 10, 21, 42, \ldots$

This confirms that the closed-form expression is consistent with the reccurence definition.

Lemma

For $k \neq \pm 1$, the closed form of the k-Ernst parametric sequence can also be given by the following equation.

$$E_{k,n} = \begin{cases} \frac{k(k^n-1)}{k^2-1}, & n \text{ is even} \\ \frac{k^{n+1}-1}{k^2-1}, & n \text{ is odd} \end{cases} \qquad (2.3)$$

Proof

We must show that this expression is equivalent to the equality (2.2) by examining it separately according to whether k is odd or even.

If n is even

$$E_{k,n} = \frac{k^{n+1}}{k^2-1} - \frac{1}{2(k+1)} - \frac{1}{2(k-1)}$$

$$= \frac{k^{n+1}}{k^2-1} \frac{-(k-1)-(k+1)}{2(k^2-1)}$$

$$= \frac{k^{n+1}-k}{k^2-1} = \frac{k(k^n-1)}{k^2-1}.$$

If n is odd

$$E_{k,n} = \frac{k^{n+1}}{k^2-1} + \frac{1}{2(k+1)} - \frac{1}{2(k-1)}$$

$$= \frac{k^{n+1}}{k^2-1} \frac{+(k-1)-(k+1)}{2(k^2-1)}$$

$$= \frac{k^{n+1}-1}{k^2-1}.$$

Therefore, (2.1) and (2.2) are equivalent.

Now, we gave generating function of the k-Ernst Sequence:

Theorem

Generating function of the k-Ernst sequence is given next equality.

$$G(x) = \frac{x}{(1-x)(1-(k-1)x-kx^2)} \qquad (2.4)$$

Proof

If we use the recurrence relation for the parametric k- Ernst sequence as

$$E_{k,n} = (k-1)E_{k,n-1} + kE_{k,n-2} + 1, \qquad E_{k,0} = 0, \ E_{k,1} = 1.$$

We goal is to find its generating function

$$G(x) = \sum_{n=0}^{\infty} E_{k,n}x^n.$$

Multiply the recurrence by x^n and then take the sum of both sides over $n \geq 2$,

Left-hand side:

$$\sum_{n=2}^{\infty} E_{k,n}x^n = G(x) - E_{k,0} - E_{k,1}x = G(x) - x$$

Right-hand side has three parts:

$$(k-1)\sum_{n=2}^{\infty} E_{k,n-1}x^n = (k-1)x\sum_{m=1}^{\infty} E_{k,m}x^m = (k-1)xG(x).$$

$$k\sum_{n=2}^{\infty} E_{k,n-2}x^n = kx^2\sum_{m=0}^{\infty} E_{k,m}x^m = kx^2G(x).$$

$$k\sum_{n=2}^{\infty} 1.x^n = \frac{x^2}{1-x}.$$

Thus,

$$G(x) - x = (k-1)xG(x) + kx^2G(x) + \frac{x^2}{1-x}.$$

Bring all $G(x)$ terms to one side:

$$G(x)(1-(k-1)x-kx^2) = x + \frac{x^2}{1-x} = \frac{x}{1-x}.$$

Therefore,

$$G(x) = \frac{x}{(1-x)(1-(k-1)x-kx^2)}.$$

This, completes the proof.

In the following theorem, Binet's formula will be found using a generating function.

Theorem

The Binet formula of the $E_{k,n}$ sequence can be obtained using the generating function $G(x)$.

Proof

Using the equality

$1-(k-1)x-kx^2 = (1-kx)(1+x),$

Then, the generating function becomes:

$$G(x) = \frac{x}{(1-x)(1-kx)(1+x)}.$$

So, we express $G(x)$ as:

$$\frac{x}{(1-x)(1-kx)(1+x)} = \frac{K}{1-x} + \frac{L}{1-kx} + \frac{M}{1+x}.$$

To find K, L, M substitute convenient values for x:

For $x = 1$, $\quad 1 = 2(1-k)K \Rightarrow K = \frac{1}{2(1-k)} = -\frac{1}{2(k-1)}.$

For $x = \frac{1}{k}$, $\quad \frac{1}{k} = \left(1 - \frac{1}{k}\right)\left(1 + \frac{1}{k}\right) L \Rightarrow L = \frac{k}{k^2-1}$.

For $x = -1$, $\quad -1 = 2(1+k)M \Rightarrow M = -\frac{1}{2(1+k)} = \frac{1}{2(k+1)}$.

Thus, $K = -\frac{1}{2(k-1)}, L = \frac{k}{k^2-1}, M = \frac{1}{2(k+1)}$.

$$G(x) = -\frac{1}{2(k-1)}\frac{1}{1-x} + \frac{k}{k^2-1}\frac{1}{1-kx} - \frac{1}{2(1+k)}\frac{1}{1+x}.$$

Using geometric series expands

$$\frac{1}{1-ux} = \sum_{n\geq 0} u^n x^n,$$

we get the coefficient of x^n as

$$E_{k,n} = -\frac{1}{2(k-1)}.1 + \frac{k}{k^2-1}.k^n - \frac{1}{2(1+k)}.(-1)^n.$$

Then, for $k \neq \pm 1$,

$$E_{k,n} = \frac{k}{k^2-1}.k^n - \frac{1}{2(1+k)}.(-1)^n - \frac{1}{2(k-1)}.$$

This is exactly the closed formula for the k-Ernst numbers.

Constructing a New Sequence Using $E_{k,n}$ and $J_{k,n}$

The homogeneous generalized Jacobsthal sequence is given in (Nilsrakoo & Nilsrakoo, 2025) as

$$J_{k,n} = (k-1)J_{k,n-1} + kJ_{k,n-2}, \qquad J_{k,0} = 0, \qquad J_{k,1} = 1.$$

We analyze homogeneous generalized Jacobsthal sequence and non-homogeneous Ernst sequence using difference sequences.

Definition

Let

$$P_n := E_{k,n} - J_{k,n}.$$

Then using $J_{k,0} = 0$, $J_{k,1} = 1$; $E_{k,0} = 0$, $E_{k,1} = 1$, we get $P_0 = 0$ and $P_1 = 0$.

Subtract the recurrences:

$$E_{k,n} - J_{k,n} = (k-1)\left(E_{k,n-1} - J_{k,n-1}\right) + k\left(E_{k,n-2} - J_{k,n-2}\right) + 1$$

$$P_n = (k-1)P_{n-1} + kP_{n-2} + 1, \quad P_0 = 0, \quad P_1 = 0. \qquad (3.1)$$

Lemma

Closed-form of P_n is defined as follows:

$$P_n = \frac{1}{k^2-1}k^n + \frac{1}{2(1+k)}.(-1)^n - \frac{1}{2(k-1)} \qquad (3.2)$$

Proof

First, we need to find a solution for the homogeneous part. The characteristic equation:

$$r^2 - (k-1)r - k = 0$$

has roots $r_1 = k, r_2 = -1$.

Then, homogeneous solution

$$P_n^{(h)} = Kk^n + L(-1)^n.$$

For Particular Solution; let $P_n^{(p)} := M$.

Then, using (3.1)

$$M - (k-1)M - kM = 1 \Rightarrow M(1 - k + 1 - k) = 1$$

So, we get $M = -\frac{1}{2(k-1)}$.

Therefore, the general solution is;

$$P_n = Kk^n + L(-1)^n - \frac{1}{2(k-1)}.$$

Now, we use initial conditions to find K and L as

For $n = 0$, $K + L + \frac{1}{2(1-k)} = E_{k,0} - J_{k,0} = 0 \Rightarrow$

$$K + L = -\frac{1}{2(1-k)}.$$

For $n = 1$, $Kk - L + \frac{1}{2(1-k)} = E_{k,1} - J_{k,1} = 0 \Rightarrow$

$$Kk - L = -\frac{1}{2(1-k)}.$$

If we add two equations above:

$$K(1+k) = -\frac{1}{1-k} \Rightarrow$$

$$K = \frac{1}{k^2-1}, \qquad L = \frac{1}{2(1+k)}.$$

So for P_n general solution;

$$P_n = \frac{1}{k^2-1}k^n + \frac{1}{2(1+k)}(-1)^n - \frac{1}{2(k-1)}.$$

Remark

From the lemma above, the relationship between $E_{k,n}$ and $J_{k,n}$ is given by the following equality

$$E_{k,n} = J_{k,n} - \frac{k^n}{1-k^2} - \frac{(-1)^n}{2(1+k)} + \frac{1}{2(1-k)} \qquad (3.3)$$

The following lemmas shows the relationship between the sequence $E_{k,n}$ and the sequence $J_{k,n}$.

Lemma

For certain specific values of n, the following relationships hold between $E_{k,n}$ and $J_{k,n}$.

- $E_{k,2} - J_{k,2} = 1,$

- $E_{k,3} - J_{k,3} = k$
- $E_{k,4} - J_{k,4} = k^2 + 1$
- $E_{k,5} - J_{k,5} = k^3 + k$
- $E_{k,6} - J_{k,6} = k^4 + k^2 + 1$

Lemma

For $k = 2$ and $k = 3$ the following relationships hold between $E_{k,n}$ and $J_{k,n}$.

- $E_{2,n} = 2\, J_{2,n} + \frac{(-1)^n - 1}{2}$,
- $E_{3,n} = \frac{6\, J_{3,n} + (-1)^n - 1}{4}$.

The two lemmas above can be proven by induction.

Conclusion

This study examines a generalization of the Ernst sequence related to the family of parameter-dependent Jacobsthal number sequences defined by linear iterations. While the Jacobsthal number sequence has a homogeneous structure, the Ernst sequence contains a constant term and therefore exhibits non-homogeneous properties. The research involves deriving implicit formulas, generating functions, and Binet-like expressions, and establishing that the denominator polynomials of the generating functions are directly related to the characteristic polynomials of the sequence. Generating functions are a powerful tool that can be used in the analysis of the combinatoric properties of sequences and in further research.

Another aim of this study is to reveal the relationship between these two sequences. To this end, the structural connection between homogeneous and non-homogeneous sequences has been clarified. The obtained Binet-like formulas and generating functions provide a solid mathematical foundation for the use of these sequences in theoretical and applied fields (e.g., algorithm analysis, coding theory, or network models). The methods applied in this study can be evaluated together with the (Çolak, Bilgin, & Soykan, 2024), (Demirci & Soykan, 2025), (Dogan & Soykan, 2025), (Çolak, Bilgin, & Soykan, 2023) and (Kalca & Soykan, 2025) studies to generate new research.

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SOME GENERALIZATIONS OF ERNST NUMBERS