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# CHAPTER I

# Parameter Estimation for Exponentiated Inverse Rayleigh Distribution

**Asuman YILMAZ**[1]

## 1.Introduction

The exponentiated inverse Rayleigh distribution (EIRD) can be used in reliability estimation and statistical quality control techniques. Some well-known statistical distributions, including the lognormal, inverse Weibull, and generalized inverted exponential distributions, behave similarly to this distribution. The EIRD has also found wide application. Especially, it can represent phenomena where components under a study indicate early failure behavior such as mechanical or electrical devices. There are different studies on EIRD in the literature. For example, (Rao and Mbwambo, 2019) estimated the unknown parameters of the EIRD using other methods such as maximum likelihood estimation (MLE), least square error

[1] Van Yüzüncü Yıl University, Faculty of Economics and Administrative Sciences, Department of Econometrics, 65080 Van, TURKEY Orcid: 0000-0002-8653-6900E-mail: asumanduva@yyu.edu.tr

(LSE), and weight least square error (WLSE) methods. They also compared these methods with an efficient simulation. The estimation of the inverted exponentiated Rayleigh distribution under a progressively first-failure censoring scheme was investigated by (Gao & Gui, 2019) and its prediction was further examined by (Maurya, Tripathi & Rastogi, 2019). Inverted exponentiated Rayleigh distributions with adaptive type-II progressive hybrid censored data were considered (Panahi, & Moradi, 2020). Also, (Fan & Gui, 2022), studied the statistical inference of inverted exponentiated Rayleigh distribution based on joint progressively type-II censored data. Some statistical properties and parameter estimates for the exponentially transformed inverse Rayleigh distribution have been investigated by (Banerjee & Bhunia, 2022). The reliability function and unknown parameters of the exponential inverse Rayleigh distribution were estimated by (İbrahim & Salih, 2024).

There is no doubt that precise and effective estimation of model parameters is crucial in many fields. Therefore, both classical and Bayesian methods are used in this study to determine the best estimation technique for the unknown parameters of EIRD. In classical parameter estimation, in addition to MLEs, LSEs, WLSEs, and percentile estimators (PEs) are also considered. Also, we used the Cramer-von-Mises (CVM) distance to define minimum distance estimators for the distribution parameters. We determine the Bayes estimators of the unknown parameters under the gamma prior distribution and the squared error loss function (SELF). We used the MCMC approximation for Bayesian computations since Bayesian estimators are not available in an explicit form. An extensive Monte-

Carlo simulation analysis is used to evaluate the effectiveness of different estimating techniques.

## 2.Exponentiated Rayleigh Distribution

The exponentiated inverse Rayleigh distribution (EIRD) was proposed by (Nadarajah & Kotz, 2006) as a generalization of the inverse Rayleigh distribution. The cumulative density function (cdf) and the probability density function (pdf) are given below:

$$f(x)=\frac{2\alpha\sigma^2}{x^3}e^{-\left(\frac{\sigma}{x}\right)^2}\left(1-e^{-\left(\frac{\sigma}{x}\right)^2}\right)^{\alpha-1} \qquad x\geq 0\ \sigma>0\ \alpha>0 \tag{2.1}$$

and

$$F(x)=1-\left(1-e^{-\left(\frac{\sigma}{x}\right)^2}\right)^{\alpha} \quad x\geq 0\ \sigma>0\ \alpha>0, \tag{2.2}$$

respectively. Here $\sigma$ is the scale parameter and the $\alpha$ is the shape parameter.

## 3. Estimation Procedures

Here, the parameter estimation methods obtained in the study are briefly discussed.

### 3.1 Maximum Likelihood Estimation

Let $\{X_1, X_2, ..., X_N\}$ be a random sample from the exponentiated inverse Rayleigh distribution. Then likelihood function of a random sample from the this distribution based on Equation (2.2) is given by

$$L=2^n\alpha^n\sigma^{2n}\prod_{i=1}^{n}x_i^{-3}e^{-\sum_{i=1}^{n}(\sigma/x_i)^2}\prod_{i=1}^{n}\left(1-e^{-(\sigma/x_i)^2}\right)^{\alpha-1} \tag{3.1}$$

After that, logarithms of Equation (3.1) are taken to obtain the log-likelihood function as follows:

$$\ln L = n\ln 2 + n\ln\alpha + 2n\ln\sigma - \sum_{i=1}^{n}\ln x_i^3 - \sum_{i=1}^{n}\left(\frac{\sigma}{x_i}\right)^2 + (\alpha-1)\sum_{i=1}^{n}\ln\left(1-e^{-(\sigma/x_i)^2}\right) \quad (3.2)$$

The estimating equations are obtained by differentiating the logarithm of likelihood function and setting it to zero.

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(1-e^{-(\sigma/x_i)^2}\right) = 0, \quad (3.3)$$

and

$$\frac{\partial \ln L}{\partial \sigma} = \frac{2n}{\sigma} - 2\sigma\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^2 + 2\sigma(\alpha-1)\sum_{i=1}^{n}\frac{e^{-(\sigma/x_i)^2}}{x_i^2\left(1-e^{-(\sigma/x_i)^2}\right)} = 0. \quad (3.4)$$

Then, the parameter $\alpha$ is found by solving equation (3.3) as follows:

$$\hat{\alpha} = \frac{-n}{\sum_{i=1}^{n}\ln\left(1-e^{-(\sigma/x_i)^2}\right)}, \quad (2.5)$$

We use iterative methods such as Newton-Raphson because Equation (3.4) does not provide an explicit estimation for the $\sigma$.

### 3.2.Least Squares and Weighted Least Squares Estimators

The LSEs and WLSE methods were first proposed by (Swain et al.,1988) to estimate the parameters of beta distributions. Let $X_1,...,X_n$ be a random sample of size $n$ from distribution function $F(.)$ and $X_{(i)};\ i=1,2,...,n$ denotes the ordered sample. The

expected value and variance of $F\left(X_{(i)}\right)$ are easily obtained from the relation between the Beta and uniform distribution as

$$E\left(F\left(X_{(i)}\right)\right)=\frac{i}{n+1} \quad \text{and} \quad Var\left(F\left(X_{(i)}\right)\right)=\frac{i\left(n-i+1\right)}{\left(n+1\right)^2\left(n+2\right)} \ .$$

A regression model $E\left(F\left(X_{(i)}\right)\right)=\frac{i}{n+1}$ can be expressed as follows:

$$F\left(X_{(i)}\right)=\frac{i}{n+1}+\varepsilon_i,\ i=1,2,...,n.$$

Then the LSEs of the unknown parameters can be obtained by minimizing the sum of squares of errors

$$\sum_{i=1}^{n}\left(F\left(X_{(i)}\right)-\frac{i}{n+1}\right)^2 \tag{3.6}$$

with respect to the unknown parameters. Therefore, the LSE of the unknown parameters of EIRD is as follows:

$$LSE=\sum_{i=1}^{n}\left(\frac{n+1-i}{n+1}-\left(1-e^{-\left(\sigma/x_{(i)}\right)^2}\right)^{\alpha}\right)^2 \tag{3.7}$$

with respect to $\alpha$ and $\sigma$. The estimates of the parameters $\alpha$ and $\sigma$ can be also obtained by solving the following nonlinear equations:

$$\frac{\partial LSE}{\partial\alpha}=-\sum_{i=1}^{n}\left[\frac{n+1-i}{n+1}-\left(1-e^{-\left(\sigma/x_{(i)}\right)^2}\right)^{\alpha}\right]\left[\left(1-e^{-\left(\sigma/x_{(i)}\right)^2}\right)^{\alpha}\ln\left(1-e^{-\left(\sigma/x_{(i)}\right)^2}\right)\right]=0 \tag{3.8}$$

and

$$\frac{\partial LSE}{\partial\sigma}=-\sum_{i=1}^{n}\left[\frac{n+1-i}{n+1}-\left(1-e^{-\left(\sigma/x_{(i)}\right)^2}\right)^{\alpha}\right]\frac{2\alpha\sigma}{x_{(i)}^2}e^{-\left(\sigma/x_i\right)^2}\left[1-e^{-\left(\sigma/x_{(i)}\right)^2}\right]^{\alpha-1}=0 \tag{3.9}$$

respectively.

The problem of heteroscedasticity emerges because the variances of errors are dependent on *i*. The performance of the

estimators is adversely affected by this problem. We employ the WLSE method to solve this problem. Therefore, the WLSEs of the unknown parameters of the EIRD are obtained by minimizing the function (Swain et. al.,1988)

$$WLSE = \sum_{i=1}^{n} w_i \left( F\left(X_{(i)}\right) - \frac{i}{n+1} \right)^2 \qquad (3.10)$$

with respect to the unknown parameters. Therefore, in case the of the EIRD, the WLSEs of the parameters $\alpha$ and $\sigma$ are found by minimizing the following function

$$WLSE = \sum_{i=1}^{n} w_i \left( \frac{n+1-i}{n+1} - \left( 1 - e^{-\left(\left(\sigma/x_{(i)}\right)^2\right)^2} \right)^{\alpha} \right)^2 . \qquad (3.11)$$

The estimates of the parameters $\alpha$ and $\sigma$ can be also obtained by solving the following nonlinear equations:

$$\frac{\partial WLSE}{\partial \alpha} = -\sum_{i=1}^{n} w_i \left[ \frac{n+1-i}{n+1} - \left( 1 - e^{-\left(\sigma/x_{(i)}\right)^2} \right)^{\alpha} \right] \left[ \left( 1 - e^{-\left(\sigma/x_{(i)}\right)^2} \right)^{\alpha} \ln\left( 1 - e^{-\left(\sigma/x_{(i)}\right)^2} \right) \right] = 0 \qquad (3.12)$$

$$\frac{\partial WLSE}{\partial \sigma} = -\sum_{i=1}^{n} w_i \left[ \frac{n+1-i}{n+1} - \left( 1 - e^{-\left(\sigma/x_{(i)}\right)^2} \right)^{\alpha} \right] \frac{2\alpha\sigma}{x_{(i)}^2} e^{-\left(\sigma/x_i\right)^2} \left[ 1 - e^{-\left(\sigma/x_{(i)}\right)^2} \right]^{\alpha-1} = 0 \qquad (3.13)$$

Here, $w_i = \dfrac{1}{\mathrm{Var}\left( F\left(X_{(i)}\right) \right)}$

and

$$\mathrm{Var}\left( F\left(X_{(i)}\right) \right) = \frac{(n+1)^2 (n+2)}{i(n-i+1)} .$$

### 3.5 The Percentile Estimators

A straight line fitted to the theoretical points derived from the distribution function and the sample percentile points can be used to approximate the unknown parameters if the data originates from a

distribution function with a closed form. (Kao,1958) was the first to propose this method. Here, we use this method for the unknown parameters of the EIRD. Assume that $X_{(i)}$, $i=1,2,...,n$ represents the corresponding order statistics and that $X_1, X_2, ..., X_n$ is a random sample of size n drawn from the distribution function $F(.)$.

Then the Percentile estimators can be obtained by minimizing the following function

$$PE = \sum_{i=1}^{n} \left\{ X_{(i)} - F^{-1}\left( \frac{i}{n+1} \right) \right\}^2, \tag{3.14}$$

Here, $F^{-1}$ is the inverse distribution function. For the EIRD, equation (3.14) reduces to

$$PE = \sum_{i=1}^{n} \left( X_{(i)} - \frac{\sigma}{\sqrt{-\ln\left(1-\left((n+1-i)/(n+1)\right)^{1/\alpha}\right)}} \right)^2 \tag{3.15}$$

with respect to $\alpha$ and $\sigma$. The estimates of the parameters $\alpha$ and $\sigma$ can be also obtained by solving the following nonlinear equations:

$$\frac{\partial PE}{\partial \sigma} = \sum_{i=1}^{n} \left( X_{(i)} - \frac{\sigma}{\sqrt{-\ln\left(1-\left((n+1-i)/(n+1)\right)^{1/\alpha}\right)}} \right) \left( \frac{1}{\sqrt{-\ln\left(1-\left((n+1-i)/(n+1)\right)^{1/\alpha}\right)}} \right) = 0 \tag{3.16}$$

and

$$\frac{\partial PE}{\partial \alpha} = \left( X_{(i)} - \frac{\sigma}{\sqrt{-\ln\left(1-(1-u)^{1/\alpha}\right)}} \right) \frac{\sigma}{\ln\left(1-(1-u)^{1/\alpha}\right)} \frac{(1-u)^{1/\alpha} \ln(1-u)}{\alpha^2 \left(1-(1-u)^{1/\alpha}\right)} = 0. \tag{3.17}$$

## 6 Cramer von Misses Method

This method empirically shows that the bias of the estimator is smaller than the bias of other minimum distance estimators, see (Macdonald,1971). It is obtained by minimizing the following function.

$$CVM = \frac{1}{12n} + \sum_{i=1}^{n}\left( F\left(X_{(i)}\right) - \frac{2i-1}{2n}\right)^2 \qquad (3.18)$$

Here, $F$ is the distribution function. For the EIRD, equation (3.18) reduces to $CVM\left(\alpha,\sigma\right) = \frac{1}{12n} + \sum_{i=1}^{n}\left(1-\left(1-e^{-\left(\sigma/x_{(i)}\right)^2}\right)^{\alpha} - \frac{2i-1}{2n}\right)^2$

$$\frac{\partial CVM}{\partial \alpha} = -\sum_{i=1}^{n}\left[1-\left(1-e^{-\left(\sigma/x_{(i)}\right)^2}\right)^{\alpha} - \frac{2i-1}{2n}\right]\left[\left(1-e^{-\left(\sigma/x_{(i)}\right)^2}\right)^{\alpha}\ln\left(1-e^{-\left(\sigma/x_{(i)}\right)^2}\right)\right] = 0 \qquad (3.19)$$

$$\frac{\partial CVM}{\partial \sigma} = -\sum_{i=1}^{n}\left[1-\left(1-e^{-\left(\sigma/x_{(i)}\right)^2}\right)^{\alpha} - \frac{2i-1}{2n}\right]\frac{2\alpha\sigma}{x_{(i)}^2}e^{-\left(\sigma/x_i\right)^2}\left[1-e^{-\left(\sigma/x_{(i)}\right)^2}\right]^{\alpha-1} = 0 \qquad (3.20)$$

## 4.Bayesian Inference

This section examines the Bayesian techniques for estimating the unknown parameters of the EIRD under the SELF. The choice of loss function is crucial in Bayesian parameter estimation. SELF is one of the most widely used loss functions. This loss function is symmetric. It assigns equal values to losses for overestimation and underestimation of equal magnitude. For more details about this loss function, see (Renjini, Abdul-Sathar, & Rajesh, 2016; Ali, Aslam, & Kazmi, 2013).

This loss function is defined as:

$L\left(\hat{\theta},\theta\right) = \left(\hat{\theta}-\theta\right)^2$,

where $\hat{\theta}$ is the estimator of the parameter $\theta$. The Bayes estimate of $\theta$ under SELF is the posterior mean of $\theta$.

We assume that the parameters $\alpha$ and σ have independent prior distributions of $Gamma(a,b)$ and $Gamma(c,d)$, respectively. Depending on the parameter values, the gamma distribution can take on a variety of shapes. So, the gamma distribution family is quite flexible and can be thought of as appropriate priors $\alpha$ and $\sigma$, see (Kundu & Pradhan, 2009).

Then Gamma prior distribution for the parameters $\alpha$ and σ are

$$\pi_1(\alpha) = \frac{a^b}{\Gamma(b)} \alpha^{a-1} e^{-b\alpha} \quad a,b,\alpha > 0 \tag{4.1}$$

and

$$\pi_2(\sigma) = \frac{c^d}{\Gamma(d)} \sigma^{c-1} e^{-d\sigma} \qquad c,d,\sigma > 0. \tag{4.2}$$

Now, the joint prior density function of $\alpha$ and σ is

$$\pi(\alpha,\sigma) = \pi_1(\alpha)\pi_2(\sigma) = \frac{a^b c^d}{\Gamma(b)\Gamma(d)} \alpha^{a-1} e^{-b\alpha} \sigma^{c-1} e^{-d\sigma} \qquad \alpha,\sigma > 0;\ a,b,c,d > 0. \tag{4.3}$$

The hyper parameters $a,b,c,d$ are assumed to be known. They are chosen in a manner to reflect the prior knowledge about the unknown quantiles.

Combining Equation (4.3) with (3.1) and using Bayes theorem, the joint posterior distribution of $\alpha$ and $\sigma$ is obtained as:

$$p(\alpha,\sigma|x) = \frac{L(x|\alpha,\sigma)\pi(\alpha,\sigma)}{\int\int L(x|\alpha,\sigma)\pi(\alpha,\sigma)d\alpha d\sigma} \propto 2^n \alpha^{n+a-1} \sigma^{2n+c-1} e^{-b\alpha - d\sigma} \prod_{i=1}^{n} x_i^{-3} e^{-\sum_{i=1}^{n} (\sigma/x_i)^2} \prod_{i=1}^{n} \left(1 - e^{-(\sigma/x_i)^2}\right)^{\alpha-1}, \tag{4.4}$$

Now, using the Bayes estimators of any function of $\alpha$ and $\sigma$ say, $g(\alpha,\sigma)$, under SELF is obtained as follows:

$$\hat{g}(\alpha,\sigma) = E\left(g(\alpha,\sigma)\right) = \int_0^\infty \int_0^\infty g(\alpha,\sigma)\, p(\alpha,\sigma|x)\, d\alpha d\sigma \tag{4.5}$$

The Bayes estimators of the parameters are the ratio of the integral, which is not in explicit form, as we can see in this case for

the posterior distribution has an integral in the denominator that cannot be solved. Therefore, it will be tedious to determine the posterior expectation to construct the Bayes estimator of α and $\sigma$. We will use the Monte Carlo Markov Chain (MCMC) method in this study. This method is briefly summarized below.

### 4.1 Markov Chain Monte Carlo Method

Here, we discuss the MCMC algorithm procedure to draw sequences of samples from the full conditional distributions of the parameters under SELF. We also analyze the MCMC approximation, namely the Gibbs sampling and M-H algorithm, to generate a sample from the posterior distribution and calculate the Bayes estimation, see (Gelfand & Smith, 1991). When it is simple to sample from the full conditional distributions, the Gibbs sampling method can be effective. In other words, if the parameters have standard forms, it is simple to model their conditional distributions. However, generating samples from full conditionals that correspond to the joint posterior distribution is not easy. Thus, the Metropolis-Hasting algorithm was taken into consideration. To complete a cycle in the Gibbs chain, exact samples from a part of the full conditional are obtained using the Metropolis step. Consequently, the Metropolis-Hastings method was examined. This method was proposed by (Metropolis et al., 1953).

By using equation (3.4), the conditional posterior distributions of the parameters $\alpha$ and σ are given by

$$\pi\left(\alpha\left|\sigma,x\right.\right)\propto\alpha^{n+a-1}e^{-b\alpha}\prod_{i=1}^{n}\left(1-e^{-\left(\sigma/x_i\right)^2}\right)^{\alpha-1} \tag{4.6}$$

and

$$\pi\left(\sigma|\alpha,x\right)\propto\sigma^{2n+c-1}e^{-d\sigma}\prod_{i=1}^{n}x_i^{-3}e^{-\sum_{i=1}^{n}(\sigma/x_i)^2}\prod_{i=1}^{n}\left(1-e^{-(\sigma/x_i)^2}\right)^{\alpha-1} \qquad (4.7)$$

The Gibbs algorithm consists of these steps:

- Start with an initial value $\left(a_0=\hat{a},\ \alpha_0=\hat{\alpha},\ \sigma_0=\hat{\sigma}\right)$ and set $j=1.$
- Use Metropolis-Hasting algorithm to generate samples from posterior density for $\alpha$ and $\sigma$.
- Repeat the above two steps M times to obtain posterior samples.
- The Bayes estimators $\alpha$ and $\sigma$ under SELF are as follows after obtaining of the posterior samples:

$\hat{\alpha}_{MCMC}=\dfrac{1}{M-M_0}\sum_{j=1}^{M-M_0}\alpha_j$ and $\hat{\sigma}_{MCMC}=\dfrac{1}{M-M_0}\sum_{j=1}^{M-M_0}\sigma_j$,

respectively. Here, $M_0$ is the Markov chain burn period.

## 5. Simulation Study

In this section, the performance of Bayesian and classical methods is compared in terms of bias and mean square error (MSE) values for different (small med sample sizes using a Monte Carlo simulation study. The sample size was taken as $n=30,50,100$ shape parameter as $\alpha=0.5,1,2.$ throughout the study, the scale parameter $\sigma$ is 1 since all estimators are scale invariant. For Bayesian inference, hyper parameters $a,b,c,d$ were taken as 0. In the classical aproximation, MLE, LSE, WLSE, PE and CVM methods were used. The MCMC approximation is used to compute the Bayes estimate of the unknown parameters under SELF based on gamma prior. For the computation, 10,000 MCMC samples are used. Matlab R 2013 is used for all calculations, with 10,000 replications for each case. Simulation results are summarized in Table 1. The Bias and MSE values are found by using the following formula:

$Bias(\theta)=\frac{1}{N}\sum_{i=1}^{n}\left(\hat{\theta}_i-\theta\right)$ and $MSE(\theta)=\frac{1}{N}\sum_{i=1}^{n}\left(\hat{\theta}_i-\theta\right)^2$, respectively.

*Table 1. The Bias and MSE values for the classical different parameter estimators of $\alpha$ and $\sigma$*

| | | | $\hat{\alpha}$ | | $\hat{\sigma}$ | |
|---|---|---|---|---|---|---|
| n | $\alpha$ | Estimator | Bias | MSE | Bias | MSE |
| 30 | 0.5 | MLE | 0.0453 | 0.0193 | 0.0551 | 0.0273 |
| | | LSE | 0.0184 | 0.0232 | 0.0120 | 0.0380 |
| | | WLSE | 0.0222 | 0.0207 | 0.0193 | 0.0328 |
| | | PE | -0.0398 | 0.0382 | 0.0761 | 0.0416 |
| | | CVM | 0.0573 | 0.0316 | 0.0678 | 0.0464 |
| | | MCMC | 0.0409 | 0.0147 | 0.0469 | 0.0174 |
| 50 | 0.5 | MLE | 0.0249 | 0.0093 | 0.0259 | 0.0157 |
| | | LSE | 0.0080 | 0.0112 | -0.0032 | 0.0213 |
| | | WLSE | 0.0121 | 0.0098 | 0.0051 | 0.0193 |
| | | PE | -0.0372 | 0.0171 | 0.0316 | 0.0256 |
| | | CVM | 0.0297 | 0.0135 | 0.0290 | 0.0233 |
| | | MCMC | 0.0155 | 0.0046 | 0.0174 | 0.0165 |
| 100 | 0.5 | MLE | 0.0074 | 0.0035 | 0.0100 | 0.0061 |
| | | LSE | -0.0009 | 0.0048 | -0.0057 | 0.0091 |
| | | WLSE | 0.0021 | 0.0041 | 0.0042 | 0.0073 |
| | | PE | -0.0359 | 0.0055 | 0.0147 | 0.0200 |
| | | CVM | 0.0091 | 0.0052 | 0.0102 | 0.0095 |
| | | MCMC | 0.0033 | 0.0015 | 0.0067 | 0.0015 |
| 30 | 1 | MLE | 0.1133 | 0.1102 | 0.0395 | 0.0185 |
| | | LSE | 0.0278 | 0.1154 | -0.0066 | 0.0225 |
| | | WLSE | 0.0404 | 0.1011 | 0.0038 | 0.0193 |
| | | PE | -0.0387 | 0.0547 | 0.0522 | 0.0239 |
| | | CVM | 0.1232 | 0.1653 | 0.0356 | 0.0256 |
| | | MCMC | 0.0853 | 0.0419 | 0.0323 | 0.0082 |
| 50 | 1 | MLE | 0.0550 | 0.0446 | 0.0210 | 0.0104 |
| | | LSE | 0.0050 | 0.0538 | -0.0082 | 0.0130 |
| | | WLSE | 0.0108 | 0.0455 | 0.0012 | 0.0112 |
| | | PE | -0.0307 | 0.0087 | 0.0265 | 0.0091 |
| | | CVM | 0.0379 | 0.0662 | 0.0168 | 0.0137 |
| | | MCMC | 0.00397 | 0.0274 | 0.0156 | 0.0037 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 100 | 1 | MLE | 0.0302 | 0.0220 | 0.0106 | 0.0048 |
| | | LSE | 0.0104 | 0.0297 | -0.0015 | 0.0063 |
| | | WLSE | 0.0167 | 0.0249 | 0.0031 | 0.0053 |
| | | PE | -0.0196 | 0.0255 | 0.0385 | 0.0059 |
| | | CVM | 0.0361 | 0.0331 | 0.0109 | 0.0065 |
| | | MCMC | 0.0196 | 0.0131 | 0.0054 | 0.0032 |
| 30 | 2 | MLE | 0.2309 | 0.5496 | 0.0265 | 0.0117 |
| | | LSE | 0.0862 | 0.7324 | -0.0079 | 0.0149 |
| | | WLSE | 0.0973 | 0.6006 | -0.0112 | 0.0128 |
| | | PE | -0.2226 | 0.8191 | 0.0291 | 0.0049 |
| | | CVM | 0.3321 | 0.8277 | 0.0277 | 0.0165 |
| | | MCMC | 0.2072 | 0.2120 | 0.0286 | 0.0055 |
| 50 | 2 | MLE | 0.1206 | 0.2529 | 0.0138 | 0.0066 |
| | | LSE | 0.0320 | 0.3794 | -0.0069 | 0.0093 |
| | | WLSE | 0.0535 | 0.3163 | -0.0008 | 0.0080 |
| | | PE | -0.0054 | 0.3129 | 0.0264 | 0.0032 |
| | | CVM | 0.1650 | 0.4829 | 0.0141 | 0.0099 |
| | | MCMC | 0.0982 | 0.1084 | 0.0135 | 0.0023 |
| 100 | 2 | MLE | 0.0781 | 0.1143 | 0.0083 | 0.0033 |
| | | LSE | 0.0237 | 0.1504 | -0.0023 | 0.0045 |
| | | WLSE | 0.0427 | 0.1204 | 0.0018 | 0.0038 |
| | | PE | 0.0161 | 0.1267 | 0.0177 | 0.0016 |
| | | CVM | 0.0871 | 0.1711 | 0.0081 | 0.0047 |
| | | MCMC | 0.0799 | 0.0917 | 0.0084 | 0.0018 |

According to the simulation results, as the sample size of all estimators increases, the bias and MSE values decrease. Also, in many cases, LSE and WLSE methods have smaller bias values for both parameters. Moreover, MCMC method has smaller MSE values in all cases. PE and CVM methods have the worst performance among the proposed estimators in terms of bias and MSE values in many cases.

## 6. Conclusion

Effective and accurate parameter estimation has an important place in many areas. In this study, the unknown parameters of EIRD

were examined with Bayesian parameter estimation methods in addition to classical parameter estimation methods. MLE, LSE, WLSE, PE, and CVM methods were used in classical parameter estimation. The Gibbs algorithm, a subclass of the MCMC method, was used in the Bayesian approximation. Loss function and prior distribution are of vital importance in Bayesian inference. Here, a gamma prior distribution was used for both shape and scale parameters. This prior distribution is very flexible and is frequently used in applications. SELF was considered as the loss function, which is symmetric and quite useful. Finally, a simulation study was conducted to compare the performance of all proposed estimation methods. It was seen from the simulation study that LSE and WLSE have smaller bias values in many cases, especially in small samples. In addition, the MCMC method has a smaller MSE value for both parameters.

This study can be expanded by considering different loss functions in future studies.

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# CHAPTER II

# Lotka-Volterra Model and Estimation of Parameters of Lynx-Hare population

**Levent ÖZBEK**[1]
**Erhan ÇETINKAYA**[2]

## Introduction

Lotka-Volterra models have been applied in different fields. It has been applied in various studies such as electricity costs, integrated circuit industry, chip manufacturing, optoelectronic industry output value and mortality risk prediction measure. The general functional form of the Lotka-Volterra model consists of strategic interactions of two related variables in a social or economic setting. Variables are sometimes described as types when they reflect growth patterns for a predator-prey population and their dynamic mode of interaction. In a competitive market environment, in addition to the natural environment of a constrained ecosystem in the

[1] Doç. Dr. Ankara Üniversitesi, İstatistik Bölümü, Ankara, Türkiye. ozbek@science.ankara.edu.tr
[2] Dr. TÜİK Başkanı, Necatibey Cd. No:114, 06420 Ankara, Türkiye, cetink@gmail.com

struggle for survival of species, a multifaceted competitive relationship can be determined between behavioral variables and their interactive parameters. The definition of the market structure is possible according to the positive or negative signs of the b and r parameters in the second components of Lotka-Volterra models equations. The perfect competition market structure provides an environment for agents to treat each other as strategic substitutes. Coexistence is the only condition for perfect competition, where the parameters have positive values. When b has a positive value and the other interaction parameter r has a negative value, the market is dominated by a predator-prey situation, similar to the natural ecosystem where the big agent preys on the small agent. Agents treat each other as strategic complements. This time, coexistence is the only condition for cooperation, where both of the relevant interaction parameters have negative values. The interactions of research and development (R&D) and gross domestic product (GDP) can be given as examples of the predator-prey model. The interactions of fixed asset investment (FAI) and consumer price index (CPI) can also be thought of in this way. When the interaction model of the fixed asset investments and the consumer price index is examined, this corresponds to the market structure of strategic complementarities. The sign of the interaction parameters $b < 0$ and $r < 0$ of the Lotka-Volterra model confirms the assumption of strategic complementarity. The Lotka-Volterra model for predicting two variables in the ecosystem can be used as a useful tool in analyzing economic variables in a competitive market mechanism. Variables that demonstrate strategic complementarity in terms of market structure definition have important economic implications for development policies.

In this work, we present a new approach to the problem of estimating parameters in Lotka-Volterra models. We present how an recursive estimator known as the adaptive extended Kalman filter (AEKF) can be used to estimate parameters in Lotka-Volterra models and how it is implemented in Lynx-Hare species.

Although various methods are used to estimate parameters in Lotka-Volterra models, existing studies have not used the AEKF method. The main contribution of this study is that it uses the AEKF method, which is an effective way to estimate time-varying model parameters simultaneously.

**The Lotka-Volterra predator-prey model**

The explanation about the predator-prey model developed by Lotka (1920) and Volterra (1926) is given in this subsection. In this study, the predator-prey model considered is used to explain how animal populations interact with each other and only takes into account how lynx-hare populations change without being exposed to any additional environmental or external factors.

**The Lotka-Volterra model of Lynx-Hare population**

The model is given by two equations; first the growth equation for hare:

$$\frac{dm(t)}{dt} = am(t) - bm(t)n(t) = \big(a - bn(t)\big)m(t) \qquad (1)$$

and other, the growth equation for the lynx,

$$\frac{dn(t)}{dt} = -cn(t) + rn(t)m(t) = \big(-c + rm(t)\big)n(t) \qquad (2)$$

$$m(0) = x_0 \quad , \quad n(0) = y_0$$

where

$m(t)$ : prey population

$n(t)$ : predator population

$a\,(a>0)$ : Prey growth rate in the absence of the prey

$b\,(b>0)$ : Predation rate by the predators on the prey

$c\ \ (c>0)$ : Decay rate of the predator in the absence of the prey

$r\,(r>0)$ : Growth rate of the predators (dependent on the prey population)

The discrete-time model of Lotka-Volterra equations is given by the following equations:

$$m(t+\Delta t) = m(t) + \big(a - bn(t)\big)m(t)\Delta t$$

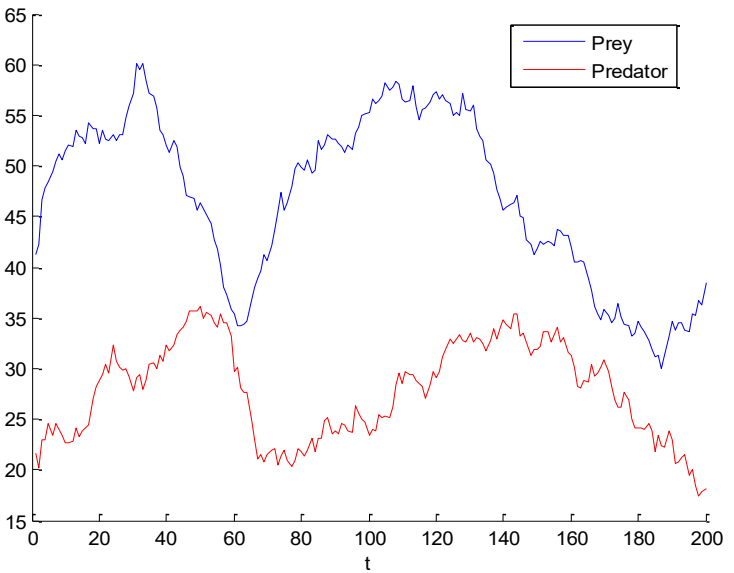
$$n(t+\Delta t) = n(t) + \big(-c - rm(t)\big)n(t)\Delta t$$

In this study we chose $\Delta t = 0.1$. In this way, the continuous model was made discrete. It becomes easier to convert this discrete model into a state-space model.

**Simulation Data for Model**

In order to see how the model behaves, the simulation dataset was created with additional noise in addition to the predator-prey model. We determined the parameters in the model as *a=0.65, b*=0.023*, c=0.65, r=0.014* and Δ=0.1.

*Figure-1: Simulated dataset from the model*




**State-Space Model and AEKF**

The discrete-time state-space model of the Lotka-Volterra model (1)-(2) can be obtained as follows. Let $x_k = [x_{1k}\ x_{2k}]'$ be the state vector containing the states to be estimated at time $k$, where the states are defined as $x_{1k} = m_k$ and $x_{2k} = n_k$

$$x_{k+1} = \begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} ax_{1,k} - bx_{1,k}x_{2,k} \\ -cx_{2,k} + rx_{2,k}x_{2,k} \end{bmatrix} + w_k \qquad (3)$$

$$z_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_k + v_k \qquad (4)$$

Equation-3. is the unknown parameters. When the matrices in the state-space models are unknown, EKF is used to estimate both

the state vector and the parameters simultaneously. The unknown parameter vector in the model is considered as a random walk process as follows.

$$\theta_{k+1} = \theta_k + \zeta_k$$

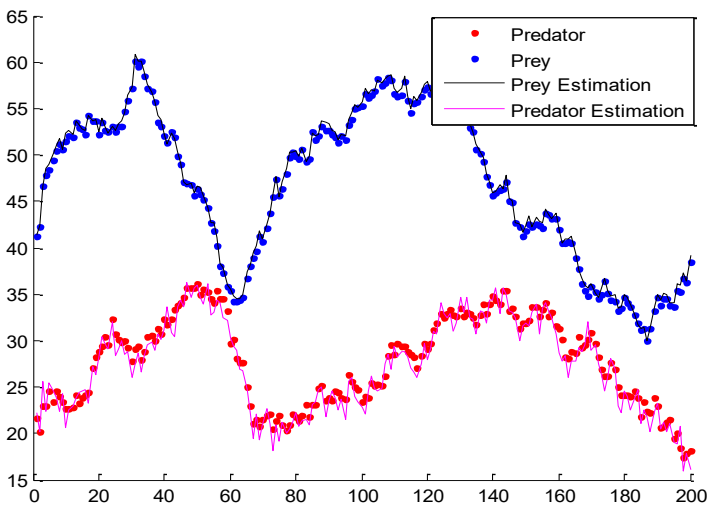
After this assumption

$$\begin{bmatrix} x_{k+1} \\ \theta_{k+1} \end{bmatrix} = \begin{bmatrix} \Phi_k(\theta_k) x_k \\ \theta_k \end{bmatrix} + \begin{bmatrix} w_k \\ \zeta_k \end{bmatrix} \tag{5}$$

$$z_k = \begin{bmatrix} H_k & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \theta_k \end{bmatrix} + v_k \tag{6}$$

The state-space model can be written in the form of when the AEKF given in the Appendix is applied, Estimated values of unobserved prey and predator values obtained from EKF are given in Figure 2.

*Figure-2: Estimated states of the simulated dataset using AEKF*



**Estimation of Parameters of Lynx-Hare population**

The Lynx-Hare population numbers observed between 1845-1903 are given in Table-1. According to these observed numbers, the EKF method was used to estimate the parameters in the model (1)-(2) and the recursive time-dependent estimated values of the parameters are given in Figure-3. The lynx and hare values estimated simultaneously recursively are given in Figure-4.

*Tablo-1: Lynx-Hare population*

| Year | Hares (x1000) | Lynx(x1000) |
|---|---|---|
| 1845 | 20 | 32 |
| 1847 | 20 | 50 |
| 1849 | 52 | 12 |
| 1851 | 83 | 10 |
| 1853 | 64 | 13 |
| 1855 | 68 | 36 |
| 1857 | 83 | 15 |
| 1859 | 12 | 12 |
| 1861 | 36 | 6 |
| 1863 | 150 | 6 |
| 1865 | 110 | 65 |
| 1867 | 60 | 70 |
| 1869 | 7 | 40 |
| 1871 | 10 | 9 |
| 1873 | 70 | 20 |
| 1875 | 100 | 34 |
| 1877 | 92 | 45 |
| 1879 | 70 | 40 |
| 1881 | 10 | 15 |
| 1883 | 11 | 15 |
| 1885 | 137 | 60 |
| 1887 | 137 | 80 |
| 1889 | 18 | 26 |
| 1891 | 22 | 18 |
| 1893 | 52 | 37 |
| 1895 | 83 | 50 |
| 1897 | 18 | 35 |
| 1899 | 10 | 12 |
| 1901 | 9 | 12 |
| 1903 | 65 | 25 |

*Figure-3: Estimation of parameters*

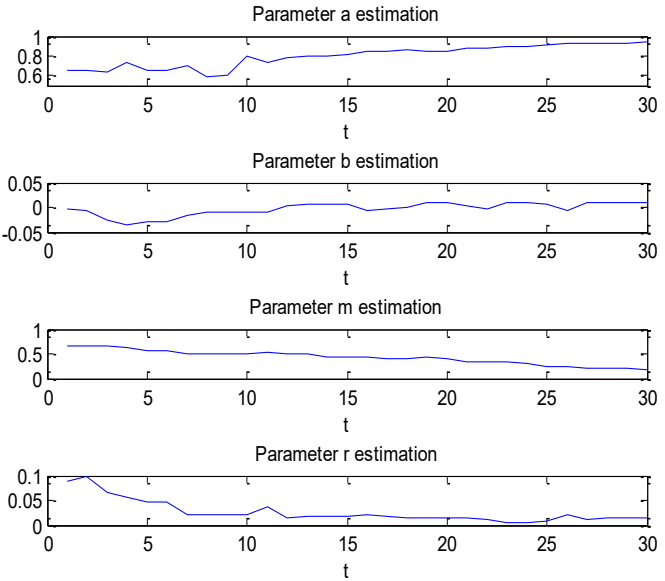



*Figure-4: Estimation of state vectors*

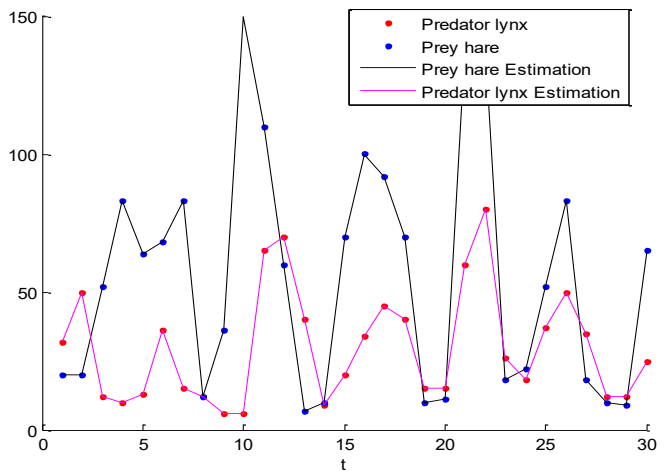


*Figure-5: Estimation error of predator*

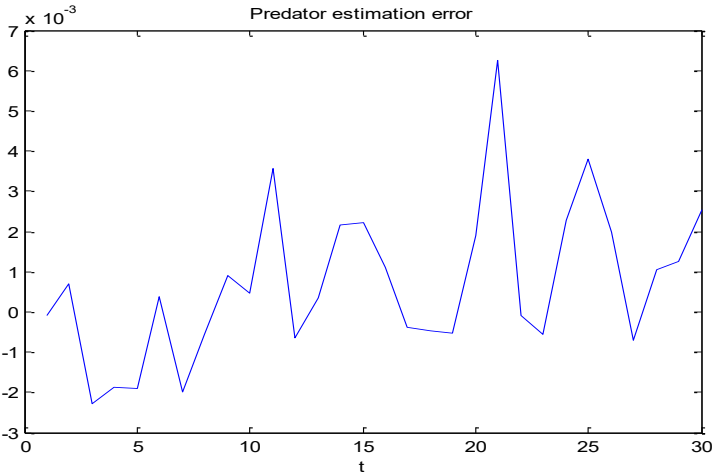
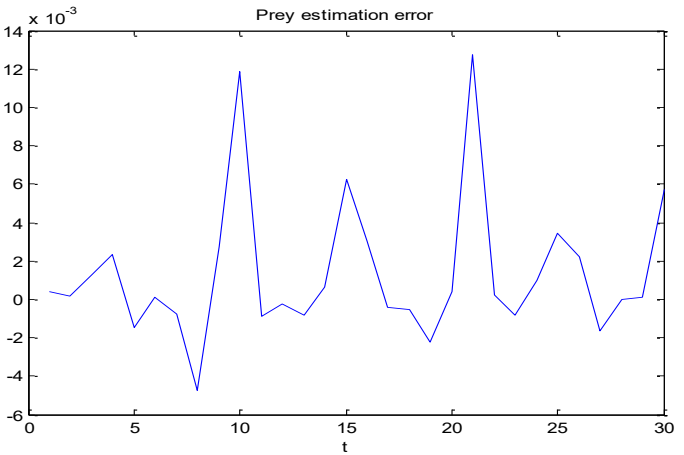



*Figure-6: Estimation error of prey*




## Appendix 1: Kalman Filter Algorithm

The Kalman filter algorithm is given by the following algorithm:

$$\hat{x}_{n|n-1} = \Phi_{n-1}\hat{x}_{n-1|n-1} + B_{n-1}u_{n-1} \quad \text{(A1)}$$

$$\hat{x}_{n|n} = \hat{x}_{n|n-1} + K_n\left[y_n - H_n\hat{x}_{n|n-1}\right] \quad \text{(A2)}$$

$$K_n = P_{n(n-1}H'_n\left[H_nP_{n(n-1}H'_n + R_n\right]^{-1} \quad \text{(A3)}$$

$$P_{n(n} = \left[I - K_nH_n\right]P_{n(n-1} \quad \text{(A4)}$$

$$P_{n|n-1} = \Phi_{n-1}P_{n-1|n-1}\Phi'_{n-1} + G_{n-1}Q_{n-1}G'_{n-1} \quad \text{(A5)}$$

## Appendix 2: Extended Kalman Filter Algorithm

Suppose that $X_n = \begin{pmatrix} x_n \\ \theta_{n-1} \end{pmatrix}, \overline{K}_n = \begin{pmatrix} K_n \\ L_n \end{pmatrix}, \overline{P}_n = \begin{pmatrix} P_1(n) & P_2(n) \\ P_2^T(n) & P_3(n) \end{pmatrix}$

where $\overline{K}$ and $\overline{P}$ are Kalman gain and the covariance matrix of the extended state, respectively, as stated in Ljung and Söderström (1985). Then, the updating equations will be:

$$\hat{x}_{n+1} = F_n\hat{x}_n + G_nu_n + K_n\left(y_n - H_n\hat{x}_n\right) \quad \text{(A6)}$$

$$\hat{x}_0 = 0$$

$$\hat{\theta}_n = \hat{\theta}_{n-1} + L_n\left(y_n - H_{n-1}\hat{x}_n\right) \quad \text{(A7)}$$

$$\hat{\theta}_0 = \theta_0$$

$$K_n = \left(F_nP_1(n)H_n^T + M_nP_1^T(n)H_n^T + F_nP_2(n)D_n^T + M_nP_2(n)D_n^T + R_{12}\right)S_n^{-1} \quad \text{(A8)}$$

$$S_n = H_nP_1(n)H_n^T + H_nP_2(n)D_n^T + D_nP_2^T(n)H_n^T + D_nP_3(n)D_n^T + R_2 \quad \text{(A9)}$$

$$L(n) = \left(P_2^T(n)H_{n-1}^T + P_3(n)D_n^{\ T}\right)S_n^{-1} \quad \text{(A10)}$$

$$P_1(n+1) = F_nP_1(n)F_n^T + F_nP_2(n)M_n^T + M_nP_2^T(n)F_n^T + M_nP_3(n)M_n^T - K_nS_nK_n^T + R_1 \quad \text{(A11)}$$

$$P_1(0) = \Pi_0(\theta_0)$$

$$P_2(n+1) = F_n P_2(n) + M_n P_3(n) - K_n S_n L_n^T \tag{A12}$$

$$P_2(0) = 0$$

$$P_3(n+1) = P_3(n) - L_n S_n L_n^T \tag{A13}$$

$$P_3(0) = P_0$$

Here, it is assumed that

$$F_n = F\left(\hat{\theta}_n\right)$$

$$G_n = G\left(\hat{\theta}_n\right) \tag{A14}$$

$$H_n = H\left(\hat{\theta}_n\right)$$

$$M_n = M\left(\hat{\theta}_n, \hat{x}_n, u_n\right)$$

and

$$M\left(\hat{\theta}, x, u\right) = \frac{\partial}{\partial\theta}\left[F(\theta)x + G(\theta)u\right]\Big|_{\theta=\theta} \tag{A15}$$

and

$$D_n = D\left(\hat{\theta}_{n-1}, \hat{x}_n\right)$$

$$D\left(\hat{\theta}, x\right) = \frac{\partial}{\partial\theta}\left[H(\theta)x\right]\Big|_{\theta=\hat{\theta}}$$

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# CHAPTER III

# An Econometric Analysis in Terms of Budget Revenues

**Levent ÖZBEK**[1]
**Erhan ÇETINKAYA**[2]

## Introduction

There is a versatile relationship between the purpose we use in defining planning, the concepts of decision making and choice. The emergence of a purpose carries a decision in itself while initiating a conscious thinking process on options. In this sense, every decision is a choice and every decision have a purpose. After determining the purpose, some decisions are needed in the process and there are planning actions where these decisions turn into systematic actions (Ersoy, 2012). In this case, planning appears as an effort to establish control over time for the future.

[1] Doç. Dr. Ankara Üniversitesi, İstatistik Bölümü, Ankara, Türkiye. ozbek@science.ankara.edu.tr
[2] Dr. TÜİK Başkanı, Necatibey Cd. No:114, 06420 Ankara, Türkiye, cetink@gmail.com

At the core of the planning, there is an effort to dominate and change the future. According to Sezen (1990: 7-8), the basis of this effort is the belief that human beings can dominate his/her future by using his/her mind and that she/he can change and manage it even if not completely.

**How to Achieve Economic Goals?**

Yılmaz (1999, pp.85) has defined the economic planning as an information-based effort to achieve results by mobilizing and effectively using resources to achieve pre-determined goals. Behavioral economics are generally based on the principle of rationality and provide a cause-effect relationship between events. Therefore, it has been possible to change the results with the interventions. In order that a plan to become a plan, it must include not only the aims but also the means to achieve them. As Soyak (2006) stated, another important issue is the necessity of data collection and forecasting activities while planning.

Although the desire to achieve medium and long-term macroeconomic goals differs according to the development level of the countries, which financial resources can be used to reach these goals, that is, financing the policies to be implemented is the most important. In order for the public sector to achieve economic development and goals of social policies, it should have the ability to create a financial resource or a fiscal area. The fiscal area is expressed by the *"fiscal space"* which allows a government to provide resources for a desired purpose, without jeopardizing the sustainability of the financial position or the stability of the economy (Heller, 2005: 3). It is very important to have an available fiscal area for financing budget deficits, especially in times of increasing

financial risks. Financial sustainability, medium and long-term macroeconomic goals, finding a resource pool and creating fiscal discipline also come to the forefront in the definitions made regarding the concept of financial field (Ünsal & Durucan, 2013: 29).

The "*fiscal discipline"* is called to ensure the consistency of the budget balance in the short and medium term.

The capacity of the public sector to implement macroeconomic goals without any debt obligations, in cases where it is financed by borrowing, the transfer of debt payment power financial capability to the future is defined as *"financial sustainability".*

Along with economic growth, the financial area created for the financing of the policies to be implemented by the public sector such as ensuring economic stability and fair income distribution is used, the capacity of the public to create fiscal areas by using fiscal policy tools such as taxes, public expenditures and borrowing contribute to achieving macroeconomic goals with available resources.

Governments can shape their policies according to the fluctuations in the economy regardless of the general rules. Governments which adopt such policy implementations determine their policy decisions according to the need of the conjuncture, rather than voluntarily (Auerbach et al., 2010).

**The Importance of MTP and MFTP Targets**

In accordance with Article 16 of the Public Financial Management and Control Law Nr. 5018, the Medium-Term Program

(MTP) and the Medium-Term Fiscal Plan (MTFP) prepared annually to cover the three-year period should comply with the Development Plan in terms of its goals. The specific macroeconomic goals, objectives and policies included in their content are very important for both domestic and foreign markets.

In our country, all plans and programs with economic qualities are prepared by going through both a technical and a political process. With planning, how and for what purpose the resources will be distributed or which strategies will be implemented are determined by the political authorities (Soyak, 2006: 5-6).

In these plans, which are prepared in accordance with the multi-year budgeting approach, although it is stated that the effects of uncertainties experienced in the world economy on the country's economy are kept to a minimum, the targets set in the programs remain only as “intent” and it is seen that the realizations are quite deviated from the target.

**Status of Budget Revenue Realizations According to the Goals in the MTP**

In general, all information and numerical data obtained are taken into account when making a plan or program. Especially in plans with economic and financial content, indicative numerical data on economic events form the backbone of these studies.

Central administration budget revenues, which are the subject of this study, are an important element that markets follow carefully as they provide financing for a significant part of public expenditures. From this point of view, the relation of the budget income target included in the program with other economic indicators should be taken into consideration and the target number

should be estimated by modeling in this technical analysis. Annex 1 includes some macroeconomic indicators and central government budget revenue targets and annual realization figures included in the annual MTP and MTFP for the period 2006-2019.

**Method**

In this study, it is aimed to obtain medium-period estimations for the target revenues of central administration budget. Kalman filter was used to obtain these estimates and the interaction of budget revenue with both its internal dynamics and national income and inflation data were also taken into account. In our estimation study of budget revenue, the state space model was used to model the current structure, and the Kalman filter was used to obtain medium-period estimations. Revenue estimates for the 2018-2021 period were obtained from the model created.

**Data**

The target data used in the study were obtained from the annual MTP, MTFP's, and the realization data were obtained from the data on the websites of the Turkish Statistical Institute (TUIK) and Ministry of Treasury and Finance of Turkey, which shared the realization figures of the relevant data with the public. The data used in the study belongs to the period 2006-2018 and is included in Annex 1. To evaluate the prospective model estimates, data from the study in MTP and MTFP for the period 2019-2021 were utilized.

**Methodology**

The model in below is called the *State-Space Model,* which shows the state of a system but cannot be observed, a state equation related $\{x_t, t = 0,1,2,...\}$ stochastic process and a

measurement(observation) equation related to the $\{y_t, t = 0,1,2,...\}$ stochastic process that can be observed.

$$x_{t+1} = F_t x_t + G_t w_t$$

$$y_t = H_t x_t + v_t$$

Here;

$x_t \in \Re^n$ is called system state vector

$y_t \in \Re^m$ is called system output vector

$u_t \in \Re^r$ is called system control vector

The *mxn* matrix $F_t$ shows system the transition matrix, , *mxn* matrix $H_t$ shows the observation matrix. $G_t$ is a matrix of properly selected dimensions. $w_t \in \Re^n$ and $v_t \in \Re^m$ indicates zero mean white noise processes i.e. error terms. For every $t, j$ value of white noise processes

$$\delta_{tj} = \begin{cases} 1, & t = j \\ 0, & t \neq j \end{cases}$$

it has been accepted that it provides the following assumptions:

$$E[v_t] = 0$$

$$E[w_t] = 0$$

$$E\left[v_t v_j^{'}\right] = R_t \delta_{tj}$$

$$E\left[w_t w_j^{'}\right] = Q_t \delta_{tj}$$

$$E\left[v_t w_j^{'}\right] = 0$$

$$E\left[x_0\right] = \overline{x}_0$$

$$E\left[(x_0 - \overline{x}_0)(x_0 - \overline{x}_0)^{'}\right] = P_0$$

$$E[x_0 w_t^{'}] = 0$$

$$E[x_0 v_t^{'}] = 0$$

In addition, it is assumed that in every $t = 0,1,2,...$ times all $F_t$, $H_t$, $G_t$, $Q_t$ and $R_t$ matrices are known. The aim here is, to create a state space model related to the system of interest and to estimation the unobservable state vector. Error terms

$$w_t \sim \mathrm{N}(0, \mathrm{Q_t})$$

$$v_t \sim \mathrm{N}(0, \mathrm{R_t})$$

and its initial state

$$x_0 \sim \mathrm{N}(\overline{x}_0, \mathrm{P_0})$$

indicate as assumed to have a normal distribution. The filtering problem is the problem of determining the best estimate of the $x_t$ state when $Y_t = (y_0, y_1, ..., y_t)$ observations are given. When $Y_t = (y_0, y_1, ..., y_t)$ observations are given, estimation of $x_t$ is shown as,

$$\hat{x}_t = E(x_t \,|\, y_0, y_1, ..., y_t) = E(x_t \,|\, Y_t)$$

and covariance matrix of error is shown as,

$$P_{t|t} = E\left[ (x_t - \hat{x}_{t|t})(x_t - \hat{x}_{t|t})' \,|\, Y_t \right]$$

and when $Y_{t-1} = (y_0, y_1, ..., y_{t-1})$ observations are given, estimation of $x_t$ is shown as,

$$\hat{x}_{t|t-1} = E(x_t \,|\, y_0, y_1, ..., y_{t-1}) = E(x_t \,|\, Y_{t-1})$$

and covariance matrix of error is shown as,

$$P_{t|t-1} = E\left[ (x_t - \hat{x}_{t|t-1})(x_t - \hat{x}_{t|t-1})' \,|\, Y_{t-1} \right]$$

Kalman Filter,

$$P_{0|-1} = P_0$$

$$\hat{x}_{0|-1} = \overline{x}_0$$

depending on the initial values, given with the following equations:

$$\hat{x}_{t|t-1} = F_{t-1} \hat{x}_{t-1}$$

$$P_{t|t-1} = F_{t-1} P_{t-1|t-1} F_{t-1}' + G_{t-1} Q_{t-1} G_{t-1}'$$

$$K_t = P_{t|t-1} H_t' (H_t P_{t|t-1} H_t' + R_t)^{-1}$$

$$P_{t|t} = [I - K_t H_t] P_{t|t-1}$$

$$\hat{x}_t = \hat{x}_{t|t-1} + K_t(y_t - H_t\hat{x}_{t|t-1})$$

Özbek (2017), Öztürk and Özbek (2015).

## Model and Findings

Model;

$$Income(k) = a(k) * Income(k-2) * (1 + inf(k-2)) + b(k) * GDP(k-2) + v(k) \quad (1)$$

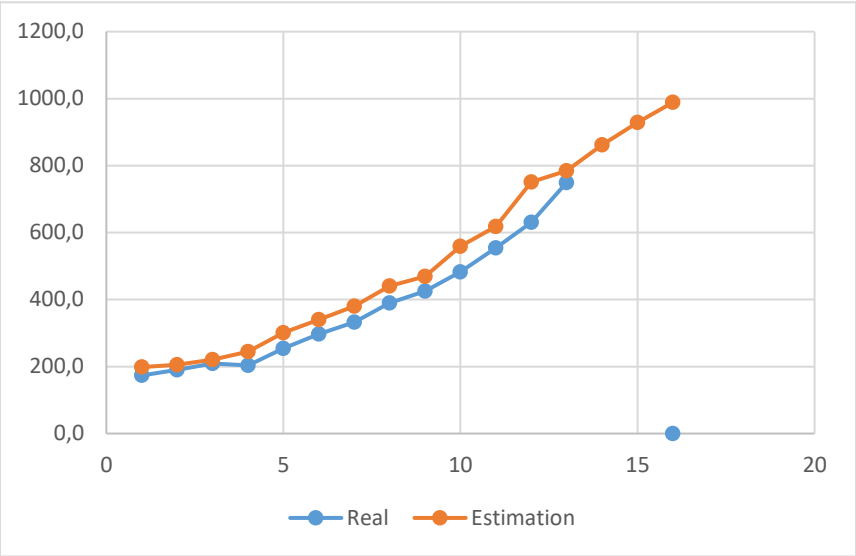
when $a(k)$ and $a(k)$ time-varying parameters

$$a(k) = a(k-1) + w1(k) \quad (2)$$

$$b(k) = b(k-1) + w2(k) \quad (3)$$

are assumed to be a random walking process (ie assuming time-varying parameters), the equations given by (1-3) can be written in the form of a state-space model. Here, $a(k)$ and $b(k)$, which cannot be observed in the state vector, are time-varying parameters. These parameters are estimated using the Kalman filter. The results of the income estimates obtained from the model are given in Figure 1.

*Figure 1. Budget Revenue Estimations from the Model*




The model we use for the estimation of budget revenues is a dynamic model using only GDP and inflation data. As it is known, the complexity of the economic events, the problems experienced due to the selection of the influencing data, or the constant change of the dynamics of the system over time appear as an important problem both in the creation of the model to be used and in the quality of the predictions to be obtained from the model. Furthermore, in our estimation study, the estimates obtained should deviate since we use a dynamic model consisting of only two data and other influencing data is excluded from the model. In Table 1 below, the target, realization and model estimation results and deviations related to these are given. 2018 realization data and 2019-2021 target data are estimates in 2019 MTP and MTFP

*Table 1. Target-Realization-Estimation Values and Deviation Rates of Budget Revenue for the 2006-2021 Period (%)*

| Date | Actual | Estimation | Target | Estimation - Actual, Deviation (%) |
|---|---|---|---|---|
| 2006 | 173,5 | 198,4 | 141,8 | 0,14 |
| 2007 | 190,4 | 205,0 | 181,1 | 0,08 |
| 2008 | 209,6 | 220,3 | 186,8 | 0,05 |
| 2009 | 203,9 | 244,7 | 244,0 | 0,20 |
| 2010 | 254,3 | 300,5 | 236,8 | 0,18 |
| 2011 | 296,8 | 339,8 | 279,0 | 0,14 |
| 2012 | 332,5 | 380,2 | 329,8 | 0,14 |
| 2013 | 389,7 | 440,3 | 370,1 | 0,13 |
| 2014 | 425,4 | 468,9 | 436,3 | 0,10 |
| 2015 | 482,8 | 559,0 | 452,0 | 0,16 |
| 2016 | 554,1 | 618,1 | 525,4 | 0,12 |
| 2017 | 630,5 | 751,3 | 598,3 | 0,19 |
| 2018* | 749,6 | 784,2 | 696,8 | 0,05 |
| 2019* | | 861,8 | 880,4 | |
| 2020* | | 928,7 | 1014,3 | |
| 2021* | | 988,9 | 1133,9 | |

## Conclusion

The aim of the study is to analyze the budget revenue targets in the light of past realizations and to estimate the values that they will receive in the medium-term period by using an econometric model. The results obtained from the model estimates are both information to policymakers and markets about the nominal values of the data in the medium term. This study is very important in terms of providing an analysis to decision makers based on reforms needed to be achieved in order to reach the 2019-2021 MTP budget revenue targets and fiscal policies that should be implemented. In their expectations and economic decisions, we think that the estimation

results obtained from the econometric model (without any political decision) created using only realization data are more effective in eliminating the current uncertainty.

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# CHAPTER IV

# Win Ratio Analysis in Clinical Trials

**Tuğçe ÖZNACAR**[1]
**Hatice Yağmur ZENGİN**[2]
**Deniz Sarp BEYAZPINAR**[3]
**Setenay ÖNER**[4]

## 1 Introduction

Life analysis allows the realization of different events that occur in t time periods and that are not the "death" event and that are also important in the study to be examined in determining the effectiveness of treatment. These different events identified in the study are called the endpoints of the study (Rauch, Schüler, & Kieser, 2017).

More than one endpoint can be identified in a study. This is because it is difficult to use a single outcome to characterize

[1] Ass. Prof., Ankara Medipol University, tugce.sencelikel@ankaramedipol.edu.tr
[2] Instructor Ph.D., Hacettepe University, yagmurzengin@hacettepe.edu.tr
[3] Ass. Prof., Baskent University, dsarpbeyazpinar@gmail.com
[4] Prof. Ph.D.., Eskişehir Osmangazi University, oners@ogu.edu.tr

treatment effects in clinical trials. The complexity of a disease cannot be adequately described by a single outcome. Therefore, in most studies, in order to find the most appropriate treatment, as much information as possible is collected by examining the area of interest in various aspects and multiple outcomes are tried to be considered (Wu & Cook, 2012).

The composite endpoint is frequently used in randomized clinical trials, primarily in the area of cardiovascular disease. An advantage of using a composite endpoint is that the event rate is higher than either of its components alone and can be used with a smaller sample size (Dong et al., 2018). In studies with such a composite endpoint, patients may experience two different types of events. In this case, a treatment may reduce the risk of both events or reduce the risk of one event while increasing the risk of the other event. At this point, it is very difficult for the physician to measure the treatment effect (Wu & Cook, 2012).

In the literature, commonly used methods for composite endpoint analysis are well-known statistical methods such as Kaplan-Meier and Cox Proportional Hazard. These methods are inadequate for analyzing the composite endpoint. All contributing endpoints are analyzed as if they are of equal importance, and only the first endpoint to occur matters. For example, after a patient has a non-fatal myocardial infarction, it is ignored whether he or she dies later. Therefore, non-fatal events that occur early are prioritized over later serious events or deaths. Furthermore, non-fatal events can occur more than once. However, this traditional approach to composite endpoints and analyses has been used in cardiovascular

trials despite critical comments (Pocock et al., 2012), (Oakes, 2016), (Abdalla et al., 2016).

Another example is in solid organ transplantation trials, where a common primary endpoint is a combination of treated biopsy-proven acute rejection (tBPAR), graft loss or death. A traditional primary endpoint analysis ignores the death of subjects who have already experienced tBPAR, but arguably death is the most clinically impactful event. Among the various methods proposed to solve this problem, which take into account the clinical significance of the different events to be considered together, the Win Ratio method stands out in particular for its simplicity and straightforward interpretation (Dong et al., 2018).

Pocock et al. proposed two different calculation methods for the Win Ratio, the matched-pairs approach and the unpaired approach. In the matched-pairs approach, patients in the two groups are matched according to their risk profile, for any pair, the time of the most important event (e.g., death) determines the "winner", if a winner cannot be determined (e.g., both patients survive), the second most important event (e.g., graft loss) is examined and so on until a winner is determined; otherwise, the pairs are tied. In the unpaired approach, the same principle is applied to all possible pairs of patients between the treatment group and the control group (every patient in the treatment group is compared to every patient in the control group), does not require a matching strategy and can therefore be more widely used in practice. Win Ratio is the ratio of winners to losers for the treatment group. Values above 1 indicate that treatment is more effective than control (Pocock et al., 2012).

The aim of this study is to evaluate the Win Ratio approach developed by Pocock et al. under different conditions and to guide researchers. For this purpose, simulation studies were carried out with scenarios created with different numbers of observations, different averages, different event/phenomenon occurrence rates, and different numbers of endpoints. After the simulation study, an application was made with a real data set and an example of the use of the Win Ratio approach in cardiology diseases was given (Pocock et al., 2012).

Pocock et al. exemplified a clinical trial comparing novel and standard therapy with a composite primary endpoint. The endpoints were cardiovascular death and hospitalization for advanced heart failure. In this example, cardiovascular death is considered more important than hospitalization. Therefore, when comparing any two patients on new and standard treatment, it determines whether one has cardiovascular death before the other. If this is unknown, it is determined which patient was hospitalized first. This is the essence of the new approach to analyzing composite endpoints. When comparing two such patients, it seems appropriate to consider the risks underlying the composite endpoint (Pocock et al., 2012).

The first method proposed by Pocock et al. is the matched pairs method. In this method, matched pairs of patients receiving new and standard treatment are created. The matching method takes individual patient risk into account.

After the match is made

(1) For each matched pair, the more important event (cardiovascular death) is examined.

(2) Is it known which patient had a shorter time to cardiovascular death?

If neither has a cardiovascular death, hospitalization is examined.

If one patient has a cardiovascular death, the other should be followed for longer to know for sure who had a cardiovascular death first.

If it is not known who had cardiovascular death first, the same principles are used to check whether it is known which patient was hospitalized first. So each matched pair fits into one of the five categories:

(a) in the new treatment group, the patient experienced cardiovascular death first;

(b) in the standard treatment group, the patient experienced cardiovascular death first;

(c) in the new treatment group, the patient experienced hospitalization first;

(d) in the standard treatment group, the patient experienced hospitalization first;

(e) none of the above.

It should be noted that categories (a) and (b) have priority over (c) and (d). Category (e) mostly includes pairs where neither the patient had a cardiovascular death nor hospitalization, but includes a few pairs where one had an event but the others had a shorter follow-up period (Pocock et al., 2012).

The composite endpoint results are shown in categories (a), (b), (c), (d) and (e) with the number of wins for the standard treatment and new treatment groups as a result of matching in categories Na, Nb, Nc, Nd and Ne.

$N_b + N_d = N_W$ shows the number of "winners" for the new treatment. It shows the results that the new treatment group achieved the endpoint later than the standard treatment group, and therefore the standard treatment was worse. Similarly $N_a + N_c = N_L$ gives the number of losers for the new treatment. It shows the results that the new treatment group met the endpoint earlier than the standard treatment group and therefore the standard treatment was better (Dong et al. 2016).

Win Ratio formula is defined as follows,

$$R_w = {}^{N_W}/_{N_L} \quad (1)$$

To calculate the confidence interval of the Win Ratio, first the win rate/probability of the new treatment should be calculated as below,

$$p_W = \frac{N_W}{N_W + N_L} \quad (2)$$

After obtaining the the win rate/probability of the new treatment, the confidence interval of the Win Ratio can be calculated as follows:

$$p_W \pm 1{,}96 \left[\frac{p_W(1-p_W)}{N_W+N_L}\right]^{1/2} = p_L, p_U \quad (3)$$

For significance test, z score is calculated using the Equation 4.

$$z = (p_W - 0{,}5)/[p_W(1 - p_W)/(N_W + N_L]^{1/2} \qquad (4)$$

Where z is the standardized value under the null hypothesis and gives the required p-value.

The second method mentioned in Pocock et al. is the unpaired pairs method. In this method, trials are conducted without any strategy for selecting matched pairs. Each patient receiving new treatment is compared with each patient receiving standard treatment and it is determined who "wins" each time. Let Ny and Ns be the numbers of patients receiving new and standard treatment (Pocock et al., 2012).

In this method, Ny x Ns pairs will be compared. As in the paired method, each pair is classified into one of the categories (a), (b), (c), (d), (e). Continuing with the same examples from the other method, for the two composite primary endpoints, cardiovascular and hospitalization $N_b + N_d = N_W$ ve $N_a + N_c = N_L$ shown as the number of "winners" and "losers" for the new treatment. Win Ratio formula $R_w = {}^{N_W}/_{N_L}$ calculated in the same way (Pocock et al., 2012).

Additionally, the distribution of the number of winners can be defines using $X_i$ (i=1, 2, …, $N_t$) information for the i.th patient in the treatment group (death, hospitalization, renal failure, etc.) and $Y_j$ (j=1, 2, …, $N_c$) get information for the j.th patient in the control group. $K_{ij}=K(X_i, Y_j)$ defined a general form of the Kernel function as (Dong et al., 2016).

$K_{ij}$ = 1 if patient i in the treatment group won against patient j in the control group;

$K_{ij}$ = 0 if patient i in the treatment group lost to patient j in the control group;

$K_{ij}$ = 0 if patient i in the treatment group is similar to patient j in the control group,

Number of the winners in the treatment groups can be calculated as follows:

$$n_t = \sum_{i=1}^{N_t} \sum_{j=1}^{N_c} K(X_i, Y_j) \quad (5)$$

## 2 Methods

The codes used in the study were created using R Studio v 1.1.463 and R v 3.5.3 package programs and R Programming Language. The survival (Therneau, 2020), gamlss, eha, dplyr packages in the R Program were used.

### 2.1 Simulation Study

Scenarios were created taking into account the number of groups, the number of composite endpoint components, the average time when the event occurred, the rate at which the event occurred and the number of observations.

In all scenarios, the time derivation was generated from the Weibull distribution using mean (d) (which varies across scenarios and groups) and standard deviation (σ=1). The event occurrence rate was generated from the Binomial distribution as high and low. The number of observations was equal across groups.

In the simulations, the number of outcomes was determined as 2, 3, 4 and the number of groups as two groups, standard treatment and new treatment. While creating the scenarios, the averages determined for the time until the event occurred in the groups; "the

new treatment group is superior to the standard treatment group", the average of the new treatment group is the same, and the averages of the standard treatment group are created by changing. The realization rate of the event was determined as high observation rate and low observation rate in the scenarios and was taken the same in the groups.

For each scenario, Win ratio, Win ratio confidence interval, p-value for Win ratio, Hazard ratio / Proportional Hazard assumption met, hazard ratio confidence interval results obtained using the semi-parametric Cox regression model are included.

In the simulation scenarios for the three groups, the outcome number was set as 2 and the number of groups was set as three groups: standard treatment 1, standard treatment 2 and new treatment. The scenarios were created to show only the superiority of the new treatment over the other treatments, with the averages determined for the time until the event occurred in the groups; "the new treatment group is superior to the standard treatment 1 group and the standard treatment 2 group". The rate of occurrence of the event was determined as high occurrence rate and low occurrence rate in the scenarios and taken the same in the groups.

In the findings, only the Win ratio value is included as a result of the scenario.

**2.2 Real Data Set Implementation**

The study was approved by the Non-Interventional Clinical Research Ethics Committee of the Eskisehir Osmangazi University [Date: 16.10.2020, Number: 16] and the real data set was obtained from Baskent University Hospital Cardiology Department.

While the number of groups was 2, the actual data set:

The results of 74 adult patients who underwent left ventricular assist device implantation for End-Stage Heart Failure at Ankara Başkent University Department of Cardiovascular Surgery were included in the study.

Patients were divided into two groups in terms of Pulmonary Vascular Resistance (PVR) values.

Group 1: PVR value above 2.5 (n=37)

Group 2: PVR value 2.5 and below (n=37)

As composite endpoints, data on the occurrence and time of nosebleeds, the occurrence and time of driveline infection, the occurrence and time of device thrombosis and finally the occurrence and time of death due to multiple organ failure were used.

Actual data set when the number of groups was 3:

The results of 69 adult patients who underwent left ventricular assist device implantation for End-Stage Heart Failure at Ankara Başkent University Department of Cardiovascular Surgery were included in the study.

Patients were divided into three groups in terms of Pulmonary Artery Pressure (PAP) value.

Group 1: Pulmonary Artery Pressure (PAP) value 40 and below (n=17)

Group 2: PABP value 41-55 (n=26)

Group 3: PABP value >55 (n=26)

Data on the occurrence and time of driveline infection and the occurrence and time of device thrombosis were used as composite endpoints.

In the calculation of the WR value, winners were calculated as those who were superior in both conditions to determine that the first group was more effective than both the second and third groups. In the calculation of the losers, each situation that was not won in both cases was taken into account and the numerator was proportioned to the denominator to obtain the WR. Win Ratio value is given in the results.

## 3 Results

### 3.1 Simulation Results

When the scenario prepared with the highest event occurrence rates and time averages of "new treatment group superior to the standard treatment group" was analyzed, Win Ratio values were found as expected. The new treatment method was found superior to the standard treatment for all four types of outcomes ($p<0.001$).

As the number of composite outcomes increased, win ratio and hazard ratio values decreased. As the observation rate of the event decreases and the number of composite outcomes increases, the hazard ratio loses its significance, while the win ratio value is found to be significant in the same situation. Both values increased as the number of observations increased. When the observation rate of the event was high and the number of observations was high, both values adapted to the scenario for each composite outcome number, but as the observation rate decreased, the win ratio value continued to adapt to the scenario, while the hazard ratio lost its value and the

p values became insignificant. Win ratio value, which is not affected by the number of observations, decreased as the number of composite outcomes increased and observation rates decreased (Table 1 and Table 2).

*Table 1. Win Ratio and Hazarda Ratio Results for 50 and 500 observations*

| n | | 50 | | 500 | |
|---|---|---|---|---|---|
| | | ER 1 | ER 2 | ER 1 | ER 2 |
| NCE 2 | Win ratio value (95% CI) | 1.95 (0.96-3.97) | 1.50 (0.79-2.84) | 1.82 (1.47-2.26) | 1.38 (1.14-1.68) |
| | Win ratio p value | **<0.001** | **<0.001** | **<0.001** | **<0.001** |
| | Hazard Ratio value (95% CI) | 2.51 (1.33-4.78) | 2.27 (1.19-4.32) | 2.3 (1.91-2.77) | 2.17 (1.79-2.63) |
| | Hazard Ratio p value | **0.033** | 0.075 | **<0.001** | **<0.001** |
| NCE 3 | Win ratio value (95% CI) | 1.74 (0.87-3.46) | 1.55 (0.79-3.05) | 1.69 (1.36-2.09) | 1.42 (1.15-1.75) |
| | Win ratio p value | **<0.001** | **<0.001** | **<0.001** | **<0.001** |
| | Hazard Ratio value (95% CI) | 3.02 (1.56-5.85) | 2.42 (1.31-4.49) | 2.85 (2.35-3.44) | 2.39 (1.98-2.89) |
| | Hazard Ratio p value | **0.027** | **0.040** | **<0.001** | **<0.001** |
| NCE 4 | Win ratio value (95% CI) | 1.54 (0.78-3.0) | 1.36 (0.70-2.62) | 1.41 (1.14-1.74) | 1.28 (1.04-1.57) |
| | Win ratio p value | **<0.001** | **0.002** | **<0.001** | **<0.001** |
| | Hazard Ratio value (95% CI) | 3.04 (1.55-5.98) | 2.18 (1.19-4.02) | 2.68 (2.22-3.25) | 2.28 (1.89-2.75) |
| | Hazard Ratio p value | **0.017** | 0.098 | **<0.001** | **<0.001** |

NCE: Number of the Composite Endpoint

ERR 1: High Event Rate

ERR 2: Low Event Rate

CI: Confidence Interval

*Table 2. Win Ratio and Hazarda Ratio Results with 100 and 250 observations*

| n | | 100 | | 250 | |
|---|---|---|---|---|---|
| | | ER 1 | ER 2 | ER 1 | ER 2 |
| NCE 2 | Win ratio value (95% CI) | 1.84 (1.14-2.97) | 1.50 (0.96-2.35) | 1.75 (1.3-2.37) | 1.43 (1.08-1.89) |
| | Win ratio p value | **<0.001** | **<0.001** | **<0.001** | **<0.001** |
| | Hazard Ratio value (95% CI) | 2.45 (1.59-3.78) | 2.32 (1.48-3.62) | 2.27 (1.74-2.95) | 2.14 (1.63-2.82) |
| | Hazard Ratio p value | **<0.001** | **0.009** | **<0.001** | **<0.001** |
| NCE 3 | Win ratio value (95% CI) | 1.76 (1.08-2.88) | 1.44 (0.90-2.30) | 1.71 (1.26-2.33) | 1.45 (1.07-1.95) |
| | Win ratio p value | **<0.001** | **<0.001** | **<0.001** | **<0.001** |
| | Hazard Ratio value (95% CI) | 2.71 (1.76-4.18) | 2.45 (1.59-3.75) | 2.86 (2.18-3.75) | 2.37 (1.81-3.10) |
| | Hazard Ratio p value | **<0.001** | **0.007** | **<0.001** | **<0.001** |
| NCE 4 | Win ratio value (95% CI) | 1.46 (0.91-2.33) | 1.31 (0.83-2.08) | 1.42 (1.06-1.91) | 1.28 (0.96-1.72) |
| | Win ratio p value | **<0.001** | **<0.001** | **<0.001** | **<0.001** |
| | Hazard Ratio value (95% CI) | 2.78 (1.79-4.33) | 2.30 (1.51-3.50) | 2.74 (2.09-3.59) | 2.67 (1.74-2.95) |
| | Hazard Ratio p value | **<0.001** | **0.015** | **<0.001** | **<0.001** |

NCE: Number of the Composite Endpoint

ER 1: High Event Rate

ER 2: Low Event Rate

CI: Confidence Interval

When the results of the three groups of artificial data sets were analyzed, the newly recommended treatment group was found

to be more effective than the other two standard treatment groups for each number of observations when the case observation rate was the highest (WR>1.00). In the case of the lowest case observation rate, the newly proposed treatment group was not effective for each number of observations compared to the other two standard treatment groups (WR < 1.00).

### 3.2 Results Of The Real Data Sets

Results for two groups of three composite outcome counts:

Outcome one (Least Significant): Driveline Infection, second outcome: Occurrence of Thrombosis, third Outcome (Most Important): Death Due to Organ Failure

The number of winners was determined according to the comparison results made over the data set.

Patients with heart failure with a PVR value above 2.5 were found to be more effective in terms of treatment (WR=1.35 p<0.001) than patients with heart failure with a PVR value of 2.5 and below, while according to the hazard ratio results, the second group was found to be 1.27 times more risky than the first group and this value was not statistically significant (p=0.394).

Results for two groups of four composite outcome counts:

Outcome one (Least Important): Nosebleed Occurrence, second outcome: Driveline Infection, third outcome: Thrombosis, the fourth and most important outcome: Death due to organ failure; patients with heart failure with a PVR value above 2.5 were found to be more effective in terms of treatment than patients with heart failure with a PVR value of 2.5 and below (WR=1.42 p<0.001), while according to hazard ratio results, the second group was found

to be 1.41 times more risky than the first group and this value was not statistically significant (p=0.189).

According to the results of the data analysis prepared in accordance with the three groups of two composite outcome numbers;

First Outcome (Least Important): Driveline Infection

Secondary Outcome (Most Important): Thrombosis Detection

The number of winners was determined according to the comparison results over the data set.

WR=0.25 was obtained. The group with low PAB was not superior to the group with medium and high PAB (p>0.05).

**4 Discussion**

Having more than one endpoint in clinical trials is important to determine treatment effects. For this reason, in most studies, more than one outcome is considered to examine the area of interest from various aspects to find the most appropriate treatment.

Although it is desirable to increase the number of endpoints, it has long been known that there are some problems that have not been resolved in the analysis of data. One of these problems is that the importance of the endpoints is not taken into account. Standard analyses for composite endpoints include the first occurrence and the time of occurrence without considering the order of importance. For this reason, in 2012, Pocock et al. proposed an approach that takes into account the order of importance in composite endpoint analysis called Win ratio (Pocock et al., 2012).

When the simulation results are analyzed, it is seen that the win ratio results are not affected by the number of observations, but they are highly affected by the number of endpoints. Win ratio values, which were higher when the number of composite endpoints was 2, decreased as the number of composite endpoints increased.

Dong et al. took three composite endpoints in his study and derived the time of occurrence of the event by giving a fixed number to a number they created while creating the artificial data set (Dong et al., 2016). In this study, the time of occurrence was derived from the Weibull distribution with a standard deviation of 1 and the mean varied according to the scenarios. The event times were derived from the binomial distribution with different probabilities. In addition, not only 3 composite outcome numbers but also two and four composite outcome numbers with different observation rates were added to the study design.

The z value calculated for the win ratio was calculated using the calculation formula given in the study by Pocock et al. The confidence interval was calculated using the Kernel function calculation codes given in the study by Dong et al (Dong et al., 2020).

Mao proposed an alternative hypothesis to the win ratio approach by extending the work of Luo et al. on confidence intervals (Mao, 2019). Luo et al. constructed weighted win ratios targeting ordered hazard alternatives. In the study, it is referred to as weighted win and loss statistics. In their study, they stated that weighted win ratios with weight functions are based on censoring distributions. They suggested that some appropriate weights could be used to get rid of the censoring distributions, so that the win ratio would depend

only on the hazard functions (Luo et al., 2017). Wang et al. pointed out the ease of interpretation of the win ratio approach and developed a more efficient estimator to estimate treatment effects after correcting for the confounding effects of other variables through a simulation study (Wang et al., 2017).

Fergusson et al. described the composite outcome and Win ratio approach in a renal transplantation study and shared both win ratio and hazard ratio results by taking two groups as treatment and control groups and three composite outcomes on the transplantation dataset (Fergusson et al., 2018). In this study, 74 (n1=n2=37) adult patients who received left ventricular assist devices for Heart Failure were included and divided into two groups as above 2.5 and 2.5 and below according to Pulmonary Vascular Resistance (PVR) values. In general studies, the Win ratio approach is used to determine the efficacy of treatment between treatment and control groups, whereas in this study, it was used to determine which of the groups divided according to PVR values the same treatment was effective. In addition, the data set was prepared as three composite endpoints and four composite endpoints and Win ratio and Hazard ratio results are given. Similar to the simulation results, win ratio values were found to be lower in the analysis of four composite endpoints. In the results with three composite endpoints and four composite endpoints, the win ratio value was found to be lower and statistically significant, while the hazard ratio results were not significant and almost close to 1.

There are very few studies with three groups in composite outcome analysis. Bebu et al. studied the three-group theoretical study for high numbers of observations. The theoretical calculations

given by Bebu et al. are used in the variance-covariance matrix calculations in this study (Bebu & Lachin, 2016). In this study, the number of observations was 750 (n1=n2=n3=250) and 375 (n1=n2=n3=125). The composite endpoint analyses with three groups consisted of standard treatment 1, standard treatment 2 and newly proposed treatment groups. Although it is a rare case, in this study, a simulation with three groups and two composite endpoints was included and win ratio values were calculated. In the win ratio calculation, scenarios were created by planning the newly recommended treatment to be more effective than both standard treatment 1 and standard treatment 2. However, this was only observed in the simulation with the highest event occurrence rate. In addition, when the averages given for the realization times of the phenomenon used in the two-group simulations are compared with the averages given in the three groups, the averages given for the three groups are kept quite high for the proposed treatment. Very low win ratio values were obtained when working with the averages given for two groups. Despite the high averages, win ratio values were similar to the two standard treatments as the observation rates of the case decreased. The highest number of observations in the study was 750 (n1=n2=n3=250). In Bebu et al.'s study, on the other hand, they worked with a much higher number of observations (N=7500) (Bebu & Lachin, 2016). This problem can be thought to be due to the number of observations. Here, the three groups should not be considered only as three different treatments. The same treatment can also be used to determine which of the three different patient groups is more effective. Therefore, more studies are needed for endpoint analyses involving three groups.

## 5 Conclusions

Win Ratio analysis, which takes into account the order of importance instead of traditional analysis in compound endpoint analysis, which is of great importance in clinical trials, was evaluated under different conditions. From the simulation results obtained with the artificial data set, it was observed that the win ratio value was not affected by whether the number of observations was small or large, but was affected by the observation rates and the number of composite endpoints. Win ratio values decreased as the number of composite endpoints increased. Similarly, as the realization rates of the phenomenon decreased, win ratio values decreased. When the number of observations is 50 and 100, the confidence interval calculated with the Kernel function based on the study of Dong et al. includes 1 even though the win ratio is significant.[5]

In the results obtained from patients from Cardiovascular Surgery, Win ratio was found to be significant in three and four composite endpoint analyses involving two groups, while Hazard model results were not significant.

In conclusion, it was found that using the Win Ratio approach instead of traditional analyses such as Hazard Ratio estimation in clinical studies with more than one endpoint provides better point of view to the treatment effect.

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# CHAPTER V

# Bayesian Estimation Of Inverse Pareto Distribution Under Different Loss Functions

**Asuman YILMAZ**[1]

## Introduction

Numerous real-world scenarios need data for a probability distribution function with an upside-down bathtub or decreasing failure rate function. For instance, during the first few days following a heart transplant, when the body adapts to the new organ, patients are at an increased risk of dying. As the patient grows healthier, the hazard rate gradually drops (Collett, 2015). In such a situation, both declining and upside-down bathtub-shaped failure rate functions can be appropriate. Depending on the true value of the parameter, the one-parameter inverse Pareto distribution (IPD) has both decreasing and upside-down bathtub-shaped failure rate functions. The failure rate and the cumulative distribution function are crucial in survival analysis. They are also nicely expressed in closed form. Recently, (Gua & Gui, 2018) studied Bayesian and classical estimation of the stress-strength reliability for IPD using complete sample data.

[1] Van Yüzüncü Yıl University, Faculty of Economics and Administrative Sciences, Department of Econometrics, 65080 Van, TURKEY Orcid: 0000-0002-8653-6900E-mail: asumanduva@yyu.edu.tr

(Kumar &Kumar, 2020) obtained MLE and Bayesian analysis of the parameters IPD using randomly censored data. The application of IPD in extreme events is discussed by (Dankunprasert, Jaroengeratikun &Talangtam, 2021). (Kumar & Kumar, 2022) investigated Bayesian and classical estimation of stress-strength reliability for IPD using progressively censored data. (Mustafa, Ijaz & Jamal, 2022) studied the distribution of order statistics of the IPD. Different estimation techniques for the unknown parameter and reliability function of the inverse Pareto distribution from both classical and Bayesian approximation have been examined by (Kumar, Jha & Kumar, 2023). (Alharbi et al., 2023) deals with the estimation of the stress-strength reliability model for $P(X < Y)$ from both classical and Bayesian methods.

One of the main benefits of Bayesian models is that they allow us to use prior knowledge to analyze unknown parameters and thus generate stronger inferences. Furthermore, Bayesian models outperform classical models, especially for small samples. Therefore, the parameter of the inverse Pareto distribution will be examined under the square error loss function (SELF), linear exponential (LINEX) loss function, and general entropy (GELF) based on Lindley and MCMC approximation methods, in this study. The study is designed as follows. In section 2, the inverse Pareto distribution is briefly mentioned. In Section 3, MLE estimators are found. Also, an asymptotic confidence interval is derived. In Section 4, SELF, LINEX, and GELF are summarized. Also, Lindley and MCMC approximations for Bayesian computations are briefly mentioned. In Section 5, an effective simulation study is conducted to compare the performance of all proposed estimators. Finally, the significant results of the study are reported.

**Inverse Pareto Distribution**

The inverse Pareto distribution can be used quite effectively in analyzing the upside-down bathtub shape hazard rate data. The shorthand $X: IPD(\alpha)$ is used to indicate that the random variable X has the inverse Pareto distribution with parameter $\alpha > 0$. The

probability density function (pdf) the cumulative density function (cdf), and the failure rate function are given by

$$f(x)=\frac{\alpha x^{\alpha-1}}{(1+x)^{\alpha}} \qquad x\geq 0 \tag{2.1}$$

$$F(x)=\left(\frac{x}{1+x}\right)^{\alpha} \qquad x\geq 0 \tag{2.2}$$

and

$$h(x)=\frac{f(x)}{1-F(x)}=\frac{\alpha x^{\alpha-1}}{(1+x)^{\alpha}\left(1-\left(\frac{x}{1+x}\right)^{\alpha}\right)} \qquad x\geq 0 \tag{2.3}$$

The monotonicity of hazard function of IPD$(\alpha)$ for different values of the parameter $\alpha$, is shown in Figure 1.

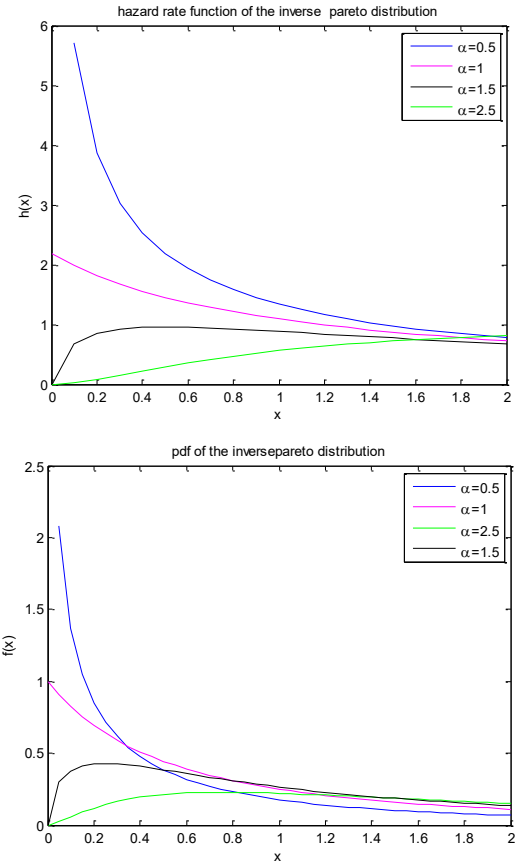


*Figure 1.Hazard rate function and pdf of inverse Pareto distribution for different values of $\alpha$, respectively.*

## Maximum Likelihood Estimation

In this section we will discuss the maximum likelihood estimation of the $\alpha$ parameter of the inverse Pareto distribution.

Let $\{X_1, X_2, ..., X_n\}$ be a random sample from $IPD(\alpha)$. Then, the likelihood function for $\{X_1, X_2, ..., X_n\}$ is as follow:

$$L(x|\alpha) = \alpha^n \prod_{i=1}^{n} \frac{x_i^{\alpha-1}}{(1+x_i)^{\alpha+1}}. \qquad (3.1)$$

Following that, the log-likelihood function is

$$lnL\left(x|\alpha\right)=n\ln\alpha+\left(\alpha-1\right)\sum_{i=1}^{n}\ln\left(x_i\right)-\left(\alpha+1\right)\sum_{i=1}^{n}\ln\left(1+x_i\right)=0 \quad (3.2)$$

The estimating equations are obtained by differentiating the log-likelihood function and setting it to zero.

$$\frac{\partial\ln L}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\ln\left(x_i\right)-\sum_{i=1}^{n}\ln\left(1+x_i\right)=0 \quad (3.3)$$

Therefore, from the solution of Equation (3.3)

$$\hat{\alpha}=\frac{n}{\sum_{i=1}^{n}\ln\left(1+\frac{1}{x_i}\right)} \quad (3.4)$$

Let us define the MLE of $\alpha$ as $\hat{\alpha}$ .

Now, we obtain the asymptotic confidence interval of the $\alpha$ parameter based on the observed Fisher information matrix. The observed Fisher information matrix of the $\hat{\alpha}$ parameter is obtained as follows:

$$I\left(\alpha\right)=-E\left(\frac{\partial^2 lnL\left(x|\alpha\right)}{\partial\alpha^2}\right)=\frac{n}{\alpha^2} \quad (3.5)$$

Thus, the observed variance become $I^{-1}\left(\alpha\right)=\frac{\alpha^2}{n}$ .

## 4.Bayesian Inference

In this section, the parameter $\alpha$ of the inverse Pareto distribution will be estimated by Bayesian methods. In the Bayesian method, the loss function and the prior distribution are very important. Firstly, the loss functions considered in this study will be discussed.

One of the popular loss functions is SELF and it is given below:

$$L\left(\hat{\theta},\theta\right)=\left(\hat{\theta}-\theta\right)^2 \quad (4.1)$$

where $\hat{\theta}$ is the estimator of the parameter $\theta$ . The posterior mean is the Bayesian estimate of the parameter θ under SELF, which is given as:

$$\hat{\theta}_{SELF} = E\left(\theta|x\right) = \int_{\theta} \theta\pi\left(\theta|x\right)d\theta \tag{4.2}$$

SELF is symmetric. That is, it assigns equal weight to overestimation and underestimation. However, the true loss is not symmetric with respect to overestimation and underestimation, in most situations. Therefore, asymmetrical loss functions can be used as an alternative to symmetric loss functions. (Varian, 1975) introduced the LINEX loss function (Linear-Exponential) and since then it has been widely used by many researchers such as (Zellner ,1986; Soliman et al., 2012). The properties of the LINEX loss function were studied by (Zelner, 1986). This function is

$$\hat{L}_{LINEX} = k\left\{e^{k\left(\hat{\theta}-\theta\right)} - k\left(\hat{\theta}-\theta\right) - 1\right\}; \quad k \neq 0. \tag{4.3}$$

Under this loss function, the Bayes estimate is given by the following equation:

$$\hat{\theta}_{LINEX} = -\frac{1}{k}\ln\left(E\left(e^{-k\theta}|x\right)\right) = \frac{-1}{k}\log\int_{\theta} e^{-k\theta}\pi\left(\theta|x\right)d\theta \tag{4.4}$$

(Calabria & Pulcini, 1996), also proposed another asymmetric loss function named as a GELF is given as follows:

$$\hat{L}_{GELF} = k_1\left\{\left(\frac{\hat{\theta}}{\theta}\right)^{k_1} - k_1\ln\left(\frac{\hat{\theta}}{\theta}\right) - 1\right\}; \quad k_1 \neq 0. \tag{4.5}$$

Under this loss function, the Bayes estimate is given by the following equation:

$$\hat{\theta}_{GELF} = E_{\theta}\left(\theta^{-k_1}|x\right)^{-1/k_1} = \left(\int_0^{\infty} \theta^{-k_1}\pi\left(\theta|x\right)d\theta\right)^{-1/k_1} \tag{4.6}$$

The sign and magnitude of $k$ in LINEX and GELF express the direction and degree of symmetry. For more details about loss functions, see (Renjini et al., 2016; Yılmaz, Kara& Aydoğdu, 2020). Since the Gamma prior distribution has a flexible structure, it will be accepted as the prior distribution in this study. Independent gamma priors for different distributions have been used by numerous researchers, see (Danish &Aslam, 2016; Kundu, 2013).

Thus, the pdf of the independent gamma prior for parameter $\alpha$ is given as:

$$\pi_{\alpha}(\alpha)=\frac{a_2^{a_1}}{\Gamma(a_1)}\alpha^{a_1-1}e^{-a_2\alpha},\quad a_1,a_2>0;\ \ \alpha>0 \tag{4.7}$$

where $a_1$ and $a_2$ are hyper parameters and they are known.

Thus, the posterior distribution is obtained as follows:

$$\pi(\alpha|x)=\frac{L(x|\alpha)\pi_{\alpha}(\alpha)}{\int_0^{\infty}L(x|\alpha)\pi_{\alpha}(\alpha)d\alpha}\propto\alpha^{a_1-1+n}e^{-a_2\alpha}\prod_{i=1}^{n}\frac{x_i^{\alpha-1}}{(1+x_i)^{\alpha+1}} \tag{4.8}$$

From Equation (4.8),

$$\pi(\alpha|x):\ \ Gamma\left(n+a_1,a_2-\sum_{i=1}^{n}\ln\left(\frac{x_i}{1+x_i}\right)\right).$$

Although Equation (4.8) under SELF can be calculated analytically, it is tedious to calculate these integrals one by one for Equations (4.4) and (4.6). Therefore, we will use MCMC and Lindley approximation methods for Bayesian computations.

### 4.1 Lindley approximation

(Lindley, 1980) developed an approximation procedure for the evaluating the ratio of the two integrals. According to this procedure, the posterior mean of arbitrary $u(\alpha)$ is given by:

$$\hat{u}=E(u(\alpha)|x)=\frac{\int_0^{\infty}u(a)\pi(\alpha|x)L(\alpha|x)d\alpha}{\int_0^{\infty}\pi(\alpha|x)L(\alpha|x)d\alpha}. \tag{4.9}$$

Here, $u(\alpha)$ denotes a function of $\alpha$ only, $\pi(\alpha)$ denotes the joint prior density function, $L(\alpha|x)$ and represents the likelihood function.

In our case, the following equation can be obtained.

$$\hat{u}(\alpha)=\left[u(\alpha)+\frac{1}{2}(u_{11}+2u_1\rho_1)\sigma_{11}+\frac{1}{2}\left(L_{111}\sigma_{11}^2u_1\right)\right]_{\hat{\alpha}} \tag{4.10}$$

where $\hat{\alpha}$ is the MLEs of $\alpha$. Here, $u_1=\frac{\partial u(\alpha)}{\partial\alpha}, u_{11}=\frac{\partial^2 u(\alpha)}{\partial\alpha^2}$,

$\rho_1=\frac{\partial \ln\pi(\alpha)}{\partial\alpha}=\frac{a_1-1}{\alpha}-a_2$, $L_{111}=\frac{\partial^3 \ln L}{\partial\alpha^3}=\frac{2n}{\alpha^3}$ and

$\sigma_{11}=\left[-E\left(\frac{\partial^2 \ln L}{\alpha^2}\right)\right]^{-1}=\frac{\alpha^2}{n}$.

From Equation (4.10), the Bayes estimator of $\alpha$ under SELF using the Lindley method is obtained as follows:

If $u(\alpha)=\alpha,\quad u_1=1, u_{11}=0,$

$$\hat{\alpha}_{SELF}=\hat{\alpha}\left(1+\frac{a_1-a_2\hat{\alpha}}{n}\right)$$

Similarly, the Bayes estimators of α under LINEX and GELF using the Lindley method are given as follows:

If $u(\alpha)=e^{-k\alpha}\ \ u_1=\frac{\partial u}{\partial\alpha}=-ke^{-k\alpha}, u_{11}=\frac{\partial^2 u}{\partial\alpha^2}=k^2e^{-k\alpha}$

$$\hat{\alpha}_{LINEX}=\frac{-1}{k}\log\left[e^{-k\hat{\alpha}}+\frac{\hat{\alpha}ke^{-k\hat{\alpha}}}{n}\left(\frac{\hat{\alpha}k}{2}-a_1+a_2\hat{\alpha}\right)\right]$$

If $u(\alpha)=\alpha^{-k}{}_1=\frac{\partial u(\alpha)}{\partial\alpha}=-k\alpha^{-(k+1)}\ \ u_{11}=\frac{\partial^2 u(\alpha)}{\partial\alpha^2}=k(k+1)\alpha^{-(k+2)}$

$$\hat{\alpha}_{GELF}=\left[\hat{\alpha}^{-k}\left(\frac{k(k+1)}{2n}+\frac{k}{n}(-a_1+a_2\hat{\alpha})+1\right)\right]^{\frac{-1}{k}},$$

respectively.

## 4.2 Markov Chain Monte Carlo Method

In this section we will use Gibbs sampling, a subclass of the MCMC method, to obtain the Bayesian estimator of the parameter $\alpha$, see (Gelfand & Smith, 1991).

The Gibbs algorithm consists of the following steps:

**Step 1:** Start with j=1.

**Step 2:** Generate $\alpha^{(j)}$ from Gamma $\left(n+a_1, a_2-\sum_{i=1}^{n}\ln\left(\frac{x_i}{1+x_i}\right)\right)$.

**Step 3:**Set $j=j+1$ .

**Step 4:** Repeat Steps 2-3, N times.

Then, the Bayesian estimators under the loss functions mentioned above are obtained as given below:

$\hat{\alpha}_{SELF}=\frac{1}{N}\sum_{j=1}^{N}\alpha_j$ , $\hat{\alpha}_{LINEX}=\frac{-1}{k}\ln\left(\frac{1}{N}\sum_{j=1}^{N}e^{-k\alpha_j}\right)$, and $\hat{\alpha}_{GELF}=\left(\frac{1}{N}\sum_{j=1}^{N}\alpha_j^{-k}\right)^{\frac{-1}{k}}$, respectively.

## 1. Simulation Study

In this Section, an effective simulation study was conducted to compare the performances of the proposed estimators. These estimators were compared in terms of bias and MSE values. The simulation study has been conducted for different sample sizes $(n=30,50,100)$ and different parameter values $(\alpha=0.5,2)$ . Bayesian estimators have been calculated with Lindley and MCMC methods under the gamma prior distribution and SELF, LINEX, and GELF loss functions. Also, for LINEX and GELF loss functions $k=0.8$ was taken. Two different cases have been considered for hyper parameters in the gamma prior distribution. In the first case $a=b=0$ , has been taken and called Prior-I. In the second case, the prior mean is chosen to be the true values of the parameters. In other words, $\alpha=a/b$ is taken and called Prior- II. All calculations were done in Matlab R 2013. The results are presented in Table 1. The Bias and MSE values are found by using the following formula:

$Bias(\alpha)=\frac{1}{N}\sum_{i=1}^{n}(\hat{\alpha}_i-\alpha)$ and $MSE(\alpha)=\frac{1}{N}\sum_{i=1}^{n}(\hat{\alpha}_i-\alpha)^2$ , respectively.

*Table 1. The Bias and MSE values for the classical different parameter estimators of $\alpha$ .*

| | | | | Prior-I | | Prior-II | |
|---|---|---|---|---|---|---|---|
| n | $\alpha$ | | Estimator | Bias | MSE | Bias | MSE |
| 30 | 0.5 | LİNDLEY | MLE | 0.0166 | 0.0098 | 0.0174 | 0.0096 |
| | | | SELF | 0.0134 | 0.0070 | 0.0160 | 0.0074 |
| | | | LINEX | 0.0101 | 0.0069 | 0.0131 | 0.0072 |
| | | | GELF | 0.0096 | 0.0056 | 0.0139 | 0.0070 |
| | | MCMC | SELF | 0.0150 | 0.0079 | 0.0153 | 0.0067 |
| | | | LINEX | 0.0124 | 0.0073 | 0.0133 | 0.0064 |
| | | | GELF | 0.0097 | 0.0068 | 0.0116 | 0.0062 |
| 50 | 0.5 | LINDLEY | MLE | 0.0065 | 0.0053 | 0.0116 | 0.0056 |
| | | | SELF | 0.0090 | 0.0036 | 0.0110 | 0.0050 |
| | | | LINEX | 0.0084 | 0.0030 | 0.0095 | 0.0048 |
| | | | GELF | 0.0086 | 0.0029 | 0.0090 | 0.0044 |
| | | MCMC | SELF | 0.0075 | 0.0032 | 0.0096 | 0.0052 |
| | | | LINEX | 0.0064 | 0.0031 | 0.0088 | 0.0049 |
| | | | GELF | -0.0025 | 0.0030 | 0.0079 | 0.0046 |
| 100 | 0.5 | LINDLEY | MLE | 0.0053 | 0.0027 | 0.0054 | 0.0027 |
| | | | SELF | 0.0039 | 0.0033 | 0.0052 | 0.0020 |
| | | | LINEX | 0.0033 | 0.0030 | 0.0048 | 0.0018 |
| | | | GELF | 0.0028 | 0.0031 | 0.0047 | 0.0015 |
| | | MCMC | SELF | 0.0022 | 0.0027 | 0.0050 | 0.0022 |
| | | | LINEX | 0.0012 | 0.0026 | 0.0043 | 0.0024 |
| | | | GELF | -0.0013 | 0.0024 | 0.0044 | 0.0016 |
| 30 | 2 | LINDLEY | MLE | 0.0710 | 0.1513 | 0.0844 | 0.1755 |
| | | | SELF | 0.0698 | 0.1371 | 0.0802 | 0.1584 |
| | | | LINEX | 0.0680 | 0.1242 | 0.0804 | 0.1351 |
| | | | GELF | 0.0648 | 0.1206 | 0.0793 | 0.1311 |
| | | MCMC | SELF | 0.0697 | 0.1310 | 0.0786 | 0.1508 |
| | | | LINEX | 0.0641 | 0.1204 | 0.7446 | 0.1309 |
| | | | GELF | 0.0679 | 0.1246 | 0.0755 | 0.1375 |
| 50 | 2 | LINDLEY | MLE | 0.0548 | 0.0885 | 0.0351 | 0.0894 |
| | | | SELF | 0.0433 | 0.0733 | 0.0416 | 0.0629 |
| | | | LINEX | 0.0382 | 0.0706 | 0.0376 | 0.0665 |
| | | | GELF | 0.0243 | 0.0709 | 0.0354 | 0.0611 |
| | | MCMC | SELF | 0.0437 | 0.0685 | 0.0391 | 0.0617 |
| | | | LINEX | 0.0327 | 0.0636 | 0.0304 | 0.0657 |
| | | | GELF | 0.0334 | 0.0634 | 0.0331 | 0.0599 |
| 100 | 2 | LINDLEY | MLE | 0.0150 | 0.0431 | 0.0144 | 0.0384 |
| | | | SELF | 0.0132 | 0.0392 | 0.0157 | 0.0365 |
| | | | LINEX | 0.0128 | 0.0349 | 0.0120 | 0.0320 |
| | | | GELF | 0.0120 | 0.0340 | 0.0123 | 0.0350 |
| | | MCMC | SELF | 0.0130 | 0.0390 | 0.0139 | 0.0367 |
| | | | LINEX | 0.0125 | 0.0314 | 0.0117 | 0.0354 |
| | | | GELF | 0.0122 | 0.0320 | 0.0114 | 0.0359 |

The simulation results can be summarized as follows:

- The bias and MSE values of all proposed estimators decrease as the sample size increases.
- Bayesian methods have smaller Bias and MSE values than MLE.
- The Gibbs sampling method has slightly better performance than Lindley approximation method in many cases.
- GELF and LINEX generally outperforms the SELF when compared with each other.
- Similarly, when comparing Prior-I with Prior-II, Prior-II performs better in many cases.

**Discussion and Results**

In this study, the parameter of the inverse Paretto distribution was estimated by MLE and Bayesian methods. SELF, LINEX, and GELF were considered for Bayesian parameter estimation. The gamma distribution was considered as a prior distribution due to its flexible structure. Also, two different prior distributions were examined according to the values given to the hyper parameters. (Prior-I and Prior-II). Lindley and MCMC approximation methods were used for Bayesian calculations. A simulation study was conducted to compare the performances of all the estimation methods considered in the study. It was seen from the simulation results that in many cases Bayesian methods have smaller bias and MSE values than MLE. Also, when Bayesian methods were compared, the MCMC method performed slightly better under prior II based on asymmetric loss functions (LINEX and GELF).

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