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Preface

In mathematics, the pursuit of understanding and innovation has always been driven by the desire to connect theoretical foundations with real-world applications. This book is a testament to that aspiration, bringing together three distinct yet complementary areas of research that span abstract algebra, applied mathematics, and differential geometry.

The first chapter, "*k-Fibonacci and k-Lucas 3-Parameter Generalized Quaternions*", explores a novel generalization of quaternions through the lens of k-Fibonacci and k-Lucas sequences. These mathematical structures offer new insights into algebraic systems and their extensions, contributing to the broader field of number theory and algebraic computation.

The second chapter, "*A New Mathematical Model for Monkeypox with Vaccination Effect and Its Quantified Analysis*", bridges mathematics and epidemiology. It introduces a mathematical model designed to analyze the dynamics of monkeypox, incorporating the effects of vaccination. By quantifying key epidemiological parameters, this study provides a valuable framework for understanding and mitigating the spread of infectious diseases.

The third chapter, "*On the Geometry of Pseudo-Slant Submanifolds in Bronze Riemannian Manifolds*", delves into the geometric properties of pseudo-slant sub manifolds within the context of bronze Riemannian manifolds. This investigation contributes to the rich tapestry of differential geometry, offering new perspectives on manifold structures and their applications.

The common thread that binds these chapters is the innovative use of mathematical tools to address complex problems, whether in pure or applied settings. Each chapter represents the culmination of rigorous research and a commitment to advancing the frontiers of mathematical knowledge.

This book is intended for mathematicians, researchers, and students who are passionate about exploring diverse mathematical disciplines. It is our hope that these studies will inspire further investigation and foster interdisciplinary collaboration.

I would like to express my gratitude to my colleagues and collaborators whose insights and contributions have been invaluable throughout this journey. I also extend my thanks to the readers, whose curiosity and engagement drive the continual evolution of mathematical thought.

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December 2024

Content

CHAPTER I

k-Fibonacci and k-Lucas 3-Parameter Generalized Quaternions

Göksal BİLGİCİ[1]

Introduction

Fibonacci and Lucas numbers, are very popular sequences among integer sequences. Fibonacci numbers and Lucas numbers are defined by the recurrence relation

$$S_r = S_{r-1} + S_{r-2}.$$

The only diffrence is initial conditions. The initial conditions of Fibonacci numbers are $F_0 = 0$ and $F_1 = 1$ whereas the initial conditions of Lucas numbers are $L_0 = 2$ and $L_1 = 1$. Binet formulas for the Fibonacci and Lucas numbers are

$$F_r = \frac{\alpha^r - \beta^r}{\alpha - \beta} \text{ and } L_r = \alpha^r + \beta^r$$

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respectively, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are roots of the characteristic equation $t^2 - t - 1 = 0$. The positive root α is known as golden ratio. The generating functions for the sequences $\{F_r\}$ and $\{L_r\}$ are

$$\sum_{r=0}^{\infty} F_r t^r = \frac{t}{1-t-t^2} \text{ and } \sum_{r=0}^{\infty} L_r t^r = \frac{2-t}{1-t-t^2}$$

respectively. Detailed information about Fibonacci and Lucas sequences and their applications can be found in /Koshy, 2019).

There are a lot of generalizations of Fibonacci and Lucas sequences. One of them is given by Falcon and Plaza (2007). They defind k-Fibonacci sequences by the recurrence relation

$$F_{k,0} = 0, F_{k,1} = 1 \text{ and } F_{k,n} = kF_{k,n-1} + F_{k,n-2} \text{ (for } n \geq 2)$$

where any integer $k \geq 1$. Following this definition Falcon (2011) introduced k-Lucas numbers by the relation

$$L_{k,0} = 2, L_{k,1} = k \text{ and } L_{k,n} = kL_{k,n-1} + L_{k,n-2} \text{ (for } n \geq 2).$$

The generating functions for the sequences $\{F_{k,r}\}$ and $\{L_{k,r}\}$ are

$$\sum_{r=0}^{\infty} F_{k,r} t^r = \frac{t}{1-kt-t^2} \text{ and } \sum_{r=0}^{\infty} L_{k,r} t^r = \frac{2-t}{1-kt-t^2}$$

respectively. Binet formulas for these numbers are

$$F_{k,r} = \frac{\alpha_k^r - \beta_k^r}{\alpha_k - \beta_k} \text{ and } L_{k,r} = \alpha_k^r + \beta_k^r$$

where $\alpha_k = \frac{k+\sqrt{k^2+4}}{2}$ and $\beta_k = \frac{k-\sqrt{k^2+4}}{2}$ are roots of the characteristic equation $t^2 - kt - 1 = 0$.

Actaually, the roots of the equation $t^2 - kt - 1 = 0$ are called metallic ratios. For example, α_1 gives the golden ratio whereas α_2 gives the silver ratio. Therefore, k-Fibonacci numbers are also a generalization of Pell-numbers which are famous as Fibonacci numbers.

Quaternions were invented by Sir William Rowan Hamilton in 1853. Some generalizations of quaternions have been given so far. Recently, Senturk and Unal (2022) introduced one of them, i.e. 3-parameter generalized quaternions. A quaternion q is shown as $q = q + q_1 i + q_2 j + q_3 k$ where q_0, q_1, q_2 and q_3 are reals and the versors satisfy the following rules

Tablo 1: Multiplication rules of versors ($\sigma_1, \sigma_2, \sigma_3$ are arbitrary reals)

	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	$-\sigma_1\sigma_2$	$\sigma_1 e_3$	$-\sigma_2 e_2$
e_2	e_2	$-\sigma_1 e_3$	$-\sigma_1\sigma_3$	$\sigma_3 e_1$
e_3	e_3	$\sigma_2 e_2$	$-\sigma_3 e_1$	$-\sigma_2\sigma_3$

Fibonacci quternions were introduced by Horadam (1963). After his definition, some authors studied Fibonacci and Lucas quaternions (Iyer, 1969; Swamy, 1973). The milestone of studies these quaternions can be regarded the study of Halici (2012), because of a systematic equation of Binet-like formula o these quaternions. There are a number of studies on Fibonacci and Lucas quaternions or their generalizations (Akyigit, Kosal and Tosun, 2013 and 2014; Nukan and Guven, 2015; Tan, Yilmaz and Sahin, 2016a, 2016b; Yuce and Aydin, 2016a and 2016b; Polatli, Kizilates and Kesim, 2016; Polatli, 2016; Ipek, 2017; Bilgici, Tokeser and Unal, 2017; Aydin, 2018; Kizilates and Kone, 2021; Gul, 2022; Dasdemir and Bilgici, 2021).

Recently, Bilgici (2022) introduced Fibonacci 3-parameter generalized quaternions and gave some properties of the numbers including generating functions, Binet-like formulas and a number of generalizations of some well-known identities. This study will be based on it.

Definitions, Generatng Functions and Binet Formulas

For any positive integer *n*, the *n*th k-Fibonacci 3-parameter generalized quaternions are defined by the equation

$$\chi_{k,n} = F_{k,n} + e_1 F_{k,n+1} + e_2 F_{k,n+2} + e_3 F_{k,n+3}$$

the *n*th k-Lucas 3-parameter generalized quaternions are defined by the equation

$$\psi_{k,n} = L_{k,n} + e_1 L_{k,n+1} + e_2 L_{k,n+2} + e_3 L_{k,n+3}.$$

By using the recurrence relations, for any positive integer *n,* it is easily to see that

$$\chi_{k,n} = k\chi_{n-1} + \chi_{n-2}$$

and

$$\psi_{k,n} = k\psi_{n-1} + \psi_{n-2}.$$

The identities $F_{k,-n} = (-1)^{n+1} F_{k,n}$ and $L_{k,-n} = (-1)^n L_n$ gives

$$\chi_{k,-n} = (-1)^{n+1}(F_{k,n} - e_1 F_{k,n+1} + e_2 F_{k,n+2} - e_3 F_{k,n+3})$$

and

$$\psi_{k,-n} = (-1)^n (L_{k,n} - e_1 L_{k,n+1} + e_2 L_{k,n+2} - e_3 L_{k,n+3})$$

respectively. By using these identities, we can expand the definitions for negative subscripts, and we have

$$\chi_{k,n} + (-1)^{n+1}\chi_{k,-n} = 2F_{k,n} + 2e_2 F_{k,n+2}$$

and

$$\psi_{k,n} + (-1)^n \psi_{k,-n} = 2L_{k,n} + 2e_2 L_{k,n+2}.$$

Theorem 1. Generating functions for $\{\chi_{k,n}\}$ and $\{\psi_{k,n}\}$ are

$$\sum_{r=0}^{\infty} \chi_{k,r} q^r = \frac{e_1 + ke_2 + (k^2+1)e_3 + (1 + e_2 + ke_3)q}{1 - kq - q^2}$$

and

$$\sum_{r=0}^{\infty} \psi_{k,r} q^r$$
$$= \frac{2 + ke_1(k^2+2)e_2(k^3+3k)e_3 + [-k + 2e_1 + ke_2 + (k^2+2)e_3]q}{1 - kq - q^2}$$

respectively.

Proof. Assume that $\chi_k(q)$ is the generating function for the k-Fibonacci 3-parameter generalized quaternions. So, we have

$$\chi_k(q) = \chi_{k,0} + \chi_{k,1} q + \sum_{r=2}^{\infty} \chi_{k,r} q^r. \quad (1)$$

If we multiply both sides of Eq.(1) by $-kq$, we have

$$-kq\chi_k(q) = -k\chi_{k,0} q - k \sum_{r=2}^{\infty} \chi_{k,r-1} q^r \quad (2)$$

and multiplying both sides of Eq.(1) by $-q^2$, we obtain

$$-q^2 \chi_k(q) = -\sum_{r=2}^{\infty} \chi_{k,r-2} q^r. \quad (3)$$

Adding Eqs. (1), (2) and (3) side by side and substituting the first four terms of k-Fibonacci sequence, we prove the first generating functions in the theorem. The other can be proved similarly. ■

Theorem 2. [Binet formulas] For any integer r, the rth k-Fibonacci and k-Lucas 3-parameter generalized quaternion are

$$\chi_{k,r} = \frac{\alpha_k^* \alpha_k^r - \beta_k^* \beta_k^r}{\alpha_k - \beta_k} \text{ and } \psi_{k,r} = \alpha_k^* \alpha_k^r + \beta_k^* \beta_k^r$$

respectively, where $\alpha_k^* = 1 + e_1\alpha_k + e_2\alpha_k^2 + e_3\alpha_k^3$ and $\beta_k^* = 1{+}e_1\beta_k + e_2\beta_k^2 + e_3\beta_k^3$.

Proof. Binet formula gives

$$\begin{aligned}
\chi_{k,r} &= F_{k,n} + e_1 F_{k,n+1} + e_2 F_{k,n+2} + e_3 F_{k,n+3} \\
&= \frac{1}{\alpha_k - \beta_k}[\alpha_k^r - \beta_k^r + e_1(\alpha_k^{r+1} - \beta_k^{r+1}) + e_2(\alpha_k^{r+2} - \beta_k^{r+2}) \\
&\qquad + e_3(\alpha_k^{r+3} - \beta_k^{r+3})] \\
&= \frac{1}{\alpha_k - \beta_k}[\alpha_k^r(1 + e_1\alpha_k + e_2\alpha_k^2 + e_3\alpha_k^3) - \beta_k^r(1{+}e_1\beta_k \\
&\qquad + e_2\beta_k^2 + e_3\beta_k^3)].
\end{aligned}$$

By the last equation, we prove the first identity. The other can be obtained similarly. ■

The next results are a need for later use.

Corollary 3. Let α_k^* and β_k^* be as given in Theorem 2, we have

$$\alpha_k^* \beta_k^* = Y_k + Z_k\sqrt{k^2+4} \tag{4}$$

and

$$\beta_k^* \alpha_k^* = Y_k - Z_k\sqrt{k^2+4} \tag{5}$$

where

$$Y_k = \phi_{k,0} - 1 + \sigma_1\sigma_2 - \sigma_1\sigma_3 + \sigma_2\sigma_3$$

and

$$Z_k = -\sigma_3 e_1 - k\sigma_2 e_2 + \sigma_1 e_3.$$

Results

We give some generalizations of well-known identities in this section and start with Vajda's identities given in the next theoem.

Theorem 4. For any integers m, n and r, the followings hold

$$\chi_{k,m+n}\chi_{k,m+r} - \chi_{k,m}\chi_{k,m+n+r} = (-1)^{m+1}F_{k,n}\left[-Y_k F_{k,r} + Z_k L_{k,r}\right]$$

and

$$\psi_{k,m+n}\psi_{k,m+r} - \psi_{k,m}\psi_{k,m+n+r} = (-1)^m (k^2+4) F_{k,n}\left[-Y_k F_{k,r} + Z_k L_{k,r}\right].$$

Proof. The Binet formula gives

$$\begin{aligned}
&\chi_{k,m+n}\chi_{k,m+r} - \chi_{k,m}\chi_{k,m+n+r} \\
&= \frac{1}{(\alpha_k - \beta_k)^2}[(\alpha_k^*\alpha_k^{m+n} - \beta_k^*\beta_k^{m+n})(\alpha_k^*\alpha_k^{m+r} - \beta_k^*\beta_k^{m+r}) \\
&\qquad - (\alpha_k^*\alpha_k^{m} - \beta_k^*\beta_k^{m})(\alpha_k^*\alpha_k^{m+r} - \beta_k^*\beta_k^{m+r})] \\
&= \frac{1}{(\alpha_k - \beta_k)^2}[\alpha_k^*\beta_k^*(\alpha_k^m\beta_k^{m+n+r} - \alpha_k^{m+n}\beta_k^{m+r}) \\
&\qquad + \beta_k^*\alpha_k^*(\alpha_k^{m+n+r}\beta_k^m - \alpha_k^{m+r}\beta_k^{m+n})] \\
&= \frac{(-1)^m}{(\alpha_k - \beta_k)^2}[\alpha_k^*\beta_k^*(\beta_k^{n+r} - \alpha_k^n\beta_k^r) + \beta_k^*\alpha_k^*(\alpha_k^{n+r} - \alpha_k^r\beta_k^n)] \\
&= \frac{(-1)^{m+1}}{(\alpha_k - \beta_k)^2}[\alpha_k^*\beta_k^*\beta_k^r(\alpha_k^n - \beta_k^n) + \beta_k^*\alpha_k^*\alpha_k^r(\alpha_k^n - \beta_k^n)] \\
&= \frac{(-1)^{m+1}F_{k,n}}{\alpha_k - \beta_k}[\alpha_k^*\beta_k^*\beta_k^r - \beta_k^*\alpha_k^*\alpha_k^r]
\end{aligned}$$

$$= \frac{(-1)^{m+1}F_{k,n}}{\alpha_k - \beta_k}\left[\beta_k^r\left(Y_k + Z_k\sqrt{k^2+4}\right) - \alpha_k^r\left(Y_k - Z_k\sqrt{k^2+4}\right)\right]$$

$$= (-1)^{m+1}F_{k,n}\left[-Y_k\left(\frac{\alpha_k^r - \beta_k^r}{\alpha_k - \beta_k}\right) + Z\_k(\alpha_k^r - \beta_k^r)\right].$$

The final equation proves the first identity. The second identity can be obtained similarly. ■

For $r \to -n$, The Vajda's identities with the identity $F_{k,2n} = F_{k,n}L_{k,n}$ give Catalan's identities given in the following theorem.

Theorem 5. For any integers m and n, the followings hold

$$\chi_{k,m+n}\chi_{k,m-n} - \chi_{k,m}^2 = (-1)^{m+n+1}\left[Y_kF_{k,n}^2 + Z_kF_{k,2n}\right]$$

and

$$\psi_{k,m+n}\psi_{k,m-n} - \psi_{k,m}^2 = (-1)^{m+n}(k^2+4)\left[Y_kF_{k,n}^2 + Z_kF_{k,2n}\right].$$

For $n \to 1$, The Catalan's identities Cassini's identities given in the next theorem.

Theorem 6. For any integers m, the followings hold

$$\chi_{k,m+1}\chi_{k,m-1} - \chi_{k,m}^2 = (-1)^m[Y_k + kZ_k]$$

and

$$\psi_{k,m+1}\psi_{k,m-1} - \psi_{k,m}^2 = (-1)^{m+1}(k^2+4)[Y_k + kZ_k].$$

Theorem 7. [d'Ocagne's identity] For any integers m and n, the followings hold

$$\chi_{k,m}\chi_{k,n+1} + \chi_{k,m+1}\chi_{k,n} = (-1)^n\left[Y_kF_{k,m-n} + Z_kL_{k,m-n}\right]$$

and

$$\begin{aligned}\psi_{k,m}\psi_{k,n+1} + \psi_{k,m+1}\psi_{k,n}\\ = (-1)^{n+1}(k^2+4)\left[Y_kF_{k,m-n} + Z_kL_{k,m-n}\right].\end{aligned}$$

Proof. The Binet formula gives

$\chi_{k,m}\chi_{k,n+1} + \chi_{k,m+1}\chi_{k,n}$

$$
\begin{aligned}
&= \frac{1}{(\alpha_k-\beta_k)^2}[(\alpha_k^*\alpha_k^m-\beta_k^*\beta_k^m)(\alpha_k^*\alpha_k^{n+1}-\beta_k^*\beta_k^{n+1}) \\
&\qquad -(\alpha_k^*\alpha_k^{m+1}-\beta_k^*\beta_k^{m+1})(\alpha_k^*\alpha_k^n-\beta_k^*\beta_k^n)] \\
&= \frac{1}{(\alpha_k-\beta_k)^2}[\alpha_k^*\beta_k^*(\alpha_k^{m+1}\beta_k^n-\alpha_k^m\beta_k^{n+1}) \\
&\qquad -\beta_k^*\alpha_k^*(\alpha_k^{n+1}\beta_k^m-\alpha_k^n\beta_k^{m+1})] \\
&= \frac{1}{(\alpha_k-\beta_k)^2}[\alpha_k^*\beta_k^*\alpha_k^m\beta_k^n(\alpha_k-\beta_k)-\beta_k^*\alpha_k^*\alpha_k^n\beta_k^m(\alpha_k-\beta_k)] \\
&= \frac{(-1)^n}{\alpha_k-\beta_k}[\alpha_k^*\beta_k^*\alpha_k^{m-n}-\beta_k^*\alpha_k^*\beta_k^{m-n}] \\
&= \frac{(-1)^n}{\alpha_k-\beta_k}\left[\alpha_k^{m-n}\left(Y_k+Z_k\sqrt{k^2+4}\right)-\beta_k^{m-n}\left(Y_k-Z_k\sqrt{k^2+4}\right)\right] \\
&= (-1)^n\left[Y_k\left(\frac{\alpha_k^{m-n}-\beta_k^{m-n}}{\alpha_k-\beta_k}\right)+Z_k(\alpha_k^{m-n}-\beta_k^{m-n})\right].
\end{aligned}
$$

The last equation proves the first identity. The other can be obtained similarly. ■

We can obtain many identities between k-Fibonacci and k-Lucas generalized quaternions by using similar methods. Some of them are given in next theorem without proofs.

Theorem 8. For any integers m, n and r, the followings hold

$\psi_{k,m} = \chi_{k,m+1} + \chi_{k,m-1}$,

$\chi_{k,m}\psi_{k,n} - \psi_{k,m}\chi_{k,n} = 2(-1)^n\left[Y_kF_{k,m-n} + Z_kL_{k,m-n}\right]$,

$\chi_{k,m}\psi_{k,n} - \chi_{k,n}\psi_{k,m} = 2(-1)^nY_kF_{k,m-n}$,

$\chi_{k,m}\chi_{k,n} - \chi_{k,n}\chi_{k,m} = 2(-1)^{n+1}Z_kF_{k,m-n}$,

$\psi_{k,m}\psi_{k,n} - \psi_{k,n}\psi_{k,m} = 2(-1)^n(k^2+4)Z_kF_{k,m-n}$,

$\chi_{k,m}^2 - \psi_{k,m}^2 = 4(-1)^{m+1}Y_k$,

$$\sum_{i=1}^{m} \chi_{k,i} = \frac{1}{k}\left(\chi_{k,m+1} + \chi_{k,m} - 1\right) - (e_1 + ke_2 + e_3 + k^2 e_3) + \frac{1}{k}(e_1 + e_2 + 2e_3).$$

Conclusion

Nowadays, hyper-complex numbers with integer sequences coefficients are very popular area. There are a lot of studies on these numbers and present stusy is one of them. k-Fibonacci gives metalic ratios and generalizes a number of integer sequences. In this study, two new definitions have been made, namely k-Fibonacci and k-Lucas 3-parameter generalized quaternions. 3-parameter generalized quaternions are a generalization of Hamilton quaternions. We give generating functions, Binet formulas and some identities. This study can fill in a gap in literatüre.

Bibliography

Akyigit, M., Kosal, H. H., & Tosun, M. (2013). Split Fibonacci quaternions. *Advances in Applied Clifford Algebras*, 23, 535-545.

Akyigit, M., Kosal, H. H., & Tosun, M. (2014). Fibonacci generalized quaternions. *Advances in Applied Clifford Algebras*, 24, 631-641.

Aydin, F. T. (2018). Bicomplex Fibonacci quaternions. *Chaos, Solitons & Fractals*, 106, 147-153.

Bilgici, G., Tokeser, U. & Unal, Z. (2017). k-Fibonacci and k-Lucas generalized quaternıons. *Konuralp Journal of Mathematics*, 5(2), 102-113

Dasdemir, A., & Bilgici, G. (2021). Unrestricted Fibonacci and Lucas quaternions. *Fundamental Journal of Mathematics and Applications*, 4(1), 1-9.

Falcon, S. (2011). On the k-Lucas numbers. *International Journal of Contemporary Mathematical Sciences*, 6(21), 1039-1050.

Falcon, S., & Plaza, A. (2007). On the Fibonacci k-numbers. *Chaos, Solitons & Fractals*, 32(5), 1615-1624.

Gul, K. (2022). Generalized k-order Fibonacci hybrid quaternions. *Erzincan University Journal of Science and Technology*, 15(2), 670-683.

Halici, S. (2012). On Fibonacci quaternions. *Advances in Applied Clifford Algebras*, *22*(2), 321-327.

Horadam, A. F. (1963). Complex Fibonacci numbers and Fibonacci quaternions. *The American Mathematical Monthly*, 70(3), 289-291.

Ipek, A. (2017). On (p, q)-Fibonacci quaternions and their Binet formulas, generating functions and certain binomial sums. *Advances in Applied Clifford Algebras*, 27, 1343-1351.

Kizilates, C., & Kone, T. (2021). On higher order Fibonacci quaternions. *The Journal of Analysis*, 1-12.

Koshy, T. (2019). *Fibonacci and Lucas Numbers with Applications*, Volume 2. John Wiley & Sons.

Nurkan, S. K., & Guven, I. A. (2015). Dual Fibonacci quaternions. *Advances in Applied Clifford Algebras*, 25, 403-414.

Polatlı, E. (2016). A generalization of Fibonacci and Lucas Quaternions. *Advances in Applied Clifford Algebras*, 26, 719-730.

Polatli, E., Kizilates, C., & Kesim, S. (2016). On split k-Fibonacci and k-Lucas quaternions. *Advances in Applied Clifford Algebras*, 26, 353-362.

Swamy, M. N. S. (1973). On Generalized Fibonacci Quarternions. The Fibonacci Quarterly, 11(5), 547-549.

Şentürk, T. D., & Ünal, Z. (2022). 3-parameter generalized quaternions. *Computational Methods and Function Theory*, 22(3), 575-608.

Tan, E., Yilmaz, S., & Sahin, M. (2016a). On a new generalization of Fibonacci quaternions. *Chaos, Solitons & Fractals*, 82, 1-4.

Tan, E., Yilmaz, S., & Sahin, M. (2016b). A note on bi-periodic Fibonacci and Lucas quaternions. *Chaos, Solitons & Fractals*, 85, 138-142.

Yuce, S., & Aydin, F. T. (2016a). A new aspect of dual Fibonacci quaternions. *Advances in Applied Clifford algebras*, 26, 873-884.

Yuce, S., & Aydin, F. T. (2016b). Generalized dual Fibonacci quaternions. *Applied Mathematics E-Notes*, 16(30), 276-289.

CHAPTER II

A New Mathematical Model for Monkeypox with Vaccination Effect and Its Quantified Analysis

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1.Introduction

Monkeypox (MPOX), one of the zoonotic diseases, is a rare infection caused by a virus belonging to the Poxviridae family. Zoonotic diseases are defined as infections that can be transmitted from animals to humans, and monkeypox falls into this category.

It was first identified in humans in 1970 in the Democratic Republic of Congo and is typically associated with the tropical rainforests of Central and West Africa (Breman, 1980).

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The monkeypox virus can be transmitted from infected animals to humans, with rodents and primates being the primary reservoirs. Transmission to humans can occur through animal bites, scratches, or the consumption of infected animal meat. Among humans, the virus spreads via direct contact with skin lesions, bodily fluids, respiratory droplets during prolonged face-to-face interactions, or contaminated objects.

Symptoms of the disease usually begin with flu-like signs, including fever, headache, muscle pain, and swollen lymph nodes. These are followed by characteristic skin rashes that appear as fluid-filled blisters and lesions. While most cases resolve within a few weeks, severe complications and high mortality rates can occur in immunocompromised individuals or in regions with limited access to healthcare (Nuzzo, Borio & Gostin, 2022).

Although monkeypox cases in developed countries remain limited, it continues to pose a significant public health threat in developing countries, where high mortality rates persist. Therefore, understanding the behavior of the disease and conducting dynamic analysis is of great importance. For this purpose, mathematical models and numerical solutions are essential. In our study, an Mpox model based on real-world data was developed, and different numerical solutions were compared for interpretation.

Mathematical modeling is the process of using mathematical structures to represent real-world systems, and it is crucial for predicting and understanding the behavior of these systems over time (Kermack & McKendrick, 1927). In epidemiological research, mathematical models are used to examine the mechanisms of infectious disease transmission and to design intervention strategies. For example, Anderson and May (1991) worked on epidemiological models to understand the dynamics of infectious diseases and develop strategies for their control. Hethcote (2000) used differential equations to analyze the spread of infectious diseases, simulating the effects of various intervention strategies. Additionally, Diekmann, Heesterbeek & Britton (2013) developed mathematical tools for preventing and controlling diseases by modeling the spread of infections. These studies highlight the importance of mathematical models in optimizing disease control and public health strategies.

In recent years, studies in mathematical modeling have become more diverse. For example, Bacaër (2015) discussed the need for multi-scale models to understand epidemiological processes and examined how infectious diseases can spread at local, regional, and global levels. Moreover, Brauer (2017) compared the use of different models to simulate the spread of infections, detailing the advantages and limitations of each model. Arenas et al. (2020) adapted a Microscopic Markov Chain Approach (MMCA) metapopulation mobility model to simulate the spread of COVID-19, accounting for age-specific incidence rates and mobility patterns. The model, applied to the epidemic in Spain, predicted a peak in infections by April 2020 without mobility restrictions and highlighted the strain on the healthcare system, especially intensive care units. The study emphasized the importance of enforcing a total lockdown to prevent a collapse of the Spanish national health system by analyzing various epidemic containment scenarios.

Usman and Adamu (2017) developed a mathematical model to study the transmission dynamics of monkeypox under vaccination and treatment interventions. They found that the disease-free equilibrium is stable when the basic reproduction number (R_0) is less than 1, while the endemic equilibrium exists when R_0 is greater than 1.

Their simulations showed that with appropriate control strategies, the infection would eventually die out in both human and non-human primate populations.

Peter et al. (2022a) developed and analysed a deterministic mathematical model for monkeypox virus (MPXV), investigating both local and global asymptotic stability for disease-free and endemic equilibria. Their results show that the model exhibits backward bifurcation, where a locally stable disease-free equilibrium can coexist with an endemic equilibrium, and that isolating infected individuals in the human population helps to reduce disease transmission. Peter et al. (2022b) present a deterministic mathematical model of monkeypox using both classical and fractional-order differential equations, incorporating all possible interactions that contribute to the spread of the disease. Their results, based on fitting the model to reported cases from Nigeria in 2019, show that the stability of the model depends on the basic reproduction number (R_0), and provide insights into the dynamics of the disease and appropriate control measures for its eradication. Elsonbaty et al. (2024) introduced a novel model to simulate the spread of monkeypox, incorporating human-rodent interactions, imperfect vaccination and nonlinear incidence rates, with the human population divided into low-risk and high-risk groups. Their analysis showed that the virus is more prevalent in the high-risk group, and through bifurcation analysis and numerical simulations they identified key parameters for controlling the virus and developing effective prevention strategies.

To determine the dynamic analysis of monkeypox disease A mathematical model was developed using current data on deaths in Africa between 1 January 2024 and 6 October 2024 (T.C. Sağlık Bakanlığı, 2024). The aim was to determine the behaviour of the model by including the effect of vaccination in the modelling. This model was used to observe the effect of vaccination on monkeypox. Four different numerical methods were used for the observations.

The solution of the modelling was expressed using Euler, Central difference, Runge-Kutta and Nonstandard finite difference (NSFD) methods and graphics. This allowed the dynamic analysis of the model system to be interpreted.

This study consists of five sections. The second section contains basic information for solving the mathematical model system. This part provides the definitions and advantages of numerical methods. The third section includes the diagram and modeling of the created system. The fourth section presents the numerical solutions of the system. Each numerical method used for the solutions is discussed. Additionally, the data obtained from the solutions are presented in figures and tables. The fifth section contains the conclusion, which includes evaluations related to the topic.

2.Basic Definitions about numerical methods

In many cases, analytical solutions to differential equations are either too complex or infeasible, especially for real-world problems with complex dynamics. This is where numerical solutions become crucial.

Numerical methods such as finite difference, finite element or Runge-Kutta provide practical approaches to approximate solutions to these equations. They allow the simulation and analysis of systems that would otherwise be impossible to solve analytically.

Through numerical methods, we can gain valuable insights into the behaviour of complex systems, make predictions and optimise solutions, which is particularly important in fields such as epidemiology, engineering and physics (Oruç & Sondaş, 2018).

One of these methods, the Euler Method, aims to obtain an approximate solution to the well-conditioned initial value problem as follows.

Definition 2.1: (Burden & Faires, 2010) Let the initial value problem can be given as follows:

$$\frac{dy}{dt} = f(t, y) \quad , \quad a \leq t \leq b \, , \quad y(a) = a. \tag{1}$$

Assuming that the grid points are evenly distributed over the interval [a,b], a positive N and grid points are chosen for computation. The distance between two points is denoted as h, and this is called the step size. To derive the Euler method, the Taylor theorem is applied, leading to the following equality:

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i). \tag{2}$$

Accordingly, if the error term is neglected, the Euler method is obtained as follows:

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)). \tag{3}$$

The Adams method makes predictions based on previous points, while Runge, in his approach, makes predictions based on just one step. For accuracy, approximate values of the interior points are calculated at each step, and exact results are obtained according to these steps (Runge, 1895; Butcher, 2000). The reason the Runge-Kutta method is the most commonly used method for ordinary differential equations is that it does not involve solving with higher-order derivatives and yields results that are very close to exact accuracy. Accordingly, the 4th-order Runge-Kutta method is defined as follows:

Definition 2.2: (Çengel &Pan, 2013) Considering the initial value problem given by Equation 1, the classical fourth-order Runge-Kutta method is derived using the fourth-order Taylor series method as follows:

$$y(t + h) = y(t) + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4). \tag{4}$$

Here, the k_1, k_2, k_3 and k_4 parameters are defined as follows.

$$k_1 = f(x_i, y_i),$$

$$k_2 = f(x_i + \frac{h}{2}, y_i + \frac{hk_1}{2}),$$

$$k_3 = f(x_i + \frac{h}{2}, y_i \frac{hk_2}{2}),$$

$$k_4 = f(x_i + h, y_i + hk_3). \quad (5)$$

Definition 2.3: (Oruç & Sondaş 2018) When the Taylor series expansion of the $g(x+h)$ function is performed for the central difference method,

$$g(x+h) = g(x) + hg'(x) + \frac{1}{2!}h^2g''(x) + \frac{1}{3!}h^3g'''(x) + \ldots \quad (6)$$

is obtained. Likewise the Taylor series expansion of the function gives the following equation:

$$g(x-h) = g(x) - hg'(x) + \frac{1}{2!}h^2g''(x) - \frac{1}{3!}h^3g'''(x) + \ldots \quad (7)$$

If Equation (7) is subtracted from Equation (6) and rearranged,

$$g(x+h) - g(x-h) = 2hg'(x) + \frac{1}{3}h^3g'''(x) + \ldots \quad (8)$$

However, since it has performed subtraction, the even-order derivatives of the $g(x_0)$ term is eliminated.

$$g'(x) = \frac{g(x+h) - g(x-h)}{2h} + \varepsilon_i$$

Here the expression represents higher order derivatives. As it approaches zero, quality becomes:

$$g'(x) = \frac{g(x+h) - g(x-h)}{2h} \quad (10)$$

This equation represents the first derivative in terms of central differences.

The Nonstandard Finite Difference method is a powerful tool for solving differential equations that involve complex or nonlinear dynamics. Unlike traditional finite difference methods, NSFD schemes are designed to address stability issues, particularly when dealing with stiff equations. By carefully selecting appropriate denominator functions, NSFD methods help avoid instability and prevent negative solutions, making them more reliable and accurate in certain scenarios. This flexibility is especially important for systems with intricate interactions, which are common in fields such as epidemiology, physics, and engineering (Mickens, 1989).

Definition 2.4: (Mickens, 1989) For a general first-order differential equation of the form: $\frac{dy}{dt} = f(t, y)$. An NSFD scheme may use a more modified form for the difference, such as:

$$\frac{(\mathrm{y}(\mathrm{t}+\mathrm{h}) - \mathrm{y}(\mathrm{t}))}{\theta(h)} \frac{dy}{dt} \tag{11}$$

where θ is denominator function and it can be chosen as $\theta(h) = \frac{e^{ph}-1}{h}$, where $\mathbf{p}$ is a parameter. The denominator function $\theta(h)$ depends on the step size $\mathbf{h}$ and the variable $\mathbf{p}$, which is calculated based on the equilibrium point. (Mickens, 2002; Ongun, Arslan & Farzi, 2017; Kocabıyık, Ongun & Çetinkaya, 2021; Çetinkaya, Kocabıyık & Ongun, 2021; Çetinkaya, 2023; Kocabiyik & Ongun, 2023 ;Kocabiyik & Ongun, 2024)

Euler method, Runge-Kutta method, Central difference method, and Nonstandard finite difference (NSFD) method each offer unique advantages in numerical analysis. The Euler method is simple and easy to implement, making it a good choice for basic problems, although it may be less accurate for stiff equations due to its first-order approximation.

In contrast, the Runge-Kutta method, especially the 4th-order version, provides higher accuracy by using multiple evaluations within each step, making it ideal for more complex problems where precision is crucial. The Central difference method is particularly effective for problems involving spatial derivatives, offering second-order accuracy in both time and space, which makes it suitable for solving partial differential equations with more precise results. Finally, the NSFD method is often used for systems with non-linear dynamics and provides flexibility in modeling complex interactions, making it particularly advantageous in systems where other methods may struggle. Each method's strength lies in its suitability for specific types of problems, with accuracy, simplicity, and efficiency being the key distinguishing factors.

3.Mathematical modelling of monkeypox disease

Mathematical modelling of monkeypox involves tracking the dynamics of susceptible, vaccinated, infected and recovered populations to understand the behaviour and spread of the disease. These models highlight the importance of vaccination in reducing the number of susceptible individuals, thereby limiting infection rates and containing outbreaks. By integrating vaccination strategies, the models provide valuable insights into effective disease control measures and public health interventions. The mathematical model of the behaviour of monkeypox disease, including the effect of vaccination, has been constructed as follows:

$$\frac{dS}{dt} = a - \mu S(t) - \beta_1 S(t)I(t) - \beta_2 S(t)$$

$$\frac{dV}{dt} = \beta_2 S(t) - a_1 V(t)I(t) - a_2 V(t) - \mu V(t)$$

$$\frac{dI}{dt} = \beta_1 S(t)I(t) + a_1 V(t)I(t) - \lambda I(t) - \mu I(t) \qquad (12)$$

$$\frac{dR}{dt} = a_2 V(t) + \lambda I(t) - \mu R(t)$$

The variables used in the mathematical model are defined as follows: S represents the susceptible individuals in the population, V represents the vaccinated individuals, I represents the infected individuals, and R represents the recovered individuals. The parameters used in the model are defined as follows: α is the recovery rate, μ is the natural mortality rate, λ is the recovery rate of infected individuals, α_1 is the transmission rate of the disease when vaccinated individuals are in contact with infected individuals, α_2 is the rate at which vaccinated individuals become recovered individuals, β_1 the transition rate from susceptible to infected individuals, and β_2 is the transition rate from susceptible to vaccinated individuals. The diagram of the model is shown in Figure 1.

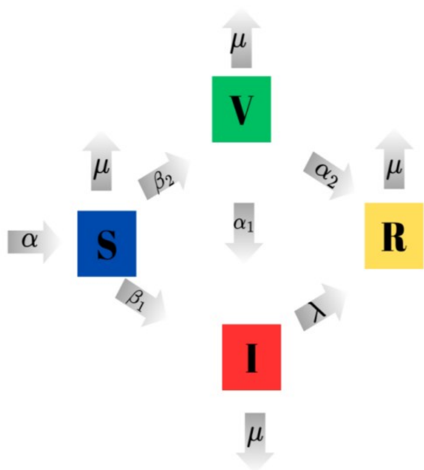

Figure 1: The diagram of the mathematical model for MPOX

4.Numerical analysis

In this section, the numerical analysis of the constructed mathematical model system has been performed using different methods. The constants and initial values used for the analysis are provided in Table 1.

Table 1:Initial conditions and constant values for the monkeypox model system.

Initial Conditions		Constant values			
S(0)	36.787	β_1	0.03	α_1	0.23
V(0)	32.004	β_2	0.07	α_2	0.21
I(0)	7.535	α	2	λ	0.27
R(0)	7.503	μ	0.02		

If the Euler method is applied to the mathematical model system expressed in Equation 12, the system takes the following form:

$$S(k+1) = S(k) + h(\alpha - \mu S(k) - \beta_1 S(k)I(k) - \beta_2 S(k))$$

$$V(k+1) = V(k) + h(\beta_2 S(k) - \alpha_1 V(k)I(k) - \alpha_2 V(k) - \mu V(k)) \quad (13)$$

$$I(k+1) = I(k) + h(\beta_1 S(k)I(k) + \alpha_1 V(k)I(k) - \lambda I(k) - \mu I(k))$$

$$R(k+1) = R(k) + h(\alpha_2 V(k) + \lambda I(k) - \mu R(k))$$

Using the values provided in Table 1 and Equation 13, the system's graphs for the Euler method are shown in Figures 2-5 (with h=0.1). Figure 2 illustrates the analysis for susceptible individuals using the Euler method. As seen in the figure, the susceptible individuals reach a disease-free equilibrium point after t=200 and continue at a constant level. In Figure 3, the analysis for vaccinated individuals shows a decrease until t=100, after which it continues at a constant level.

Figure 4, showing the analysis for infected individuals, demonstrates a rapid increase and decrease until t=100, followed by a constant progression. Figure 5 represents recovered individuals, showing a rapid increase until t=100, after which it continues to rise steadily. Table 2 shows the changes in populations at different time intervals using the Euler method. Thus, the character of the system under vaccination has been determined.

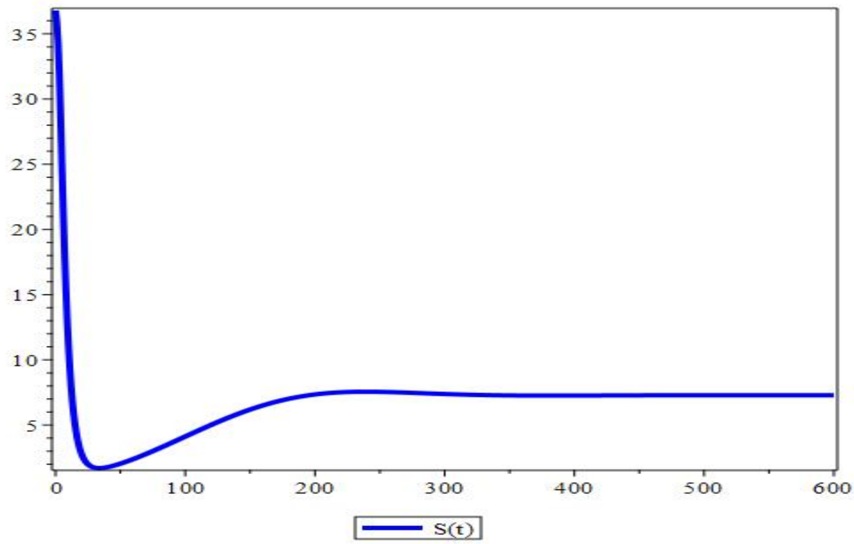

Figure 2: The change graph for susceptible individuals using the Euler method.

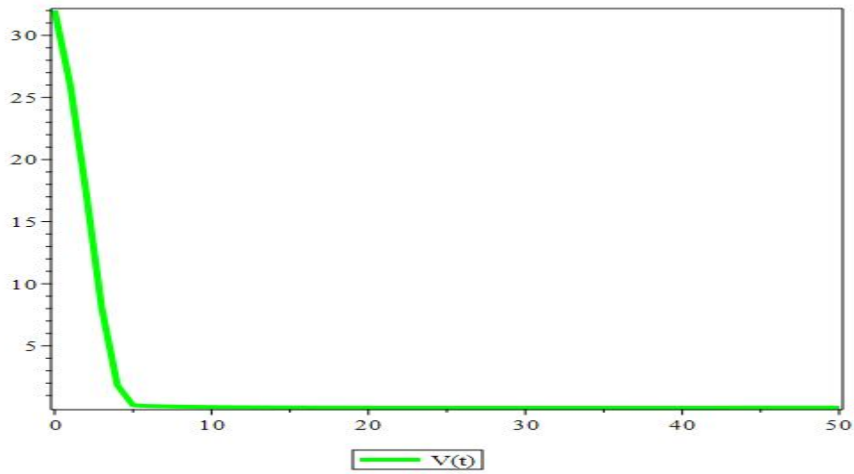

Figure 3: The change graph for vaccinated individuals using the Euler method.

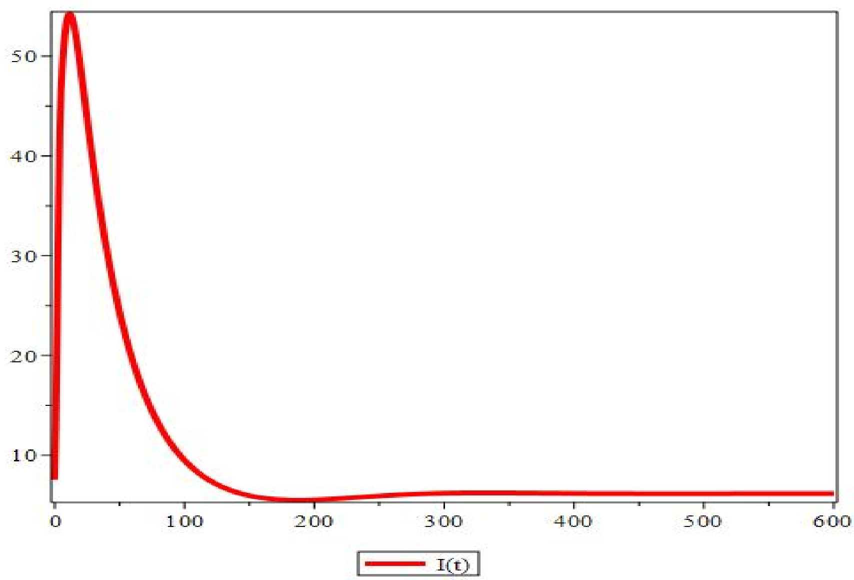

Figure 4: The change graph for infected individuals using the Euler method.

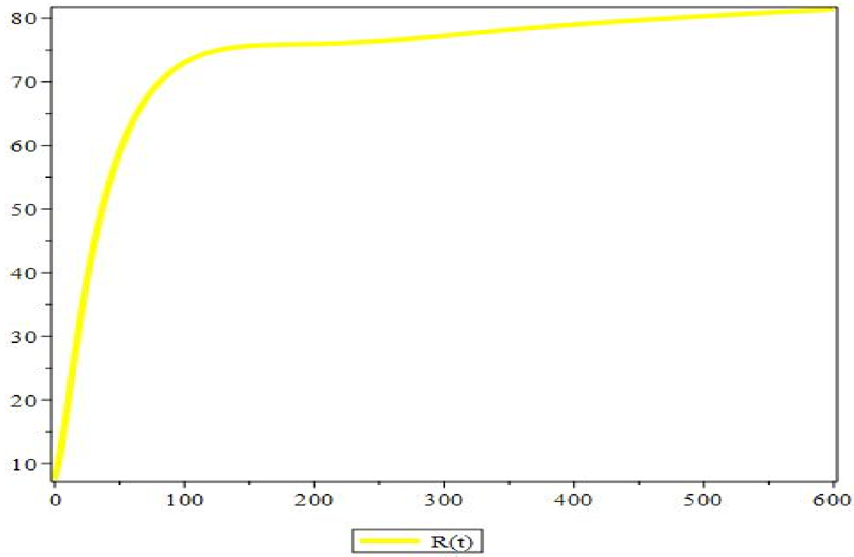

Figure 5: The change graph for recovered individuals using the Euler method.

Table 2: Population numbers over time with the Euler method

t	S(t)	V(t)	I(t)	R(t)
0	36.787	32.004	7.535	7.503
1	35.824	25.978	13.694	8.363
2	34.230	17.449	22.951	9.262
3	31.765	8.076	33.854	10.229
4	28.453	1.824	42.387	11.292
5	24.778	0.202	46.555	12.453
6	21.295	0.154	48.883	13.689
7	18.180	0.126	50.762	14.985
8	15.448	0.103	52.206	16.328
9	13.089	0.085	53.235	17.707
10	11.081	0.070	53.886	19.111
50	2.041	0.023	24.360	58.944
100	4.120	0.108	9.498	73.035
200	7.347	0.338	5.532	75.946
300	7.381	0.314	6.180	77.253

When the Runge-Kutta method is applied to the SVIR mathematical model system with the given parameter values, the graphs in Figures 6-9 are obtained. Figure 6 illustrates the analysis for susceptible individuals using the Runge-Kutta method. As seen in the figure, the number of susceptible individuals decreases rapidly until t=20, after which it progresses steadily. In Figure 7, the analysis for vaccinated individuals shows a rapid decrease followed by steady progression. Figure 8, showing the analysis for infected individuals, demonstrates a sharp increase and decrease until t=20, after which it continues at a constant rate. Figure 9 represents recovered individuals, showing a rapid increase until t=20, after which it continues to rise steadily. Table 3 presents the population change values at different time points t, which have been determined using the Runge-Kutta method.

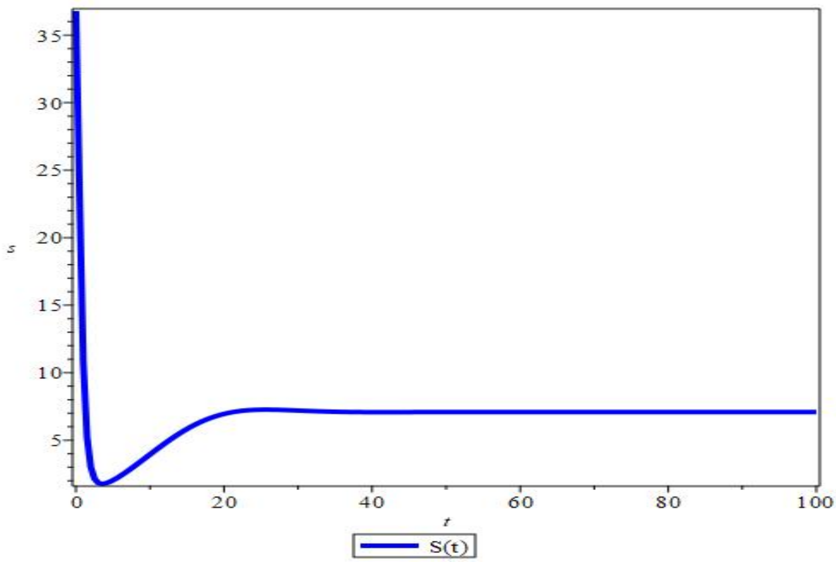

Figure 6: The variation in the number of susceptible individuals using the Runge-Kutta method.

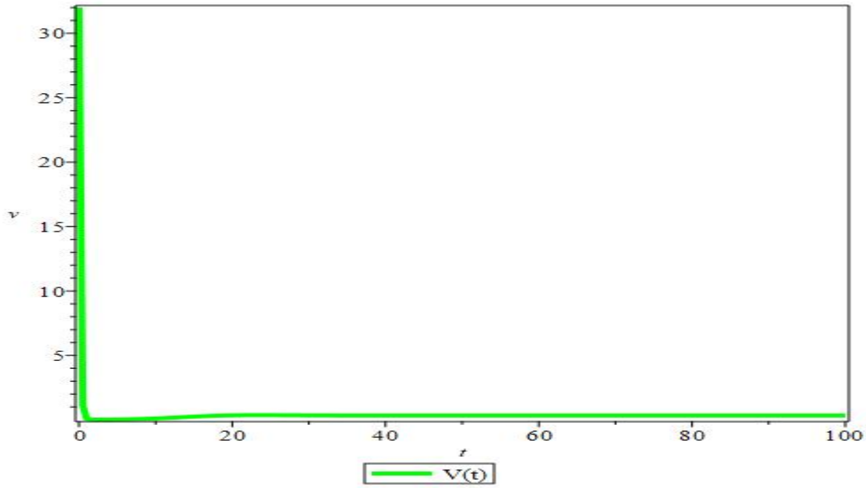

Figure 7: The variation in the number of vaccinated individuals using the Runge-Kutta method.

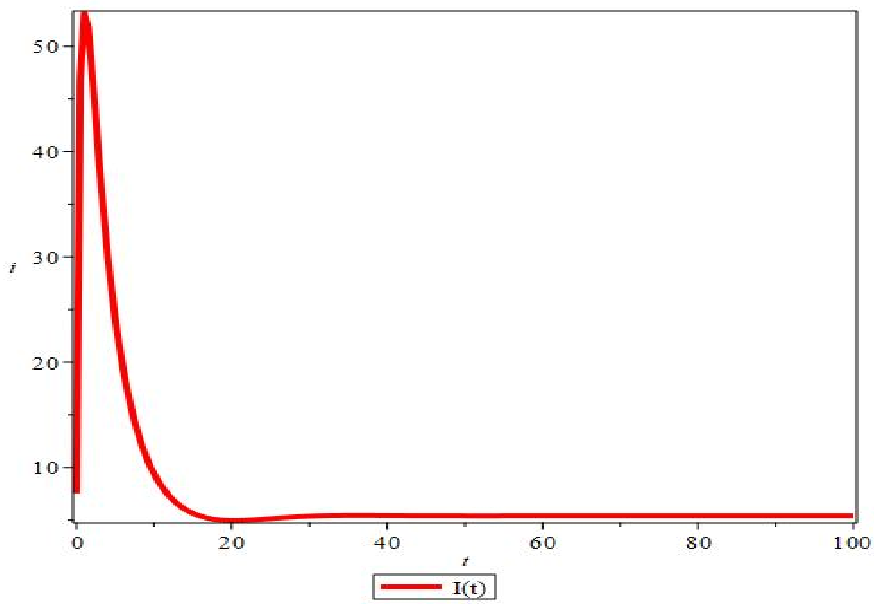

Figure 8: The variation in the number of infected individuals using the Runge-Kutta method.

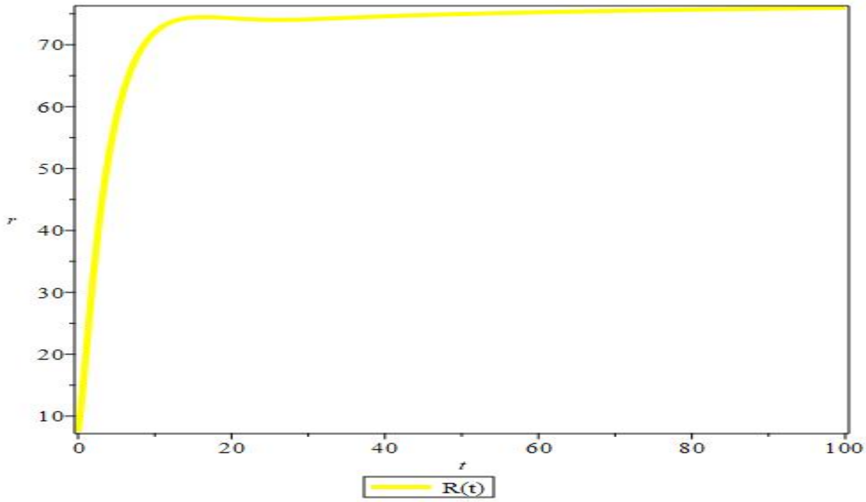

Figure 9: The variation in the number of recovered individuals using the Runge-Kutta method.

Table 3: Population numbers over time with the Runge-Kutta method

t	S(t)	V(t)	I(t)	R(t)
0	36.787	32.004	7.535	7.503
1	10.731	0.072	53.112	19.536
2	3.030	0.020	47.681	32.876
3	1.838	0.014	38.307	43.705
4	1.799	0.016	30.342	51.974
5	2.027	0.023	24.146	58.193
6	2.349	0.032	19.414	62.837
7	2.718	0.0454	15.813	66.286
8	3.113	0.061	13.073	68.828
9	3.524	0.081	10.984	70.682
10	3.941	0.104	9.392	72.015
50	7.084	0.336	5.405	74.959
100	7.086	0.336	5.407	75.957
200	7.086	0.336	5.407	76.460
300	7.086	0.336	5.407	76.529

If the Central Difference method is applied to the monkeypox mathematical model system, the system takes the following form:

$$S(k+1) = S(k-1) + 2h(\alpha - \mu S(k) - \beta_1 S(k) I(k) - \beta_2 S(k))$$

$$V(k+1) = V(k-1) + 2h(\beta_2 S(k) - \alpha_1 V(k) I(k) - \alpha_2 V(k) - \mu V(k)) \quad (14)$$

$$I(k+1) = I(k-1) + 2h(\beta_1 S(k) I(k) + \alpha_1 V(k) I(k) - \lambda I(k) - \mu I(k))$$

$$R(k+1) = R(k-1) + 2h(\alpha_2 V(k) + \lambda I(k) - \mu R(k))$$

The system's graphs using the Central difference method are shown in Figures 10-13. Figure 10 illustrates the analysis for susceptible individuals using the Central Difference method. As seen in the figure, the number of susceptible individuals decreases rapidly until t=200, after which it progresses steadily. In Figure 11, the analysis for vaccinated individuals shows mobility until t=50, after which it continues at a constant level.

Figure 12, showing the analysis for infected individuals, demonstrates a sharp increase and decrease until t=200, followed by a steady progression. Figure 13 represents recovered individuals, showing a rapid increase until t=100, after which it continues to rise steadily. Table 4 presents the population change values at different time points t.

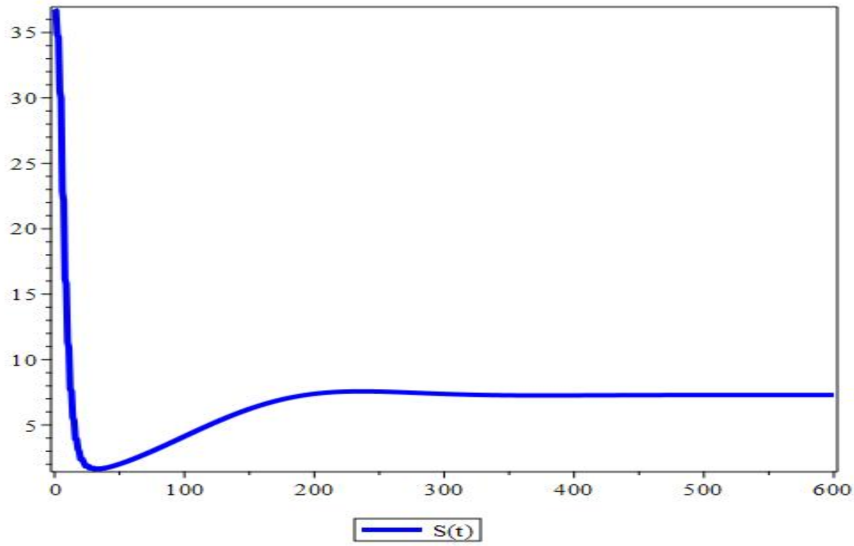

Figure 10: The population values for susceptible individuals using the Central difference method.

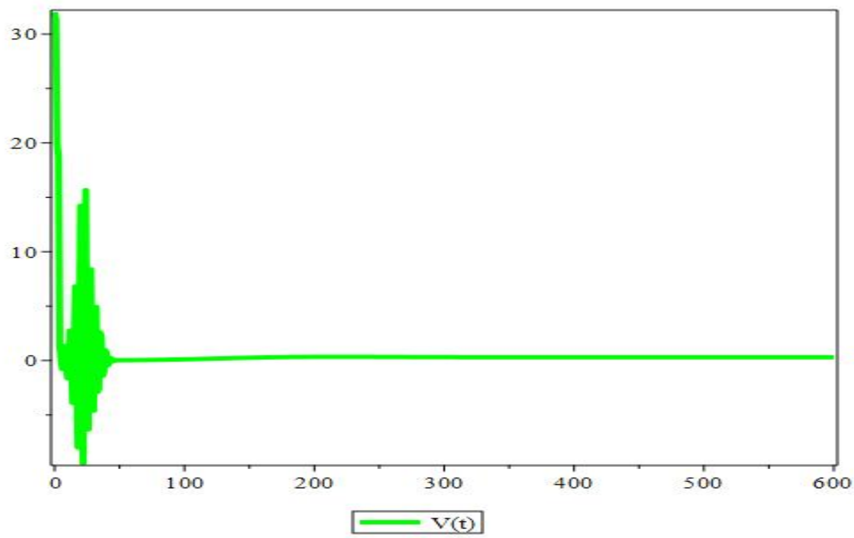

Figure 11: The population values for vaccinated individuals using the Central difference method.

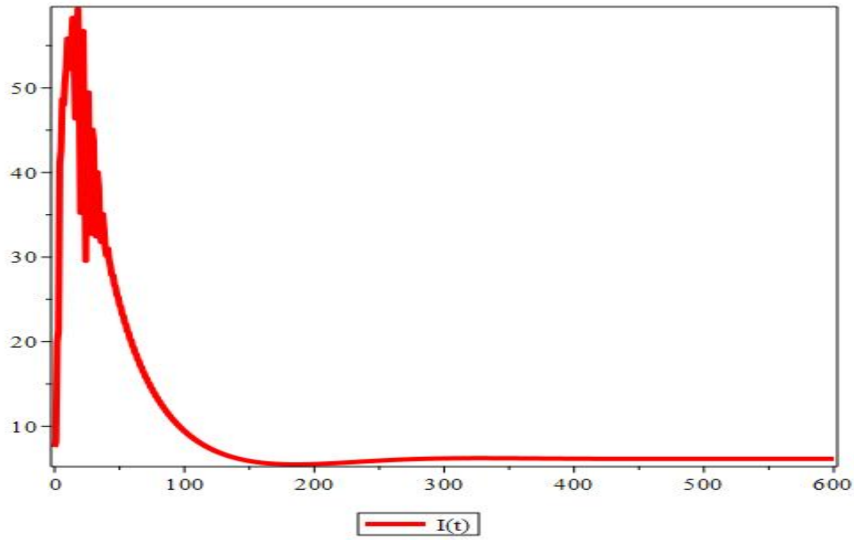

Figure 12: The population values for infected individuals using the Central difference method.

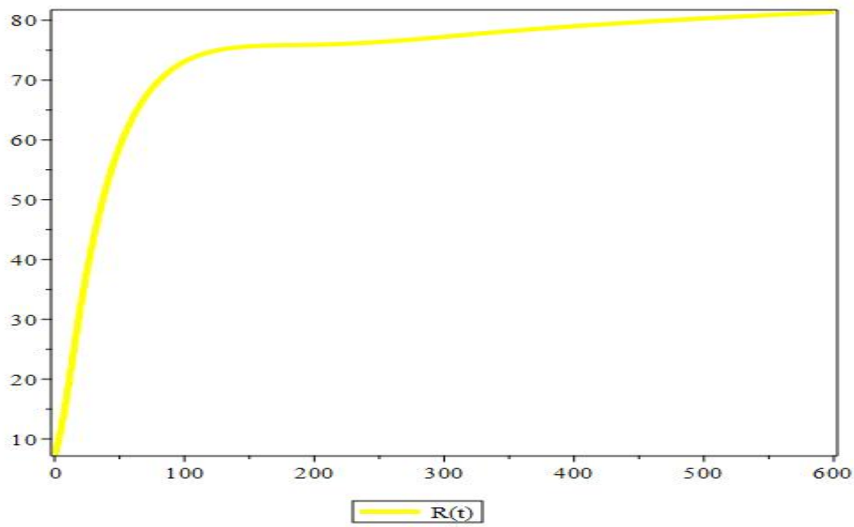

Figure 13: The population values for recovered individuals using the Central difference method.

Table 4: Population numbers over time with the Central difference method

t	S(t)	V(t)	I(t)	R(t)
0	36.787	32.004	7.535	7.503
1	36.690	31.401	19.854	7.589
2	34.861	19.953	21.243	9.224
3	34.635	18.697	41.078	9.317
4	30.481	1.300	42.698	11.097
5	29.997	0.050	48.666	11.212
6	22.819	-0.790	48.006	13.325
7	22.172	0.368	48.006	13.475
8	16.145	1.334	50.738	15.867
9	15.786	-0.151	52.422	16.029
10	11.339	-1.615	55.825	18.599
50	2.030	0.020	24.317	59.002
100	4.145	0.110	9.370	73.139
200	7.386	0.341	5.519	75.921
300	7.373	0.313	6.194	77.254

Finally, in the MPOX system, the system takes the following form when the nonstandard finite difference method is applied:

$$S(k+1) = S(k) + (\theta_1(h))(\alpha - (\mu + \beta_2)S(k) - \beta_1 S(k)I(k))$$

$$V(k+1) = V(k) + (\theta_2(h))(\beta_2 S(k) - \alpha_1 V(k)I(k) - (\mu + \alpha_2)V(k)) \quad (15)$$

$$I(k+1) = I(k) + (\theta_3(h))(\beta_1 S(k)I(k) + \alpha_1 V(k)I(k) - (\mu + \lambda)I(k))$$

$$R(k+1) = R(k) + (\theta_4(h))(\alpha_2 V(k) + \lambda I(k) - \mu R(k))$$

where, $\theta_i(h), i = 1,2,3,4$ are denominator functions, and they are selected as follows:

$$\theta_1(h) = \frac{e((\mu + \beta_2)h - 1}{\mu + \beta_2}, \theta_2(h) = \frac{e((\mu + \alpha_2)h - 1}{\mu + \alpha_2},$$

$$\theta_3(h) = \frac{\mathrm{e}((\mu+\lambda)\mathrm{h}-1}{\mu+\lambda}, \theta_4(h) = \frac{\mathrm{e}(\mu\mathrm{h})-1}{\mu}$$

Using the given parameter values and Equation 15, the system's graphs obtained with the NSFD method are shown in Figures 14-17. Figure 14 illustrates the analysis for susceptible individuals using the NSFD method. As seen in the figure, the number of susceptible individuals decreases rapidly until t=200, after which it progresses steadily. In Figure 15, the analysis for vaccinated individuals shows a rapid decrease followed by a steady progression. Figure 16, showing the analysis for infected individuals, demonstrates a sharp increase and decrease until t=200, followed by a constant progression. Figure 17 represents recovered individuals, showing a rapid increase until t=100, after which it continues to rise steadily. Table 5 presents the population values over time, obtained and expressed using the NSFD method.

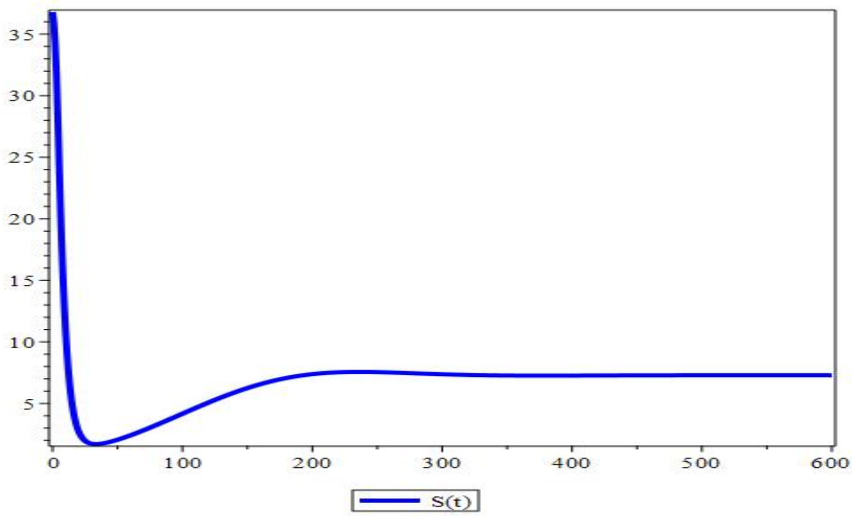

Figure 14: The behavior of the monkeypox model for susceptible individuals using the NSFD method

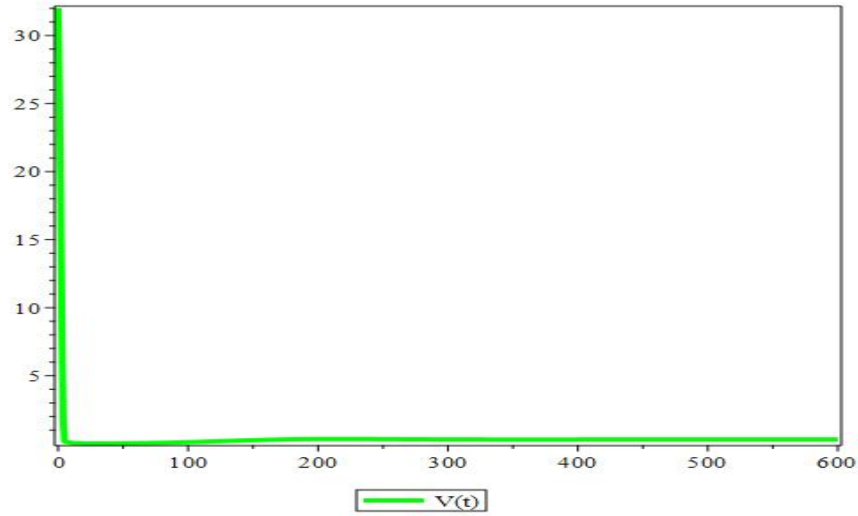

Figure 15: The behavior of the monkeypox model for vaccinated individuals using the NSFD method

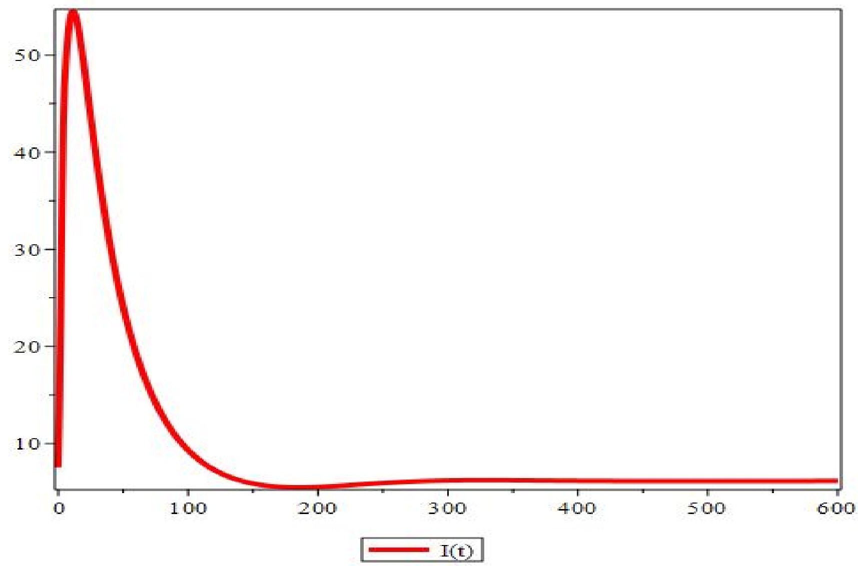

Figure 16: The behavior of the monkeypox model for infected individuals using the NSFD method

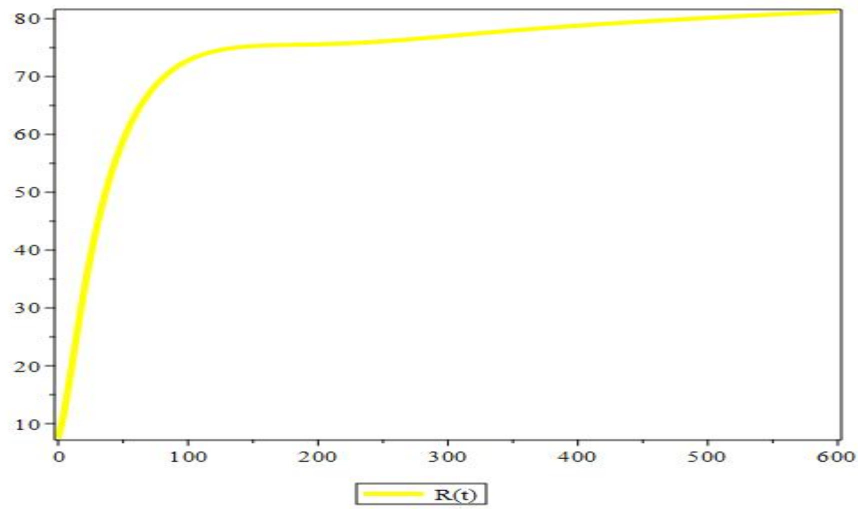

Figure 17: The behavior of the monkeypox model for recovered individuals using the NSFD method

Table 5: Population numbers over time with the NSFD method

t	S(t)	V(t)	I(t)	R(t)
0	36.787	32.004	7.535	7.503
1	35.820	25.909	13.784	8.364
2	34.209	17.250	23.216	9.264
3	31.707	7.773	34.297	10.236
4	28.344	1.613	42.819	11.306
5	24.631	0.169	46.867	12.474
6	21.130	0.155	49.187	13.720
7	18.008	0.123	51.081	15.025
8	15.274	0.101	52.526	16.378
9	12.919	0.083	53.546	17.767
10	10.918	0.069	54.181	19.180
50	2.053	0.023	24.103	58.957
100	4.167	0.111	9.326	72.768
200	7.379	0.340	5.531	75.565
300	7.371	0.313	6.190	76.980

All calculations and figures in this section have been obtained using the Maple program.

5.Conclusions

In our study, a mathematical model incorporating vaccination data for the epidemiological disease monkeypox was developed. The SVIR mathematical model, which includes susceptible, vaccinated, infected, and recovered individuals, was designed. The immune systems of individuals after vaccination and infection were analyzed using numerical methods and graphs. For the numerical analysis, Euler, Runge-Kutta, Central difference, and Nonstandard finite difference methods were used. When comparing the data obtained from the methods, it was observed that all solutions of the system were similar. This indicates that in such an epidemiological disease, the proposed system with appropriate parameter values can assist in the analysis. This work makes a significant contribution to the literature in this field.

According to the current data for the first six months of 2024 in Africa, it was observed that susceptible individuals in this study reached a disease-free equilibrium point with the given parameter values. The number of infected individuals, however, approached zero as the vaccination rate increased, which was evident in all numerical solutions. Thus, as seen from the recovered individuals' graph, it is predicted that vaccination could prevent deaths in a short time interval.

References

Anderson, R. M. (1991). Infectious diseases of humans: dynamics and control. Oxford University Press.

Arenas, A., Cota, W., Gómez-Gardenes, J., Gómez, S., Granell, C., Matamalas, J. T., ... & Steinegger, B. (2020). A mathematical model for the spatiotemporal epidemic spreading of COVID19. MedRxiv, 2020-03.

Bacaër, N. (2015). On the stochastic SIS epidemic model in a periodic environment. Journal of mathematical biology, 71(2), 491-511.

Brauer, F. (2017). Mathematical epidemiology: Past, present, and future. Infectious Disease Modelling, 2(2), 113-127.

Breman, J. G., Steniowski, M. V., Zanotto, E., Gromyko, A. I., & Arita, I. (1980). Human monkeypox, 1970-79. Bulletin of the World Health Organization, 58(2), 165.

Burden, R. L., & Faires, J. D. (2010). Numerical analysis. Brooks/Cole Cengage Learning, Boston, MA.

Butcher, J. C. (2000). Numerical methods for ordinary differential equations in the 20th century. Journal of Computational and Applied Mathematics, 125(1-2), 1-29.

Çengel, Y. & Pan, J. (2013). Diferansiyel Denklemler. (Tahsin Engin, Çev. Ed.). İzmir: İzmir Güven Kitapevi.

Çetinkaya, İ. T., Kocabıyık, M., & Ongun, M. Y. (2021). Stability Analysis of Discretized Model of Glucose–Insulin

Homeostasis. Celal Bayar University Journal of Science, 17(4), 369-377.

Çetinkaya, İ. T. (2023). An Application of Nonstandard Finite Difference Method to a Model Describing Diabetes Mellitus and Its Complications. Journal of New Theory, (45), 105-119.

Diekmann, O., Heesterbeek, H., & Britton, T. (2013). Mathematical tools for understanding infectious disease dynamics (Vol. 7). Princeton University Press.

Elsonbaty, A., Adel, W., Aldurayhim, A., & El-Mesady, A. (2024). Mathematical modeling and analysis of a novel monkeypox virus spread integrating imperfect vaccination and nonlinear incidence rates. Ain Shams Engineering Journal, 15(3), 102451.

Hethcote, H. W. (2000). The mathematics of infectious diseases. SIAM review, 42(4), 599-653.

Kermack, W. O., & McKendrick, A. G. (1927). A contribution to the mathematical theory of epidemics. Proceedings of the royal society of london. Series A, Containing papers of a mathematical and physical character, 115(772), 700-721.

Kocabıyık, M., Ongun, M. Y., & Çetinkaya, İ. T. (2021). Standart olmayan sonlu fark metodu ile dağılımlı mertebeden SVIR modelinin nümerik analizi. Balıkesir Üniversitesi Fen Bilimleri Enstitüsü Dergisi, 23(2), 577-591.

Kocabıyık, M., & Ongun, M. Y. (2023). Discretization and Stability Analysis for a Generalized Type Nonlinear Pharmacokinetic Models. Gazi University Journal of Science, 36(4), 1675-1691.

Kocabiyik, M., & Ongun, M. Y. (2024). Distributed order hantavirus model and its nonstandard discretizations and stability analysis. Mathematical Methods in the Applied Sciences.

Mickens, R. E. (1989). Exact solutions to a finite-difference model of a nonlinear reaction-advection equation: Implications for numerical analysis. Numerical Methods for Partial Differential Equations, 5(4), 313-325.

Mickens, R. E. (2002). Nonstandard finite difference schemes for differential equations. Journal of Difference Equations and Applications, 8(9), 823-847.

Nuzzo, J. B., Borio, L. L., & Gostin, L. O. (2022). The WHO declaration of monkeypox as a global public health emergency. Jama, 328(7), 615-617.

Ongun, M. Y., Arslan, D., & Farzi, J. (2017). Numerical solutions of fractional order autocatalytic chemical reaction model. Süleyman Demirel Üniversitesi Fen Bilimleri Enstitüsü Dergisi, 21(1), 165-172.

Oruç, B. & Sondaş, A. (2018). Sayısal Çözümleme. İstanbul: Umuttepe yayınları.

Peter, O. J., Kumar, S., Kumari, N., Oguntolu, F. A., Oshinubi, K., & Musa, R. (2022a). Transmission dynamics of Monkeypox virus: a mathematical modelling approach. Modeling Earth Systems and Environment, 1-12.

Peter, O. J., Oguntolu, F. A., Ojo, M. M., Olayinka Oyeniyi, A., Jan, R., & Khan, I. (2022b). Fractional order mathematical model of monkeypox transmission dynamics. Physica Scripta, 97(8), 084005.

Runge, C. (1895). Über die numerische Auflösung von Differentialgleichungen. Mathematische Annalen, 46(2), 167-178.

T.C. Sağlık Bakanlığı (2024). Birden fazla ülkede görülen Maymun Çiçek hastalığı (MPOX) salgını 2024. (28/11/2024 tarihinde https://www.seyahatsagligi.gov.tr/Site/HaberDetayi/3963 adresinden ulaşılmıştır.)

Usman, S., & Adamu, I. I. (2017). Modeling the transmission dynamics of the monkeypox virus infection with treatment and vaccination interventions. Journal of Applied Mathematics and Physics, 5(12), 2335.

CHAPTER III

On The Geometry Of Pseudo-Slant Submanıfolds In Bronze Rıemannıan Manıfolds

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An $f-$ structure on a manifold is a $(1,1)-$ tensor field of constant rank, first intoduced by Yano in (Yano & Kon, 1984). It satisfying the equation $f^3+f=0$, and this concept generalizes both almost contact and almost complex structures. Lather, Goldberg and yano extended this concept by examining a polynomial structure of degree l for a $(1,1)$ tensor field f of constant rank on $\widetilde{M}$ satisfied the equation: (Goldberg & Yano,1970).

$$\Theta(f)=f^l+a_l f^{l-1}+a_2 f+a_1 I=0.$$

Here I is identity tensor of $(1,1)-$type, and $a_l,\ldots,a_2,a_1$ are real numbers.

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Inspired by the intriguing characteristics of the golden ratio, $\varphi = \frac{1+\sqrt{5}}{2} = 1.618$ which is a positive root of the quadratic equation $x^2 - x - 1 = 0$, a novel concept called the golden structure on manifolds was introduced and examined by Hretcanu (Hretcanu,2007).

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This approach involves the development of an associated almost product structure. Subsequently, Crasmareanu and collaborators (Crasmareanu & Hreţcanu, 2008) delved into the field of golden differential geometry, uncovering several significant findings. Inspired by the silver ratio $\theta = 1 + \sqrt{2} = 2.414$ a positive root of $x^2 - 2x - 1 = 0$. In 2016, Ozkan et al. (Özkan & Peltek,

2016) introduced a new structure on manifolds known as the silver structure.

Building on the ideas of golden and silver structures on manifolds, we have recently explored the concept of a bronze structure on manifolds (Pandey & Sameer, 2018) inspired by the bronze ratio $\psi = \frac{3+\sqrt{13}}{2} = 3.302$, which is the positive root of the equation $x^2 - 3x - 1 = 0$. Notably, the golden ratio is recognized for its exceptionally slow convergence, making it the most "irrational" of all irrational numbers (Spinadel,Vera & Jose, 1996) This unique property enhances the appeal of studying the silver and bronze ratios, as their faster convergence characteristics introduce intriguing mathematical properties and make their exploration particularly compelling.

In this paper, we assume that all manifolds, connections, and tensor fields are differentiable and belong to the specified class.

In the late 20th century, Chen developed the concept of slant submanifolds within the framework of almost Hermitian manifolds, as detailed in references (Chen,1990) A. Lotta later expanded this idea to contact metric manifolds (Lotta, 1996), and Cabrerizo et al. extended it further to include slant submanifolds of K-contact and Sasakian manifolds (Cabrerizo et all, 2000). Moreover, semi-slant and slant submanifolds of metallic Riemannian manifolds were examined in (Hretcanu & Blaga, 2018).

The concept of semi-slant submanifolds within an almost Hermitian manifold was initially introduced by Papagiuc (Papaghuic 2009). Similarly, hemi-slant submanifolds were first presented by A. Carrizo. These submanifolds were also described as pseudo-slant submanifolds. More recently, Dirik and his colleagues have studied pseudo-slant submanifolds in various manifolds (Dirik & Atçeken, 2014), (Dirik & Atçeken, 2016).

This paper investigates pseudo-slant submanifolds in the context of Bronze Riemannian manifolds. Section 2 introduces fundamental definitions and concepts. Section 3, explores key findings regarding submanifolds in Riemannian manifolds endowed with a Bronze structure. In Section 4, we provide a detailed characterization of pseudo-slant submanifolds within Bronze Riemannian manifolds. The paper concludes with illustrative examples of non-trivial pseudo-slant submanifolds in these manifolds.

2.Preliminers

In this section, we introduce certain definitions and notations related to Bronze Riemannian manifolds.

Definition 1. Let $\widetilde{M}$ be a C^{∞} -manifold. If a tensor field φ of type (1,1) satisfies the equation

$$\varphi^2 = \varphi + I$$

then φ is called a golden structure on $\widetilde{M}$ and , $(\widetilde{M}, \varphi)$ is the golden manifold (Hretcanu, 2007).

Definition 2. Let $\widetilde{M}$ be a C^{∞} -manifold. A $(1,1)$- tensor field θ that satisfies the equation

$$\theta^2 = 2\theta + I$$

is referred to as a Bronze structure on $\widetilde{M}$ and $(\widetilde{M}, \theta)$ is the silver manifold. (Özkan, 2016).

Definition 3. Let $\widetilde{M}$ be a C^{∞} -manifold. A $(1,1)$- tensor field ψ that satisfies the equation

$$\psi^2 = 3\psi + I \qquad (2.1)$$

is referred to as a Bronze structure on $\widetilde{M}$ and $(\widetilde{M}, \psi)$ is the Bronze manifold. Where I denotes the identity map (Pandey & Sameer, 2018).

Proposition 1.

i) ψ and $3-\psi$ are the eigenvalues of the bronze structure .

ii) The bronze structure ψ is an isomorphism on $T_p\widetilde{M}$,∀p ∈$\widetilde{M}$.

iii) Consequently, ψ is invertible and its inverse $\psi^{-1} = \widehat{\Psi}$ verifies the following:

$$\widehat{\Psi}^2 = -3\widehat{\Psi} + I$$

We now present the following theorem, which demonstrates a connection between the bronze structure and the almost product structure of the manifold $\widetilde{M}$ (Pandey & Sameer, 2018).

If ψ represents a Bronze structure on a manifold $\widetilde{M}$, then the expression

Theorem 1. Let P represent an almost product structure. In this case, P defines a bronze structure on the manifold as follows.

$$\psi = \frac{1}{2}(3I + \sqrt{13}P)$$

moreover, if ψ denotes a bronze structure on , then

$$P = \frac{1}{\sqrt{13}}(2\psi - 3I)$$

gives an almost product structure on manifold $\widetilde{M}$ (Pandey & Sameer, 2018).

Consider P as an almost product structure on a manifold $\widetilde{M}$, and g as a Riemannian metric satisfying:

$$g(PX, PY) = g(X, Y) \tag{2.2}$$

for any $X, Y \in \Gamma(T\widetilde{M})$.

Alternatively, P can be considered as a g-symmetric tensor, defined as:

$$g(PX, Y) = g(X, PY) \tag{2.3}$$

for any $X, Y \in \Gamma(T\widetilde{M})$. Here, (g, P) is called a Riemannian almost product structure .

ψ is referred to as the Bronze structure. If the Riemannian metric g is ψ harmonious, then $(\widetilde{M}, g, \psi)$ is called a bronze Riemannian manifold (Pandey & Sameer, 2018). For $\psi -$ harmonious metric, we get

$$g(\psi X, Y) = g(X, \psi Y) \tag{2.4}$$

for any $X, Y \in \Gamma(T\ \widetilde{M})$. If the interchange X and ψX in (2.4), then , we have

$$g(\psi X, \psi Y) = g(\psi^2 X, Y) = g(3\psi X + X, Y)$$

$$= 3g(\psi X, Y) + g(X, Y). \tag{2.5}$$

Example 1. Let $\mathbb{R}^4$ denote the Euclidean 4-space with standard coordinates $(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4)$. Consider ψ a (1,1)-tensor field defined on $\mathbb{R}^4$.

$$\psi(u_1, \mathrm{u}, u_3, u_4) = (\psi u_1, \psi u_2, (3 - \psi)u_3, (3 - \psi)u_4)$$

for any vector field $(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4) \in \mathbb{R}^4$, where $\psi = \frac{3+\sqrt{13}}{2}$ and $3 - \psi = \frac{3-\sqrt{13}}{2}$ are the roots of $x^2 - 3x - 1 = 0$. To understand the structure of this tensor, we can look at its matrix representation. The tensor field ψ maps the vector field as follows, corresponding to the matrix B:

$$B=\begin{pmatrix} \psi & 0 & 0 & 0 \\ 0 & \psi & 0 & 0 \\ 0 & 0 & 3-\psi & 0 \\ 0 & 0 & 0 & 3-\psi \end{pmatrix}$$

The eigenvalues of this matrix are ψ and $3-\psi$. Then we obtain Thus, we have $\psi^2 - 3\psi - I = 0$. Moreover, we get

$$\langle \psi(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4), (t_1, t_2, t_3, t_4) = \langle (\psi u_1, \psi u_2, (3-\psi)u_3, (3-\psi)u_4), (t_1, t_2, t_3, t_4)\rangle$$

$$= \psi u_1 t_1 + \psi u_2 t_2 + (3-\psi)u_3 t_3 + (3-\psi)u_4 t_4$$

$$= \psi t_1 u_1 + \psi t_2 u_2 + (3-\psi)t_3 u_3 + (2-\psi)t_4 u_4$$

$$= \langle (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4), (\psi t_1, \psi t_2, (3-\psi)t_3, (3-\psi)t_4)\rangle$$

$$= \langle (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4), \psi(t_1, t_2, t_3, t_4)\rangle$$

On the other hand,

$$\begin{aligned} \langle \psi X, \psi Y\rangle = \langle \psi^2 X, Y\rangle = \langle 3\psi X + X, Y\rangle &= \langle 3\psi(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4) + (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4), (t_1, t_2, t_3, t_4)\rangle \\ &= \Big\langle \big(3(\psi u_1, \psi u_2, (3-\psi)u_3, (3-\psi)u_4) + (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4)\big), (t_1, t_2, t_3, t_4)\Big\rangle, \end{aligned}$$

$$=\langle (3(\psi u_1, \psi u_2, (3-\psi)u_3, (3-\psi)u_4), (t_1, t_2, t_3, t_4))\rangle + \langle (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4), (t_1, t_2, t_3, t_4)\rangle$$

$$=\langle 3\psi(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4), (t_1, t_2, t_3, t_4)\rangle + \langle (\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4), (t_1, t_2, t_3, t_4)\rangle$$

$$= \langle 3\psi X, Y\rangle + \langle X, Y\rangle.$$

for each vector fields $(\mathrm{u}_1, \mathrm{u}_2, \mathrm{u}_3, \mathrm{u}_4)$, $(t_1, t_2, t_3, t_4) \in \mathbb{R}^4$. Hence, $(\mathbb{R}^4, \langle , \rangle, \psi)$ is a Bronze Riemannian manifold.

Theorem 2. Let $(\widetilde{\mathrm{M}}, g, \psi)$ represent a Bronze Riemannian manifold. The Bronze structure ψ is said to be integrable $\Leftrightarrow \widetilde{\nabla}\psi = 0$,

3. Submanifolds of a Bronze Riemannian Manifold

Submanifolds of a Bronze Riemannian manifold are structures that preserve the geometric and metric properties of the manifold, characterized by a special tensor structure related to the Bronze ratio.

Let M be a submanifold of a Beonze Riemannian manifold $(\widetilde{M}, g, \psi)$, here g metric on M. Furthermore, let ∇ and $\nabla^{\perp}$ be the connections on TM and $T^{\perp}M$ of M, respectively. In this context, the Gauss and Weingarteen formulas can be stated as follows:

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{3.1}$$

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^{\perp} V, \tag{3.2}$$

for all $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$.

σ and A_V are connected by the following relationship.

$$g(A_V X, Y) = g(V, \sigma(X, Y)) \tag{3.3}$$

for all $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$. The mean curvature vector H of M is given by

$$H = \frac{1}{m}\sum_{i=1}^{m} \sigma(e_i, e_i) \tag{3.4}$$

Here m =dim(M), $sp\{e_1, e_2, \dots, e_m\}$ is a local orthonormal frame of M.

Let (M, g) be a submanifold of a Bronze Riemannian manifold $(\widetilde{M}, g, \psi)$. The submanifold M is said to be totally umbilical if σ satisfies

$$\sigma(X, Y) = g(X, Y)H, \tag{3.5}$$

for all $X, Y \in \Gamma(TM)$, here H is the mean curvature vector. A submanifold M is said to be totally geodesic if the second fundamental form $\sigma = 0$, and the manifold M is said to be minimal if $H = 0$.

Let (M, g) be a submanifold of a Bronze Riemannian manifold $(\widetilde{M}, g, \psi)$. Then, we get

$$\psi X = TX + NX, \tag{3.6}$$

In this context, TX represents the tangential part, while NX denotes the normal part of ψX, for all $X \in \Gamma(TM)$.

Similary , we get

$$\psi V = tV + nV, \tag{3.7}$$

In this context, tV represents the tangential part, while nV denotes the normal part of ψ, for all $V \in \Gamma(T^{\perp}M)$.

Proposition 2. Let M be a submanifold of Bronze Riemannian manifold $(\widetilde{\mathrm{M}}, g, \psi)$. Then, we get

$$g(TX, Y) = g(X, TY), \tag{3.8}$$

$$g(nW, V) = g(W, nV), \tag{3.9}$$

$$g(NX, V) = g(X, tV)$$

for any $X, Y \in \Gamma(TM)$ and for $W,\ V \in \Gamma(TM)^{\perp}$.

From (2.5) , we easily see that

$$g(TX, TY) + g(NX, NY) = g(X, Y) + 3g(TX, Y). \qquad (3.10)$$

Thus by using (2.1), (3.6) and (3.7), we obtain

$$T^2X = 3TX + X - tNX,\ 3NX = NTX + nNX \qquad (3.11)$$

and

$$3tV = TtV + tnV,\ n^2V = 3nV + V - NtV. \qquad (3.12)$$

If M is $\psi -$ invariant, thus $N = 0$. Thus from (3.11) and (3.12), we obtain

$$T^2 = 3T + I,\ n^2 = 3n + I. \qquad (3.13)$$

Therefore, (T, g) and (n, g) forms a bronze structure on M.

Here, the covariant derivatives of $T,\ N,\ t$ and n are defined as follows:

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \qquad (3.14)$$

$$(\nabla_X N)Y = \nabla_X^{\perp} NY - N\nabla_X Y, \qquad (3.15)$$

$$(\nabla_X t)V = \nabla_X tV - t\nabla_X^{\perp} V \qquad (3.16)$$

and

$$(\nabla_X n)V = \nabla_X^{\perp} nV - n\nabla_X^{\perp} V. \qquad (3.17)$$

For any $X, Y \in \Gamma(TM)$.

Through direct calculations, the following formulas are obtained:

$$(\nabla_X T)Y = A_{NY}X + t\sigma(X,Y) \tag{3.18}$$

and

$$(\nabla_X N)Y = n\sigma(X,Y) - \sigma(X,TY). \tag{3.19}$$

Similary, for all $V \in \Gamma(T^{\perp}M)$, we have

$$(\nabla_X t)V = A_{nV}X - TA_V X \tag{3.20}$$

and

$$(\nabla_X n)V = -\sigma(tV,X) - NA_V X. \tag{3.21}$$

Corollary 1. Let M be a submanifold of a Bronze Riemannian manifold $(\widetilde{M}, g, \psi)$. If M are $\psi -$ anti invariant and invariant submanifold, the following properties are satisfied:

If M is $\psi -$ invariant submanifold	If M is $\psi -$anti$-$ invariant submanifold
$N = 0$	$T = 0,$
$(\nabla_X T)Y = t\sigma(X,Y),$	$(\nabla_X N)Y = n\sigma(X,Y),$
$(\nabla_X n)V = -\sigma(tV,X)$	$(\nabla_X t)V = A_{nV}X,$
$n\sigma(X,Y) = \sigma(X,TY)$	$A_{NY}X = -t\sigma(X,Y)$
$A_{nV}Y = A_V TY$	$A_{NY}Z = -A_{NZ}Y$

for all $X, Y, Z \in \Gamma(TM)$, for any $V \in \Gamma(T^{\perp}M)$.

4. Pseudo-Slant Submanifolds of a Bronze Riemannian Manifold.

Some characterizations of pseudo-slant submanifolds in a Bronze Riemannian manifold have been provided.

Definition 2. Let (M, g) be a submanifold of a Bronze Riemannian manifold $(\widetilde{M}, g, \psi)$. For each $X \neq 0$ tangential to M at x, the angle $\beta(x) \in \left[0, \frac{\pi}{2}\right]$, between ψX and $T_x M$ is called the slant angle of M. If this slant is constant, the submanifold is known as a slant submanifold. When $\beta = 0$ the submanifold is called an invariant submanifold, and when $\beta = \frac{\pi}{2}$, it is called an anti-invariant submanifold. If the slant angel $\beta(x) \in (0, \frac{\pi}{2})$ then the submanifold is classified as a proper-slantsubmanifold (Cabrerizo & et al, 2000).

Theorem 3. Let (M, g) be a submanifold of a Bronze Riemannian manifold $(\widetilde{M}, g, \psi)$. M is considered a slant submanifold ⟺there exists a constant $\varrho \in [0,1]$ such that:

$$T^2 = \varrho(3\psi + I), \tag{4.1}$$

and

$$\psi^2 = \frac{1}{\varrho} T^2 \tag{4.2}$$

furthermore, if β slant angel of M , then $\varrho = cos^2\beta$ (Cabrerizo & et al, 2000).

Lemma 1. Let (M, g) be a submanifold of a Bronze Riemannian manifold $(\widetilde{M}, g, \psi)$. Then, we have

$$g(TX, TY) = cos^2\beta\{g(X, Y) + 3g(X, TY)\} \tag{4.3}$$

and

$$g(NX, NY) = sin^2\beta\{g(X, Y) + 3g(TX, Y)\}. \qquad (4.4)$$

for all X, Y ∈ Γ(TM).

Proof. From (3.8) and (4.1), we can conclude that

$$g(TX, TY) = g(X, T^2Y) = \varrho g(X, 3\psi Y + Y$$
$$= cos^2\beta\{g(X, Y) + 3g(X, TY)\}$$

The equations in (3.10) and (4.3) result in

$$g(NX, NY) = g(X, Y) + 3g(X, TY) - g(TX, TY)$$

$$= g(X, Y) + 3g(X, TY)$$

$$-cos^2\beta\{g(X, Y) + 3g(X, TY)\}$$

$$= sin^2\beta\{g(X, Y) + 3g(TX, Y)\}.$$

Definition 3 . Let M be a submanifold of a Bronze Riemannian manifold $(\widetilde{M}, g, \psi)$. M is pseudo-slant submanifold if there exist two orthogonal distributions D_β and $D^\perp$, exist on M such that

(1) The tangent bundle TM has an orthogonal direct sum decomposition expressed as

$$TM = D^\perp \oplus D_\beta,$$

(2) $D^\perp$is anti-invariant, which means that $\psi D^\perp \subset T^\perp M$,

(3) D_β is a slant, $\beta \neq \frac{\pi}{2}$, implying that the angle between D_β and $\psi(D_\beta)$ remains constant (Khan & Khan, 2007).

Remartk 1. Let us asume that M is a pseudo slant submanifold of a Bronze Riemannian manifold $(\tilde{M}, g, \psi)$.

Let p =dim($D^{\perp}$) and q=dim(D_{β}). We can distinguish the following six cases:

(1) When $\mathrm{q} = 0$, M is anti-invariant,
(2) If $p = 0$ and $\beta = 0$, then M is invariant.
(3) If $\mathrm{p} = 0$ and $\beta \in (0, \frac{\pi}{2})$, then M is classified as proper slant.
(4) When $\beta = \frac{\pi}{2}$, M is anti-invariant.
(5) If $\mathrm{p} \neq 0$ and q≠0 with $\beta = 0$, then M is a semi-invariant.
(6). If $\mathrm{p} \neq 0$ and q≠0 with $\beta \in (0, \frac{\pi}{2})$, then M is consi dered pseudo-slant.

Let μ, represent the orthogonal complement of ψTM in $T^{\perp}M$. In this case, $T^{\perp}M$ can be expressed as the following decomposition:

$$T^{\perp}M = \psi TM \oplus \mu = ND^{\perp} \oplus ND_{\beta} \oplus \mu, \; ND_{\beta} \perp ND^{\perp}. \tag{4.5}$$

Definition 4. Let (M, g) be a submanifold of a Bronze Riemannian manifold $(\tilde{\mathrm{M}}, g, \psi)$. The submanifold is called D_{β}-geodesic (or $\mathrm{D}^{\perp}$-geodesic) if $\sigma(\mathrm{X}, \mathrm{Y}) = 0$ for any $\mathrm{X}, \mathrm{Y} \in \Gamma(\mathrm{D}_{\beta})$ (or $\sigma(\mathrm{Z}, \mathrm{U}) = 0$ for any $\mathrm{Z}, \mathrm{U} \in \Gamma(\mathrm{D}^{\perp})$, respectively). If for any $\mathrm{X} \in \Gamma(\mathrm{D}_{\beta})$ and $\mathrm{U} \in \Gamma(\mathrm{D}^{\perp})$, $\sigma(\mathrm{X}, \mathrm{U}) = 0$, then M is called $D^{\perp} - D_{\beta}$ mixed geodesic submanifold.

In the following sections, we will use "PS" instead of the term "pseudo-slant.

Theorem 4. Let M be a PS submanifold of a locally Bronze Riemannian manifold $(\widetilde{M}, g, \psi)$. The distribution D_β −integrable, and its leaves are D_β −geodesic in M if the following condition holds:

$$g(\sigma(X, Y), \psi U) = 0$$

for any $X, Y \in \Gamma(D_\beta)$ and $Z \in \Gamma(D^\perp)$.

Proof. Asume that the distribution D_β is integrable, and each leaf of D_β is D_β- geodesic in M. Additionally, $\nabla_X Y \in \Gamma(D_\beta)$ for all $X, Y \in \Gamma(D_\beta)$ and $U \in \Gamma(D^\perp)$. Using the result from (3.1), we obtain the following:

$$\begin{aligned} g(\sigma(X, Y), \psi U) &= g(\tilde{\nabla}_X Y - \nabla_X Y, \psi U) \\ &= g(\tilde{\nabla}_X Y, \psi U) = g(\psi \tilde{\nabla}_X Y, U) \\ &= g(\tilde{\nabla}_X \psi Y - (\tilde{\nabla}_X \psi) Y, U). \end{aligned}$$

Using the result from Theorem 1, we obtain the following:

$$g(\sigma(X, Y), \psi U) = g(\tilde{\nabla}_X \psi Y, U)$$

using equation (3.1), we derive the following:

$$\begin{aligned} g(\sigma(X, Y), \psi U) &= g(\nabla_X \psi Y + \sigma(X, \psi Y), U) = g(\nabla_X \psi Y, U) \\ &= g(\nabla_X Y, \psi U) = 0 \end{aligned}$$

for all $X, Y \in \Gamma(D_\theta)$ and $U \in \Gamma(D^\perp)$.

Theorem 5. Let M be a PS submanifold of a locally Bronze Riemannian manifold $(\widetilde{M}, g, \psi)$. D_β is integrable if he following condition hold:

$$g(\nabla_X U, 3\psi Y) - g(\nabla_Y U, 3\psi X$$
$$= g(\sigma(X, \psi Y), \psi U) - g(\sigma(Y, \psi X), \psi U)$$

for any $X, Y \in \Gamma(D_\theta)$ and $U \in \Gamma(D^\perp)$. ψ

Proof. From (2.1), (3.1), (3.2) and (3.3), we can conclude the following:

$$g(\sigma(X, \psi Y), \psi U) = g\big(A_{\psi U} X, \psi Y\big)$$

$$= g(\nabla_X^\perp \psi U, \psi Y) - g(\tilde{\nabla}_X \psi U, \psi Y)$$

$$= g(\nabla_X^\perp \psi U, \psi Y) - g\left(\left(\tilde{\nabla}_X \psi\right) U, \psi Y\right)$$

$$-g(\psi \tilde{\nabla}_X U, \psi Y)$$

$$= -g(\tilde{\nabla}_X U, \psi^2 Y) = -g(\tilde{\nabla}_X U, (3\psi\ + I) Y)$$

$$= -g(\nabla_X U, 3\psi Y) - g(\nabla_X U, Y).$$

Thus,

$$g(\nabla_X Y, U) = g(\sigma(X, \psi Y), \psi U) + g(\nabla_X U, 3\psi Y)$$

in the equation above, if we replace X with $Y,$ we obtain the following:

$$g(\nabla_Y X, U) = g(\sigma(Y, \psi X), \psi U) + g(\nabla_Y U, 3\psi X).$$

If we subtract the two equations side by side, we get the following:

$$g(\nabla_X Y, U) - g(\nabla_Y X, U) = g(\sigma(X, \psi Y), \psi U) + g(\nabla_X U, 3\psi Y)$$
$$-g(\sigma(Y, \psi X), \varphi U) - \ g(\nabla_Y U, 3\psi X),$$

$$g(U,[X,Y]) = g(\nabla_X U, 3\psi Y) + g(\sigma(X,\psi Y),\psi U)$$
$$-g(\nabla_Y U, 3\psi X) - g(\sigma(Y,\psi X),\psi U).$$

Since D_β is integrable, it follows that:

$$g(\nabla_X U, 3\psi Y) - g(\nabla_Y U, 3\psi X)$$
$$= g(\sigma(X,\psi Y),\psi U) - g(\sigma(Y,\psi X),\psi U).$$

Theorem 6. Let M be a PS submanifold of a locally Bronze Riemannian manifold $(\widetilde{\mathrm{M}}, g, \psi)$. n is parallel ⇔ A_{V} satisfies the condition:

$$\mathrm{A_V tU} = -\mathrm{A_U tV}$$

for any $\mathrm{V}, \mathrm{U} \in \Gamma(\mathrm{T}^{\perp}\mathrm{M})$.

Proof. If n is parallel, then $\nabla n = 0$. From (3.3) and (3.21) , we obtain the following:

$$0 = g(\sigma(tV,X) + NA_V X, U)$$
$$= g(A_U tV, X) + g(A_V X, tU)$$
$$= g(A_U tV + A_V tU, X)$$

for any $V,\ U \in \Gamma(T^{\perp}M)$ and for any $X \in \Gamma(TM)$.

Theorem 7. Let M be a PS submanifold of a locally Bronze Riemannian manifold $(\widetilde{\mathrm{M}}, g, \psi)$. If N is parallel, In this case, M is either a mixed geodesic or an anti-invariant submanifold.

Proof. t is parallel ⟺ if N is parallel, If t is parallel, then $\nabla t = 0$. which means M is invariant. For all $X \in \Gamma(D_\beta), Z \in \Gamma(D^\perp)$, $V \in \Gamma(T^\perp M)$. We can conclude this from (3.19) and (3.20).

$$A_{nV}X - TA_V X = 0,$$

$$\begin{aligned} 0 &= g(A_{nV}X - TA_V X, Z) \\ &= g(\sigma(X,Z), nV) - g(TA_V X, Z) \\ &= g(\sigma(X,Z), nV) - g(\sigma(X,TZ), V) \\ &= g(n\sigma(X,Z), V) - g(\sigma(X,TZ), V), \end{aligned}$$

so

$n\sigma(X,Z) = \sigma(X,TZ)$

for $Z \in \Gamma(D^\perp)$, we have $TZ = 0$. Therefore, it follows that:

$n\sigma(X,Z) = 0$

By replacing X with TX in the above equation, we obtain

$n\sigma(TX,Z) = 0.$

By replacing X with TX in the above equation and using (4.1) , we have

$$n\sigma(T^2X,Z) = ncos^2\beta\sigma((3\psi + I)X,Z) = 0.$$

Thus, we conclude that either $\sigma = 0$ (indicating that M is mixed geodesic) or $cos\beta = 0$ which leads to $\beta = \frac{\pi}{2}$ (indicating M is anti-invariant).

Theorem 8 . Let M be a totally umbilical PS submanifold of a locally Bronze Riemannian manifold $(\widetilde{M}, g, \psi)$. If N is parallel, In this case, M can be classified as either a minimal, an anti-invariant submanifold.

Proof. N is parallel $\Leftrightarrow$ t is parallel, If t is parallel, then $\nabla t = 0.$ which means M is invariant. For all $X \in \Gamma(D_\theta), Y \in D^\perp$, $W \in \Gamma(T^\perp M)$. We can conclude this from (3.19) and (3.20).

$$A_{nW}X - TA_W X = 0,$$

so

$$\begin{aligned} 0 &= g(A_{nW}X - TA_W X, Y), \\ &= g(\sigma(X,Y), nW) - g(TA_W X, Y) \\ &= g(\sigma(X,Y), nW) - g(\sigma(X,TY), W) \\ &= g(n\sigma(X,Y), W) - g(\sigma(X,TY), W) \end{aligned}$$

for $Y \in \Gamma(D^\perp)$, we have $TY = 0$. Therefore, it follows that:

$$n\sigma(X,Y) = 0.$$

By replacing X with TX in the above eq. we obtain

$$n\sigma(TX,Y) = 0.$$

Therefore, since M is totally umbilical submanifold, we can refer to the findings in (3.5)

$$ng(TX,Y)H = 0.$$

Replacing X by TX in the above eq. and using (4.1) , we have

$$ng(T^2X,Y)H = ncos^2\beta g((3\psi + I)X,Y)H$$

$$= ncos^2\beta\{g(3\psi X,Y) + g(X,Y)\}H$$

$$= ncos^2\beta\{g(X,Y)\}H = 0.$$

Hence we conclude that either $H = 0$ (indicating that M is minimal), or $cos\beta = 0$ which leads to $\beta = \frac{\pi}{2}$ (indicating that M is anti-invariant).

Theorem 9. Let M be a PS submanifold in a locally Bronze Riemannian manifold $(\widetilde{\mathrm{M}}, g, \psi)$. $\mathrm{D}^{\perp}$ is integrable $\Leftrightarrow \mathrm{A}_{\mathrm{ND}^{\perp}}\mathrm{D}^{\perp} = 0$,

for all $\mathrm{Z}, \mathrm{U} \in \Gamma(\mathrm{D}^{\perp})$.

Proof. If M is a pseudo-slant submanifold in a locally Bronze Riemannian manifold (M, g, ψ). Then, for all $Z, U \in \Gamma(D^{\perp})$, we have $TZ = TU = 0$, which implies $\nabla_Z TU = \nabla_U TZ = 0$.

By using (3.14), we get $T([Z,U]) = 0$ if and only if $A_{NZ}U = A_{NU}Z$ holds, for all $U \in \Gamma(D^{\perp})$. From (3.14), for all $X \in \Gamma(TM)$ and $Z, U \in \Gamma(D^{\perp})$, we get

$$g\big((\nabla_X T)Z, U\big) = g(A_{NZ}X, U) + g(th(X,Z), U) = -g(\nabla_X Z, TU)$$
$$= 0,$$

which implies $g(A_{NZ}X, U) = -g(th(X,Z), U)$. From

$$g(A_{NZ}X, U) = g(A_{NZ}U, X) = g(A_{NU}Z, X) = g(h(X,Z),$$
$$= g(th(X,Z), U),$$

we obtain $g(A_{NZ}U, X) = 0$ for all $X \in \Gamma(TM)$ and $Z, U \in \Gamma(D^{\perp})$. so, $A_{NZ}U = 0$, for all $Z, U \in \Gamma(D^{\perp})$.

Conversely, if $A_{NZ}U = 0$, for all $Z, U \in \Gamma(D^{\perp})$ then from

$$g(th(X,Z),U) = g(h(X,Z),NU) = g(A_{NU}Z,X) = 0$$

and (3.14), we get

$$0 = g\big((\nabla_Z T)U, X\big) = -g(T\nabla_Z U, X) = -g(\nabla_Z U, TX),$$

for any $Z, U \in \Gamma(D^{\perp})$, $X \in \Gamma(D_\theta)$. From $T(D_\theta) = D_\theta$, we obtain $\nabla_Z U \in \Gamma(D^{\perp})$ which implies $[Z, U] \in \Gamma(D^{\perp})$.

Theorem 10. Let M be a PS submanifold in a locally Bronze Riemannian manifold $(\widetilde{M}, g, \psi)$. In this case, $D^{\perp}$ is integrable $\Leftrightarrow$

$$(\nabla_W T)U = (\nabla_U T)W$$

for all $W, U \in \Gamma(D^{\perp})$.

Proof. For all $W, U \in \Gamma(D^{\perp})$. Using (3.18) , we obtain

$$(\nabla_W T)U = A_{NU}W + t\sigma(W,U) \tag{4.6}$$

Replacing W by U in the above equation, we have

$$(\nabla_U T)W = A_{NW}U + t\sigma(U,W) \qquad (4,7)$$

Then, (4.6), (4.7) and from Theorem 9, we arrive at the conclusion.

Finally, let's support the topic with an example.

Example 2 . To construct a PS submanifold of a Bronze Riemannian manifold based on the provided parametrization $\chi(u,v)$, we first need to analyze the given mapping and then define the associated Riemannian structure.the mapping is defined as:

$$\chi(u,v) = (u\sin\alpha, -v\sin\alpha, (3-\cos\alpha)u, (3+\cos\alpha)v)$$

this mapping $\chi: \mathbb{R}^2 \rightarrow \mathbb{R}^4$ defines a submanifold M in $\mathbb{R}^4$. To ensure M is a submanifold of $\mathbb{R}^4$,we will consider the tangent vectors and the embedding The parametriziation consist of two parameters (u, v). Next, we calculate the tangent vector of M by differentiating χ with respect to each parameter:

$$\mathrm{E}_1 = \frac{\partial \chi}{\partial \mathrm{u}} = (\sin\alpha, 0, \ 3 - \cos\alpha, 0)$$

$$\mathrm{E}_2 = \frac{\partial \chi}{\partial \mathrm{v}} = (0, -\cos\alpha, 0, 3 + \cos\alpha)$$

For the Bronze Riemannian structure ψ of $\mathbb{R}^4$, the coordinat system is given by (x_1, y_1, x_2, y_2).

$$\psi(\frac{\partial}{x_i}) = \frac{\partial}{y_i}, \psi(\frac{\partial}{y_j}) = \frac{\partial}{x_j}, \qquad 1 \leqslant i, j \leqslant 2$$

then we obtain

$$\psi E_1 = (0, sin\alpha, 0, 3 - cos\alpha)$$

$$\psi E_2 = (-cos\alpha, 0, 3 + cos\alpha, 0),$$

$$g(\psi \mathrm{E}_1, \mathrm{E}_2) = g(\mathrm{E}_1, \psi \mathrm{E}_2),$$

$$g(\psi \mathrm{E}_1, \psi \mathrm{E}_2) = 3g(\psi \mathrm{E}_1, \mathrm{E}_2) + g(\mathrm{E}_1, \mathrm{E}_2).$$

Thus, this structure is observed to be a bronze structure.

Through direct calculations, we determine that $\mathrm{D}_\beta = \mathrm{Sp}\{\mathrm{E}_1, \mathrm{E}_2\}$ defines a slant distribution with a slant angle of

$$\cos\beta = \frac{g(E_{1,}\varphi E_2)}{\|E_{1,}\|\|\varphi E_2\|} = \frac{8}{\sqrt{10-6\cos\alpha}.\sqrt{10+6\cos\alpha}}$$
$$= \frac{8}{\sqrt{100-36cos\alpha}}$$

$$\beta = \arccos\left(\frac{8}{\sqrt{100-36cos\alpha}}\right).$$

$-1 \leq cos\alpha \leq 1,\ 64 \leq 100 - 36cos\alpha \leq 136,$

Consequently, M is a 2-dinensionel invariant or proper PS submanifold of $\mathbb{R}^4$ endowed with its standard Bronze Riemannian structure.

Conclusion . In this study of PS submanifolds within the framework of a Bronze Riemannian manifold, we have explored their unique geometric properties and the intrinsic connections to the Bronze ratio. PS submanifolds reveal fascinating characteristics, particularly in how their tangent spaces interact with the ambient manifold's structure.

REFERENCES

Atceken, M., & Dirik, S. (2014). On the geometry of pseudo-slant submanifolds of a Kenmotsu manifold. *Gulf journal of Mathematics*, *2*(2).

Akpinar, R. C. (2020). On bronze Riemannian ian structures. *Tbilisi Mathematical Journal*, *13*(3), 161-169.

Acet, B. E., & Acet, T. (2022).Some Types of Slant Submanifold of Bronze Riemaniann Manifolds. *The Pure and Applied Mathematics, 29*(4), 277-291.

Cabrerizo, J. L., Carriazo, A., Fernandez, L. M., & Fernandez, M. (2000). Slant submanifolds in Sasakian manifolds. *Glasgow Mathematical Journal*, *42*(1), 125-138.

Chen, B. Y. (1990). Slant immersions. *Bulletin of the Australian Mathematical Society*, *41*(1), 135-147.

Chen, B. Y. (1990). Geometry of slant submanifolds, Katholieke Univ. *Leuven, Louvain*, *1*, 990.

Crasmareanu, M., & Hreţcanu, C. E. (2008). Golden differential geometry. *Chaos, Solitons & Fractals*, *38*(5), 1229-1238..

Dirik, S., & Atçeken, M. (2016). Pseudo-slant submanifolds in Cosymplectic space forms. *Acta Universitatis Sapientiae, Mathematica*, *8*(1), 53-74.

Goldberg, S. I., & Yano, K. (1970). Polynomial structures on manifolds. In *Kodai Mathematical Seminar Reports* (Vol. 22, No.

2, pp. 199-218). Department of Mathematics, Tokyo Institute of Technology.

Hretcanu, C. E. (2007). Submanifolds in Riemannian ian manifold with Golden structure. In *Workshop on Finsler geometry and its applications, Hungary*..

Hretcanu, C. E., & Blaga, A. M. (2018). Slant and Semi-Slant Submanifolds in Metallic Riemannian ian Manifolds. *Journal of Function Spaces*, *2018*(1), 2864263.

Khan, V. A., & Khan, M. A. (2007). Pseudo-slant submanifolds of a Sasakian manifold. *Indian Journal of pure and applied Mathematics*, *38*(1), 31.

Lotta, A. (1996). Slant submanifolds in contact geometry. *Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie*, 183-198.

Özkan, M., & Peltek, B. (2016). A new structure on manifolds: silver structure. *International Electronic Journal of Geometry*, *9*(2), 59-69.

Özkan, M. (2014). Prolongations of golden structures to tangent bundles. *Differential Geometry-Dynamical Systems*, *16*.

Pandey, P. K. Sameer (2018). Bronze differential geometry. Journal of Science and Arts Year 18, No. 4(45), pp. 973-980, 2018

Pandey, P.K., Sameer, (2018). Bronze differential geometry, National Conference on Differential Geometry and Its Applications, India, Papaghuic N., (2009). Semi-slant submanifolds of a Kaehlarian manifold, *An. St. Univ. Al. I. Cuza. Univ. Iasi.* 40,55-61

Spinadel, D., Vera, W., Jose, P.M., (1996). Humanistic Mathematics Network Journal, 1(19), article no. 14.

Yano, K. (1982). On a Structure Defined by a Tensor Field f of Type (1, 1) Satisfying $f^3 + f = 0$. In *North-Holland Mathematics Studies* (Vol. 70, pp. 251-261). North-Holland.

CHAPTER IV

Separation Axioms on Fuzzy Parameterized Fuzzy Hypersoft Topological Spaces

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1. Introduction

In disciplines such as sociology, economics, climate science, and engineering, traditional mathematical methods often fail to address complex problems due to their inherent intricacies. To overcome these challenges, Fuzzy Set Theory, introduced by Zadeh (1965), has proven to be a highly effective approach. It provides a robust framework for representing vague concepts through partial membership. The theory has been extensively explored by both mathematicians and computer scientists, leading to the development

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of fuzzy control systems, fuzzy logic, fuzzy topology, and other related fields. Numerous applications of fuzzy set theory have been developed over time. In parallel, Probability Theory and Rough Set Theory, proposed by Pawlak (1982), also aim to tackle similar issues. Molodtsov (1999) introduced Soft Set Theory, offering an entirely novel approach to modeling uncertainty, with each of these theories carrying its own set of complexities. Molodtsov laid the groundwork for Soft Set Theory, applying it in areas such as function smoothness, operations analysis, Riemann integration, and game theory. The foundational aspects of this theory were further explored by Maji et al. (2001), with subsequent developments by Pei and Miao (2005), Feng et al. (2008), Chen et al. (2005), Aktaş and Çağman (2007), Ali et al. (2009), and Ozturk and Bayramov (2014). The exploration of both fuzzy sets and soft sets was initiated by Maji et al. (2001). The combination of soft set and fuzzy set structures, referred to as fuzzy soft sets, has been widely adopted by researchers, contributing significantly to the literature (Maji et al., 2001; Pei, 2005). Pei (2005) emphasized that scholars from diverse fields have studied information systems and highlighted the close relationships between soft sets and information systems. Moreover, Pei demonstrated that soft sets can be viewed as a class of specialized information systems, termed fuzzy information systems, and that research on soft sets and information systems can be integrated, potentially leading to the discovery of new outcomes and methodologies. Kharal and Ahmad (2009) introduced the concept of mapping fuzzy soft sets to enhance fuzzy soft theory, focusing on the properties of fuzzy soft images and inverse images of fuzzy soft sets. Additionally, Çağman et al. (2010) investigated fuzzy parameterized fuzzy soft set theory and its applications. Tanay and Kandemir (2011) explored fuzzy soft topology on a given initial universe, defining various notions of fuzzy soft topological spaces and examining their properties. Roy and Samanta (2012) defined fuzzy soft topology over the initial universe, introducing bases and subbases for this space and providing characterizations. Smarandache (2018) advanced new methods for uncertainty management by extending soft set theory to hypersoft sets, which involve a transformation into multi-decision methods. The hypersoft set structure represents a more generalized form of soft sets, consisting of elements selected from different attributes. Due to its practical applications, substantial research (Smarandache, 2018; Maji et al., 2009; Yolcu & Ozturk, 2021) has emerged in a short time. Yolcu and Ozturk (2024) further developed the fuzzy hypersoft set structure by merging fuzzy and hypersoft set structures.

In this paper, the concepts of fuzzy parameterized fuzzy hypersoft adherent points and fuzzy parameterized fuzzy hypersoft interior points are defined within fuzzy parameterized fuzzy hypersoft topological spaces. Their relationships with the fuzzy parameterized fuzzy hypersoft closure set and the fuzzy parameterized fuzzy hypersoft interior set are explored. Additionally, hereditary properties are presented, with examples, in relation to the separation axioms in fuzzy parameterized fuzzy hypersoft topological spaces.

2. Preliminaries

Definition 2.1 *(Zadeh,1965) Let Π be a initial universe. A fuzzy set Λ in Π, $\Lambda = \{(\pi, \mu_\Lambda(\pi)) : \pi \in \Pi\}$, where $\mu_\Lambda : \Pi \to 0,1]$ is the membership function of the fuzzy set Λ; $\mu_\Lambda(\pi) \in 0,1]$ is the membership $\pi \in \Pi$ in Λ. The set of all fuzzy sets ove Π will be denoted by $FP(\Pi)$.*

Definition 2.2 *(Molodtsov, 1999) Let Π be an initial universe and E be a set of parameters. A pair $(\aleph, E)$ is called a soft set over Π, where F is a mapping $\aleph: E \to \mathcal{P}(\Pi)$. In other words, the soft set is a parameterized family of subsets of the set Π.*

Definition 2.3 *(Maji et.al,2001) Let Π be a initial universe, E be a set of parameters and $FP(\Pi)$ be the set of all fuzzy sets in Π. Then a pair $(\aleph, E)$ is called a fuzzy soft set over Π, where $\aleph: E \to FP(\Pi)$ is a mapping.*

Definition 2.4 *(Smarandache,2018) Let Π be the universal set and $P(\Pi)$ be the power set of Π. Consider $e_1, e_2, e_3, \ldots, e_n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attribute values are resspectively the sets $E_1, E_2, \ldots, E_n$ with $E_i \cap E_j = \emptyset$, for $i \neq j$ and $i, j \in \{1,2, \ldots, n\}$, then the pair $(\aleph, E_1 \times E_2 \times \ldots \times E_n)$ is said to be Hypersoft set over Π where $\aleph: E_1 \times E_2 \times \ldots \times E_n \to P(\Pi)$.*

Definition 2.5 *(Rahman et.al.,2022) Let Π be the universal set and $FP(\Pi)$ be a family of all fuzzy set over Π and $E_1, E_2, \ldots, E_n$ the pairwise disjoint sets of parameters. Let A_i be the nonempty subset of E_i for each $i = 1,2, \ldots, n$. A fuzzy parameterized fuzzy hypersoft set defined as the pair $(\aleph, A_1 \times A_2 \times \ldots \times A_n)$ where; $\aleph: A_1 \times A_2 \times \ldots \times A_n \to FP(\Pi)$ and*

$$\aleph(A_1 \times A_2 \times \ldots \times A_n) = \left\{ \begin{matrix} < \dfrac{\alpha}{\mu(\alpha)}, \dfrac{\pi}{\aleph(\alpha)(\pi)}, >: \pi \in \Pi, \\ \alpha \in A_1 \times A_2 \times \ldots \times A_n \subseteq E_1 \times E_2 \times \ldots \times E_n \end{matrix} \right\}$$

For sake of simplicity, we write the symbols Σ for $E_1 \times E_2 \times \ldots \times E_n$, for $A_1 \times A_2 \times \ldots \times A_n$ and α for an element of the set . The set of all fuzzy hypersoft sets over Π will be denoted by $FPFHS(\Pi, \Sigma)$. Here after, FHS will be used for short instead of fuzzy hypersoft sets.

Definition 2.6 *(Rahman et.al.,2022) Let Π be the universal set and $FP(\Pi)$ be a family of all fuzzy set over Π and $E_1, E_2, \ldots, E_n$ the pairwise disjoint sets of parameters. Let A_i be the nonempty subset of E_i for each $i = 1,2,\ldots,n$. A fuzzy parameterized fuzzy hypersoft set defined as the pair $(\aleph, A_1 \times A_2 \times \ldots \times A_n)$ where; $\aleph: A_1 \times A_2 \times \ldots \times A_n \to FP(\Pi)$ and*

$$\aleph(A_1 \times A_2 \times \ldots \times A_n) = \begin{Bmatrix} < \dfrac{\alpha}{\mu(\alpha)}, \dfrac{\pi}{\aleph(\alpha)(\pi)}, >: \pi \in \Pi, \\ \alpha \in A_1 \times A_2 \times \ldots \times A_n \subseteq E_1 \times E_2 \times \ldots \times E_n \end{Bmatrix}$$

Definition 2.7 *(Rahman et.al.,2022)*

i) A fuzzy hypersoft set $(\aleph, \Delta)$ over the universe Π is said to be null fuzzy parameterized fuzzy hypersoft set and denoted by $0^{fp}_{(\Pi_{FH}, \Delta)}$ if for all $\pi \in \Pi$ and $\alpha \in \Delta$, ,$\mu(\alpha) = 0, \aleph(\alpha)(\pi) = 0$.

ii) A fuzzy hypersoft set $(\aleph, \Delta)$ over the universe Π is said to be absolute fuzzy parameterized fuzzy hypersoft set and denoted by $1^{fp}_{(\Pi_{FH}, \Delta)}$ if for all $\pi \in \Pi$ and $\alpha \in \Delta$, ,$\mu(\alpha) = 1, \aleph(\alpha)(\pi) = 1$.

Definition 2.8 *(Rahman et.al.,2022) Let Π be an initial universe set $(\aleph_{1,1})$, $(\aleph_{2,2})$ be two fuzzy parameterized fuzzy hypersoft sets over the universe Π. We say that $(\aleph_{1,1})$ is a fuzzy parameterized fuzzy hypersoft subset of $(\aleph_{2,2})$ and denote $(\aleph_{1,1}) \widetilde{\subseteq} (\aleph_{2,2})$ if*

i) $_1 \subseteq _2$

ii) For any $\alpha \in _1$, ,$\mu_1(\alpha) \subset \mu_2(\alpha), \aleph_1(\alpha) \subseteq \aleph_2(\alpha)$.

Definition 2.9 *(Rahman et.al.,2022) The complement of fuzzy parameterized fuzzy hypersoft set* $(\aleph,)$ *over the universe* Π *is denoted by* $(\aleph,)^c$ *and defined as* $(\aleph,)^c = (\aleph^c,)$, *where* $\aleph^c(n)$ *is complement of the set* $\aleph(n)$, *for* $n \in$.

$$\aleph(A_1 \times A_2 \times \ldots \times A_n)^c = \begin{Bmatrix} < \dfrac{\alpha}{1-\mu(\alpha)}, \dfrac{\pi}{1-\aleph(\alpha)(\pi)}, >: \pi \in \Pi, \\ \alpha \in A_1 \times A_2 \times \ldots \times A_n \subseteq E_1 \times E_2 \times \ldots \times E_n \end{Bmatrix}$$

Definition 2.10 *(Rahman et.al.,2022) Let* Π *be an initial universe set and* $(\aleph_{1,1})$, $(\aleph_{2,2})$ *be two fuzzy parameterized fuzzy hypersoft sets over the universe* Π. *The union of* $(\aleph_{1,1})$ *and* $(\aleph_{2,2})$ *is denoted by* $(\aleph_{1,1}) \tilde{\cup} (\aleph_{2,2}) = (\aleph_{3,3})$ *where each* $n \in_3, \aleph_3(n)(\pi) = max\{\ \aleph_1(n)(\pi), \aleph_2(n)(\pi)\}, \mu_3(n) = max\{\mu_1(n), \mu_2(n)\}$

Definition 2.11 *(Rahman et.al.,2022) Let* Π *be an initial universe set and* $(\aleph_{1,1})$, $(\aleph_{2,2})$ *be fuzzy parameterized fuzzy hypersoft sets over the universe* Π. *The intersection of* $(\aleph_{1,1})$ *and* $(\aleph_{2,2})$ *is denoted by* $(\aleph_{1,1}) \tilde{\cap} (\aleph_{2,2}) = (\aleph_{3,3})$ *where* $_3 =_1 \cap_2$ *and each* $n \in_3, \aleph_3(n)(\pi) = min\{\ \aleph_1(n)(\pi), \aleph_2(n)(\pi)\}, \mu_3(n) = min\{\mu_1(n), \mu_2(n)\}$.

Definition 2.12 *(Yolcu and Ozturk,2024a) Let* $FPFHS(\Pi, \Delta)$ *be the set of all fuzzy parameterized fuzzy hypersoft subsets of* (Π, Δ) *over the universe* Π *and* $\tilde{\tau}$ *be a subfamily of* $FPFHS(\Pi, \Delta)$. *Then* $\tilde{\tau}$ *is called a fuzzy parameterized fuzzy hypersoft topology on* Π *if the following condition are satisfied.*

1. $0^{fp}_{(\Pi_{FH}, \Delta)}$ and $1^{fp}_{(\Pi_{FH}, \Delta)}$ belongs to $\tilde{\tau}$,

2. The union of any number of fuzzy parameterized fuzzy hypersoft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$,

3. The intersection of any two fuzzy parameterized fuzzy hypersoft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$.

The triple $(\Pi, \tilde{\tau}, \Delta)$ is called a fuzzy parameterized fuzzy hypersoft topological space over Π. Every member of $\tilde{\tau}$ is called a fuzzy parameterized fuzzy hypersoft open set (FPFHOS) in Π.

Definition 2.13 *(Yolcu and Ozturk,2024a) Let $FPFHS(\Pi, \Delta)$ be the set of all fuzzy hypersoftsets over the universe Π. Then,*

1. If $\tilde{\tau} = \{0^{fp}_{(\Pi_{FH},\Delta)}, 1^{fp}_{(\Pi_{FH},\Delta)}\}$, then $\tilde{\tau}$ is called to be fuzzy parameterized fuzzy hypersoft indiscrete topology and $(\Pi, \tilde{\tau}, \Delta)$ is called to be fuzzy parameterized fuzzy hypersoft indiscrete topological space over the universe Π.

2. If $\tilde{\tau} = FPFHS(\Pi, \Delta)$, then $\tilde{\tau}$ is called to be fuzzy parameterized fuzzy hypersoft discrete topology and $(\Pi, \tilde{\tau}, \Delta)$ is called to be fuzzy parameterized fuzzy hypersoft discrete topological space over the universe Π.

Definition 2.14 *(Yolcu and Ozturk,2024a) Let $(\Pi, \tilde{\tau}, \Delta)$ be a fuzzy parameterized fuzzy hypersoft topological space over Π and $(\aleph,)$ be a fuzzy parameterized fuzzy hypersoft set over Π. Then $(\aleph,)$ is said to be a fuzzy parameterized fuzzy hypersoft closed set (FPFHCS) if its complement $(\aleph,)^c$ belongs to $\tilde{\tau}$.*

Definition 2.15 *(Yolcu and Ozturk,2024a) Let $(\Pi, \tilde{\tau}, \Delta)$ be a fuzzy parameterized fuzzy hypersoft topological space over Π and $(\aleph,)$ be a fuzzy parameterized fuzzy hypersoft set over Π. The fuzzy parameterized fuzzy hypersoft closure of $(\aleph,)$ denoted by $cl_{FPFH}(\aleph,)$ is the intersection of all fuzzy parameterized fuzzy hypersoft closed super sets of $(\aleph,)$.*

It is clear that $cl_{FPFH}(\aleph,)$ is the smallest fuzzy parameterized fuzzy hypersoft closed set over Π which contain $(\aleph,)$.

Definition 2.16 *(Yolcu and Ozturk,2024a) Let $(\Pi, \tilde{\tau}, \Delta)$ be a fuzzy parameterized fuzzy hypersoft topological space over Π and $(\aleph,)$*

be a fuzzy parameterized fuzzy hypersoft set over Π. The fuzzy parameterized fuzzy hypersoft interior of $(\aleph,)$ denoted by $int_{FPFH}(\aleph,)$ is the union of all fuzzy parameterized fuzzy hypersoft open subsets of $(\aleph,)$.

It is clear that $int_{FPFH}(\aleph,)$ is the largest fuzzy parameterized fuzzy hypersoft open set contained in $(\aleph,)$.

Definition 2.17 *(Yolcu and Ozturk,2024b) Let $(\Pi, \tilde{\tau}, \Delta)$ be a fuzzy parameterized fuzzy hypersoft topological space over Π and $\tilde{B} \subseteq \tilde{\tau}$. $\tilde{B}$ is called a fuzzy parameterized fuzzy hypersoft basis for the fuzzy parameterized fuzzy hypersoft topology $\tilde{\tau}$ if every element of $\tilde{\tau}$ can be written as the fuzzy parameterized fuzzy hypersoft union of elements of $\tilde{B}$.*

3. Fuzzy Parameterized Fuzzy Hypersoft Point

In this section, we will present the structure on fuzzy parameterized fuzzy hypersoft topological (FPFHT) structures such as interior, closure by using fuzzy parameterized fuzzy hypersoft points (FPFHPs) and their neighbourhoods.

Definition 3.1 *Let $\subset \Delta$, $\alpha \in$ and $\pi \in \Pi$. A FHS $(\aleph,)$ is said to be a fuzzy parameterized fuzzy hypersoft point (briefly, FPFHP) if $\aleph(\alpha')$ is a null fuzzy set for every $\alpha' \in \backslash\{\alpha\}$ and $\aleph(\alpha)(y) = 0$ for all $y \neq \pi$. We will denote $(\aleph,)$ simply by $P_{\mu(\alpha)}^{(\alpha,\pi)}$ and denote all the fuzzy parameterized fuzzy hypersoft points over Π simply by $FPFHP(\Pi, \Delta)$.*

Definition 3.2 *A FPFHP $P_{\mu(\alpha)}^{(\alpha,\pi)}$ is said to belong to a FHS $(\aleph,)$ if $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\subseteq} (\aleph,)$. We write it as $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\aleph,)$.*

It is clear that the fuzzy parameterized fuzzy hypersoft union of FPFHPs of a FHS $(\aleph,)$ returns the FHS $(\aleph,)$, that is,

$$(\aleph,) = \tilde{\cup}\, \{P_{\mu(\alpha)}^{(\alpha,\pi)} : P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\aleph,)\}.$$

Definition 3.3 *Let $(\Pi, \tilde{\tau}, \Delta)$ be a fuzzy parameterized fuzzy hypersoft topological space over Π. A FHS $(\aleph,)$ over Π is called a fuzzy parameterized fuzzy hypersoft neighbouthood of the FPFHP $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\subseteq} (\aleph,)$, if there exist a FHOS (Ξ, Δ) such that $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\Xi, \Delta) \tilde{\subseteq} (\aleph,)$. The neighbourhood system of a FPFHP $P_{\mu(\alpha)}^{(\alpha,\pi)}$, denoted by $\vartheta\left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)$, is the family of all its neighbourhoods.*

Theorem 3.1 *Let $(\Pi, \tilde{\tau}, \Delta)$ be a fuzzy parameterized fuzzy hypersoft topological space over Π and $(\aleph,)$ be a fuzzy parameterized fuzzy hypersoft set over Π. Then $(\aleph,)$ is a FHOS iff $(\aleph,)$ is a fuzzy parameterized fuzzy hypersoft neighbourhood of its each FHPs.*

Proof. Suppose that $(\aleph,)$ be a FHOS over Π and $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\aleph,)$. Then $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\aleph,) \tilde{\subseteq} (\aleph,)$. Hence, $(\aleph,)$ is a fuzzy parameterized fuzzy hypersoft neighbourhood of $P_{\mu(\alpha)}^{(\alpha,\pi)}$.

Conversely, let $(\aleph,)$ be a fuzzy parameterized fuzzy hypersoft neighbourhood of its each FHPs and $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\aleph,)$. Then, there exist $(\Xi, \Delta) \tilde{\in} \tilde{\tau}$ such that $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\Xi, \Delta) \tilde{\subseteq} (\aleph,)$. Since $(\aleph,) = \tilde{\cup}\, \{P_{\mu(\alpha)}^{(\alpha,\pi)} : P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\aleph,)\}$, it follows that $(\aleph,)$ is a union of FPFHOSs and hence $(\aleph,)$ is a FHOS.

Theorem 3.2 *The neighbourhood system $\vartheta\left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)$ at $P_{\mu(\alpha)}^{(\alpha,\pi)}$ in a FPFHTS $(\Pi, \tilde{\tau}, \Delta)$ has the following properties:*

1. If $(\aleph,) \tilde{\in} \vartheta\left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)$, then $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\aleph,)$,

2. If $(\aleph,) \tilde{\in} \vartheta\left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)$ and $(\aleph,) \tilde{\subseteq} (\Xi,\Delta)$, then $(\Xi,\Delta) \tilde{\in} \vartheta\left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)$,

3.If $(\aleph_{1,1}), (\aleph_{2,2}) \tilde{\in} \vartheta\left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)$, then $(\aleph_{1,1}) \tilde{\cap} (\aleph_{2,2}) \tilde{\in} \vartheta\left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)$,

4. If $(\aleph,) \tilde{\in} \vartheta\left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)$, then there exist a $(\aleph_{1,1}) \tilde{\in} \vartheta\left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)$ such that $(\aleph_{1,1}) \tilde{\in} \vartheta\left(P_{FH}^{(\beta,y)}\right)$ for every $P_{FH}^{(\beta,y)} \tilde{\in} (\aleph_{1,1})$.

Proof. The proof of (1),(2) and (3) is obvious from the Definition 20.

(4) If $(\aleph,) \tilde{\in} \vartheta\left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)$ then there exist a fuzzy parameterized fuzzy soft open set $(\aleph_{1,1})$ such that $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\aleph_{1,1}) \tilde{\subseteq} (\aleph,)$. Therefore, $(\aleph_{1,1}) \tilde{\in} \vartheta\left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)$, so for each $P_{\mu(\beta)}^{(\beta,y)} \tilde{\in} (\aleph_{1,1})$, $(\aleph_{1,1}) \tilde{\in} \vartheta\left(P_{\mu(\beta)}^{(\beta,y)}\right)$ is obtained.

Definition 3.4 *Let $P_{\mu(\alpha)}^{(\alpha,\pi)}$ and $P_{\mu(\beta)}^{(\beta,y)}$ be two FPFHPs over the common universe Π. Then, we say that the FPFHPs are distinct FPFHPs if $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\cap} P_{\mu(\beta)}^{(\beta,y)} = 0_{(\Pi_{FH},\Delta)}$.*

It is clear that $P_{\mu(\alpha)}^{(\alpha,\pi)}$ and $P_{\mu(\beta)}^{(\beta,y)}$ are dinstinct FPFHPs if and only if $\pi \neq y$ or $\alpha \neq \beta$.

Definition 3.5 *Let $(\Pi, \tilde{\tau}, \Delta)$ be a fuzzy parameterized fuzzy hypersoft topological space over Π. Let $(\aleph,)$ be a FHS over Π and $P_{\mu(\alpha)}^{(\alpha,\pi)}$ be a FPFHP over Π.*

1. $P_{\mu(\alpha)}^{(\alpha,\pi)}$ is a fuzzy parameterized fuzzy hypersoft interior point of $(\aleph,)$, if $(\Xi,\Delta) \tilde{\subseteq} (\aleph,)$ for some $(\Xi,\Delta) \tilde{\in} \vartheta\left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)$,

2. $P_{\mu(\alpha)}^{(\alpha,\pi)}$ is a fuzzy parameterized fuzzy hypersoft adherent point of $(\aleph,)$, if $(\Xi,\Delta) \tilde{\cap} (\aleph,) \neq 0_{(\Pi_{FH},\Delta)}$ for any $(\Xi,\Delta) \tilde{\in} \vartheta\left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)$.

Theorem 3.3 *Let* $(\Pi, \tilde{\tau}, \Delta)$ *be a fuzzy parameterized fuzzy hypersoft topological space over* Π *and* $(\aleph,)$ *be a FPFHS over* Π *.*

1. $int_{FH}(\aleph,) = \tilde{\cup}\{P_{\mu(\alpha)}^{(\alpha,\pi)} : P_{\mu(\alpha)}^{(\alpha,\pi)}$ is a fuzzy parameterized fuzzy hypersoft interior point of $(\aleph,)\}$,

2. $cl_{FH}(\aleph,) = \tilde{\cup}\{P_{\mu(\alpha)}^{(\alpha,\pi)} : P_{\mu(\alpha)}^{(\alpha,\pi)}$ is a fuzzy parameterized fuzzy hypersoft adherent point of $(\aleph,)\}$.

Proof. Straightforward.

Theorem 3.4 *Let* $(\Pi, \tilde{\tau}, \Delta)$ *be a fuzzy parameterized fuzzy hypersoft topological space over* Π*,* $(\aleph,)$ *be a FPFHS over* Π *and* $\tilde{B}$ *be a basis for* $(\Pi, \tilde{\tau}, \Delta)$. *Then,*

$$(\aleph,) \tilde{\in} \tilde{\tau}$$
$$\Leftrightarrow \forall P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} FPFHP(\Pi, \Delta), \exists (\Xi, \Delta) \tilde{\in} \tilde{B} \text{suchthat} P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\Xi, \Delta) \tilde{\subseteq} (\aleph,).$$

Proof. $(\Rightarrow)$ Suppose that $(\aleph,) \tilde{\in} \tilde{\tau}$ and $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} FPFHP(\Pi, \Delta)$. Since $\tilde{B}$ is a basis for $(\Pi, \tilde{\tau}, \Delta)$, there exists $\tilde{B}' \tilde{\subseteq} \tilde{B}$ such that $(\aleph,) = \tilde{\cup}\{(\Xi, \Delta) : (\Xi, \Delta) \in \tilde{B}'\}$. Moreover, there exists $(\Xi, \Delta) \in \tilde{B}'$ such that $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\Xi, \Delta)$ for $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\aleph,)$. Hence $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\Xi, \Delta) \tilde{\subseteq} (\aleph,)$.

$(\Leftarrow)$ Assume that sufficient conditions of the theorem are provided. So,

$$(\aleph,)$$
$$= \tilde{\cup}\{P_{\mu(\alpha)}^{(\alpha,\pi)} : P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\subseteq} (\aleph,)\} \tilde{\subseteq} \tilde{\cup}\{(\Xi, \Delta) : P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\Xi, \Delta) \tilde{\subseteq} (\aleph,)\} \tilde{\subseteq} \tilde{\cup}(\aleph,).$$

Thus, $(\aleph,) \tilde{\in} \tilde{\tau}$.

4. Fuzzy Parameterized Fuzzy Hypersoft Separation Axioms

Definition 4.1 *a) Let* $(\Pi, \tilde{\tau}, \Delta)$ *be a FPFHTS over* Π *and* $P_{\mu(\alpha)}^{(\alpha,\pi)}$ *and* $P_{\mu(\beta)}^{(\beta,y)}$ *be two distinct FPFHPs over the common universe* Π. *If there exist FPFHOSs* $(\aleph_{1,1})$ *and* $(\aleph_{2,2})$ *such that*

$$P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\aleph_{1,1}) \quad \text{and } P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\cap} (\aleph_{2,2}) = 0_{(\Pi_{FH},\Delta)} \text{or}$$

$$P_{\mu(\beta)}^{(\beta,y)} \tilde{\in} (\aleph_{2,2}) \quad \text{and } P_{\mu(\beta)}^{(\beta,y)} \tilde{\cap} (\aleph_{1,1}) = 0_{(\Pi_{FH},\Delta)}$$

then $(\Pi, \tilde{\tau}, \Delta)$ is called a fuzzy parameterized fuzzy hypersoft T_0 −spaces.

b)

$$P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\aleph_{1,1}) \quad \text{and } P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\cap} (\aleph_{2,2}) = 0_{(\Pi_{FH},\Delta)} \text{and}$$

$$P_{\mu(\beta)}^{(\beta,y)} \tilde{\in} (\aleph_{2,2}) \quad \text{and } P_{\mu(\beta)}^{(\beta,y)} \tilde{\cap} (\aleph_{1,1}) = 0_{(\Pi_{FH},\Delta)}$$

then $(\Pi, \tilde{\tau}, \Delta)$ is called a fuzzy parameterized fuzzy hypersoft T_1 −spaces.

c) If there exist FPFHOSs $(\aleph_{1,1})$ and $(\aleph_{2,2})$ such that $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\aleph_{1,1})$, $P_{\mu(\beta)}^{(\beta,y)} \tilde{\in} (\aleph_{2,2})$ and $(\aleph_{1,1}) \tilde{\cap} (\aleph_{2,2}) = 0_{(\Pi_{FH},\Delta)}$ then $(\Pi, \tilde{\tau}, \Delta)$ is called a fuzzy parameterized fuzzy hypersoft T_2 −spaces.

Example 4.1 *Let* $\{\pi_1, \pi_2\}$ *be a universe set,* $E_1 = \{l_1, l_2\}$ *and* $E_2 = \{l_3, l_4\}$ *be two attributes set. Suppose that*

$$P_{0.2}^{(\alpha_1,\pi_1)} = \{< \frac{(l_1,l_3)=\alpha_1}{0.2}, \{\frac{\pi_1}{0,1}\} >\},$$

$$P_{0.2}^{(\alpha_1,\pi_2)} = \{< \frac{(l_1,l_3)=\alpha_1}{0.2}, \{\frac{\pi_2}{0,3}\} >\},$$

$$P_{0.3}^{(\alpha_2,\pi_1)} = \{< \frac{(l_1,l_4)=\alpha_2}{0.3}, \{\frac{\pi_1}{0{,}4}\} >\},$$

$$P_{0.3}^{(\alpha_2,\pi_2)} = \{< \frac{(l_1,l_4)=\alpha_2}{0.3}, \{\frac{\pi_2}{0{,}2}\} >\},$$

$$P_{0.4}^{(\alpha_3,\pi_1)} = \{< \frac{(l_2,l_3)=\alpha_3}{0.4}, \{\frac{\pi_1}{0{,}6}\} >\},$$

$$P_{0.4}^{(\alpha_3,\pi_2)} = \{< \frac{(l_2,l_3)=\alpha_3}{0.4}, \{\frac{\pi_2}{0{,}4}\} >\},$$

$$P_{0.6}^{(\alpha_4,\pi_1)} = \{< \frac{(l_2,l_4)=\alpha_4}{0.6}, \{\frac{\pi_1}{0{,}8}\} >\},$$

$$P_{0.6}^{(\alpha_4,\pi_2)} = \{< \frac{(l_2,l_4)=\alpha_4}{0.6}, \{\frac{\pi_2}{0{,}7}\} >\}$$

such that $\alpha_1 = (l_1, l_3), \alpha_2 = (l_1, l_4), \alpha_3 = (l_2, l_3), \alpha_4 = (l_2, l_4)$ for $\alpha_i \in E_1 \times E_2 = \Delta$. The fuzzy parameterized fuzzy hypersoft topology that accepts the family $\tilde{B}$,

$$\tilde{B} = \{P_{0.2}^{(\alpha_1,\pi_1)}, P_{0.2}^{(\alpha_1,\pi_2)}, P_{0.3}^{(\alpha_2,\pi_1)}, P_{0.3}^{(\alpha_2,\pi_2)}, P_{0.4}^{(\alpha_3,\pi_1)}, P_{0.4}^{(\alpha_3,\pi_2)}, P_{0.6}^{(\alpha_4,\pi_1)}\},$$

as the basis is

$$\tilde{\tau} = \{0_{(\Pi_{FH},\Delta)}, 1_{(\Pi_{FH},\Delta)}, (\aleph_1, \Delta), (\aleph_2, \Delta), (\aleph_3, \Delta), \ldots, (\aleph_{128}, \Delta)\}$$

where $(\aleph_1, \Delta) = \{P_{0.2}^{(\alpha_1,\pi_1)}\}$, $(\aleph_2, \Delta) = \{P_{0.2}^{(\alpha_1,\pi_2)}\}$, $(\aleph_3, \Delta) = \{P_{0.3}^{(\alpha_2,\pi_1)}\}$, $(\aleph_4, \Delta) = P_{0.3}^{(\alpha_2,\pi_2)}\}$, $(\aleph_5, \Delta) = \{P_{0.4}^{(\alpha_3,\pi_1)}\}$, $(\aleph_6, \Delta) = \{P_{0.4}^{(\alpha_3,\pi_2)}\}$, $(\aleph_7, \Delta) = \{P_{0.6}^{(\alpha_4,\pi_1)}\}$,

$(\aleph_8, \Delta) = (\aleph_1, \Delta) \tilde{\cup} (\aleph_1, \Delta), \ldots, (\aleph_{128}, \Delta) =$
$(\aleph_1, \Delta) \tilde{\cup} (\aleph_2, \Delta) \tilde{\cup} \ldots \tilde{\cup} (\aleph_7, \Delta)$. Then $\tilde{\tau}$ is a FHT over Π. It is clear that $(\Pi, \tilde{\tau}, \Delta)$ is fuzzy parameterized fuzzy hypersoft T_0 −space but not a fuzzy parameterized fuzzy hypersoft T_1 −space. Because there does not exist each FPFHOSs consisting $P_{0.6}^{(\alpha_4,\pi_2)}$ and other FPFHPs.

Example 4.2 *Let $\Pi = \mathbb{N}$ be a natural numbers set and $E_1 = \{l_1\}$, $E_2 = \{l_2\}$ be two attribute sets. Then $P_{FH}^{(\alpha,n)}$ are FPFHPs for $n \in \mathbb{N}$, $\alpha = (l_1, l_2)$. Hence, the FPFHPs $P_{FH}^{(\alpha,n)}$ and $P_{FH}^{(\alpha,m)}$ are distinct FPFHPs iff $n \neq m$. Obvious that there is one to one compatibility between the seet of natural numbers and the set of FPFHPs. Then we give cofinite topology on this set. Then FPFHS $(\aleph, \Delta)$ is a FHOS iff the finite FPFHP is discorded from the set of FPFHPs. Therefore, $(\Pi, \tilde{\tau}, \Delta)$ is a fuzzy parameterized fuzzy hypersoft T_1 −space but not a fuzzy parameterized fuzzy hypersoft T_2 −space.*

Definition 4.2 *We consider Example-4.1. The fuzzy parameterized fuzzy hypersoft topology that accepts the family $\tilde{B}$,*

$$\tilde{B} = \{P_{0.2}^{(\alpha_1,\pi_1)}, P_{0.2}^{(\alpha_1,\pi_2)}, P_{0.3}^{(\alpha_2,\pi_1)}, P_{0.3}^{(\alpha_2,\pi_2)}, P_{0.4}^{(\alpha_3,\pi_1)}, P_{0.4}^{(\alpha_3,\pi_2)}, P_{0.6}^{(\alpha_4,\pi_1)}, P_{0.6}^{(\alpha_4,\pi_2)}\},$$

as the basis is

$$\tilde{\tau} = \{0_{(\Pi_{FH},\Delta)}, 1_{(\Pi_{FH},\Delta)}, (\aleph_1, \Delta), (\aleph_2, \Delta), (\aleph_3, \Delta), \dots, (\aleph_{256}, \Delta)\}$$

where $(\aleph_1, \Delta) = \{P_{0.2}^{(\alpha_1,\pi_1)}\}$, $(\aleph_2, \Delta) = \{P_{0.2}^{(\alpha_1,\pi_2)}\}$, $(\aleph_3, \Delta) = \{P_{0.3}^{(\alpha_2,\pi_1)}\}$, $(\aleph_4, \Delta) = P_{0.3}^{(\alpha_2,\pi_2)}\}$, $(\aleph_5, \Delta) = \{P_{0.4}^{(\alpha_3,\pi_1)}\}$, $(\aleph_6, \Delta) = \{P_{0.4}^{(\alpha_3,\pi_2)}\}$, $(\aleph_7, \Delta) = \{P_{0.6}^{(\alpha_4,\pi_1)}\}$,

$(\aleph_8, \Delta) = \{P_{0.6}^{(\alpha_4,\pi_2)}\}$, $(\aleph_9, \Delta) = (\aleph_1, \Delta) \tilde{\cup} (\aleph_1, \Delta), \dots, (\aleph_{256}, \Delta) = (\aleph_1, \Delta) \tilde{\cup} (\aleph_2, \Delta) \tilde{\cup} \dots \tilde{\cup} (\aleph_8, \Delta)$. Then $\tilde{\tau}$ is a FHT over Π. It is clear that $(\Pi, \tilde{\tau}, \Delta)$ is fuzzy parameterized fuzzy hypersoft T_2 −space.

Theorem 4.1 *Let $(\Pi, \tilde{\tau}, \Delta)$ be a FPFHTS over Π. Then $(\Pi, \tilde{\tau}, \Delta)$ is a fuzzy parameterized fuzzy hypersoft T_1 −space iff each FPFHP is a FHCS.*

Proof. Suppose that $(\Pi, \tilde{\tau}, \Delta)$ is a fuzzy parameterized fuzzy hypersoft T_1 −space and $P_{\mu(\alpha)}^{(\alpha,\pi)}$ be an arbitrary FPFHP over Π. We should show that $\left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)^c$ is a FHOS. For $P_{\mu(\beta)}^{(\beta,y)} \tilde{\in} \left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)^c$; then

$P_{\mu(\alpha)}^{(\alpha,\pi)}$ and $P_{\mu(\beta)}^{(\beta,y)}$ are distinct FPFHPs. Therefore $\pi \neq y$ or $\alpha \neq \beta$. Since $(\Pi, \tilde{\tau}, \Delta)$ is a fuzzy parameterized fuzzy hypersoft T_1 −space, there exists a FHOS $(\aleph,)$ such that

$$P_{\mu(\beta)}^{(\beta,y)} \tilde{\in} (\aleph,) \text{and} P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\cap} (\aleph,) = 0_{(\Pi_{FH},\Delta)}$$

Then $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\cap} (\aleph,) = 0_{(\Pi_{FH},\Delta)}$. We have $P_{\mu(\beta)}^{(\beta,y)} \tilde{\in} (\aleph,) \subseteq \left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)^c$. Therefore, $\left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)^c$ is a FHOS, i.e. $P_{\mu(\alpha)}^{(\alpha,\pi)}$ is a FHCS.

Conversely, let each FPFHP $P_{\mu(\alpha)}^{(\alpha,\pi)}$ is a FHCS. Then $\left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)^c$ is a FHOS. Suppose that $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\cap} P_{\mu(\beta)}^{(\beta,y)} = 0_{(\Pi_{FH},\Delta)}$, then $P_{\mu(\beta)}^{(\beta,y)} \tilde{\in} \left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)^c$ and $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\cap} \left(P_{\mu(\alpha)}^{(\alpha,\pi)}\right)^c = 0_{(\Pi_{FH},\Delta)}$. So, $(\Pi, \tilde{\tau}, \Delta)$ is a fuzzy parameterized fuzzy hypersoft T_1 −space over Π.

Theorem 4.2 *Let $(\Pi, \tilde{\tau}, \Delta)$ be a FPFHTS over Π. $(\Pi, \tilde{\tau}, \Delta)$ is a fuzzy parameterized fuzzy hypersoft T_2 −space iff for distinct FPFHPs $P_{\mu(\alpha)}^{(\alpha,\pi)}$ and $P_{\mu(\beta)}^{(\beta,y)}$, there exists a FHOS $(\aleph,)$ containing $P_{\mu(\alpha)}^{(\alpha,\pi)}$ but not $P_{\mu(\beta)}^{(\beta,y)}$ such that $P_{\mu(\beta)}^{(\beta,y)} \notin cl_{FH}(\aleph,)$.*

Proof. Let $P_{\mu(\alpha)}^{(\alpha,\pi)}$ and $P_{\mu(\beta)}^{(\beta,y)}$ be two FPFHPs in fuzzy parameterized fuzzy hypersoft T_2 − space $(\Pi, \tilde{\tau}, \Delta)$. Then there exist disjoint FPFHOSs $(\aleph_{1},_{1})$ and $(\aleph_{2},_{2})$ such that $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\aleph_{1},_{1})$ and $P_{\mu(\beta)}^{(\beta,y)} \tilde{\in} (\aleph_{2},_{2})$. Since $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\cap} P_{\mu(\beta)}^{(\beta,y)} = 0_{(\Pi_{FH},\Delta)}$ and $(\aleph_{1},_{1}) \tilde{\cap} (\aleph_{2},_{2}) = 0_{(\Pi_{FH},\Delta)}$, $P_{\mu(\beta)}^{(\beta,y)} \notin (\aleph_{1},_{1})$. It implies that $P_{\mu(\beta)}^{(\beta,y)} \notin cl_{FH}(\aleph,)$.

Conversely, Suppose that for distinct FPFHPs $P_{\mu(\alpha)}^{(\alpha,\pi)}$ and $P_{\mu(\beta)}^{(\beta,y)}$, there exists a FHOS $(\aleph,)$ containing $P_{\mu(\alpha)}^{(\alpha,\pi)}$ but not $P_{\mu(\beta)}^{(\beta,y)}$ such that

$P_{\mu(\beta)}^{(\beta,y)} \notin cl_{FH}(\aleph,)$. Then $_{FH}^{(\beta,y)} \widetilde{\in} (cl_{FH}(\aleph,))^c$, i.e. $(\aleph,)$ and $(cl_{FH}(\aleph,))^c$ are disjoint FPFHOSs containing $P_{\mu(\alpha)}^{(\alpha,\pi)}$ and $P_{\mu(\beta)}^{(\beta,y)}$, respectively.

Theorem 4.3 *Let $(\Pi,\tilde{\tau},\Delta)$ be a fuzzy parameterized fuzzy hypersoft T_1 −space for every FPFHP $P_{\mu(\alpha)}^{(\alpha,\pi)} \widetilde{\in} (\aleph_{1,1}) \widetilde{\in} \tilde{\tau}$. If there exist a FHOS $(\aleph_{2,2})$ on $(\Pi,\tilde{\tau},\Delta)$ such that*

$$P_{\mu(\alpha)}^{(\alpha,\pi)} \widetilde{\in} (\aleph_{2,2}) \widetilde{\subseteq} cl_{FH}(\aleph_{2,2}) \widetilde{\subseteq} (\aleph_{1,1})$$

then $(\Pi,\tilde{\tau},\Delta)$ is a fuzzy parameterized fuzzy hypersoft T_2 −space.

Proof. Assume that $P_{\mu(\alpha)}^{(\alpha,\pi)} \widetilde{\cap} P_{\mu(\beta)}^{(\beta,y)} = 0_{(\Pi_{FH},\Delta)}$. Since $(\Pi,\tilde{\tau},\Delta)$ is a fuzzy parameterized fuzzy hypersoft T_1 −space, $P_{\mu(\alpha)}^{(\alpha,\pi)}$ and $P_{\mu(\beta)}^{(\beta,y)}$ are FPFHCSs on $(\Pi,\tilde{\tau},\Delta)$. Thus $P_{\mu(\alpha)}^{(\alpha,\pi)} \widetilde{\in} \left(P_{\mu(\beta)}^{(\beta,y)}\right)^c \widetilde{\in} \tilde{\tau}$. Then ehere exist a FHOS $(\aleph_{2,2}) \widetilde{\in} \tilde{\tau}$ such that

$$P_{\mu(\alpha)}^{(\alpha,\pi)} \widetilde{\in} (\aleph_{2,2}) \widetilde{\subseteq} cl_{FH}(\aleph_{2,2}) \widetilde{\subseteq} \left(P_{\mu(\beta)}^{(\beta,y)}\right)^c$$

So, we have $P_{\mu(\beta)}^{(\beta,y)} \widetilde{\in} (cl_{FH}(\aleph_{2,2}))^c$, $P_{\mu(\alpha)}^{(\alpha,\pi)} \widetilde{\in} (\aleph_{2,2})$ and $(\aleph_{2,2}) \widetilde{\cap} (cl_{FH}(\aleph_{2,2}))^c = 0_{(\Pi_{FH},\Delta)}$, i.e. $(\Pi,\tilde{\tau},\Delta)$ is a fuzzy parameterized fuzzy hypersoft T_2 −space.

Definition 4.3 *Let $(\Pi,\tilde{\tau},\Delta)$ be a FPFHTS over Π, $(\aleph,)$ be a FHCS on $(\Pi,\tilde{\tau},\Delta)$ and $P_{\mu(\alpha)}^{(\alpha,\pi)} \widetilde{\cap} (\aleph,) = 0_{(\Pi_{FH},\Delta)}$. If there exist FPFHOSs (Ξ_1,Δ_1) and (Ξ_2,Δ_2) such that $P_{\mu(\alpha)}^{(\alpha,\pi)} \widetilde{\in} (\Xi_1,\Delta_1)$, $(\aleph,) \widetilde{\subseteq} (\Xi_2,\Delta_2)$ and $(\Xi_1,\Delta_1) \widetilde{\cap} (\Xi_2,\Delta_2) = 0_{(\Pi_{FH},\Delta)}$, then $(\Pi,\tilde{\tau},\Delta)$ is called a fuzzy parameterized fuzzy hypersoft regular space. $(\Pi,\tilde{\tau},\Delta)$ is said to be a fuzzy parameterized fuzzy hypersoft T_3 −space if it is both fuzzy parameterized fuzzy hypersoft regular and fuzzy parameterized fuzzy hypersoft T_1 −space.*

Theorem 4.4 *Let $(\Pi,\tilde{\tau},\Delta)$ be a FPFHTS over Π. $(\Pi,\tilde{\tau},\Delta)$ is a fuzzy parameterized fuzzy hypersoft T_3- space iff for every $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\aleph,) \tilde{\in} \tilde{\tau}$, there exists $(\Xi,\Delta) \tilde{\in} \tilde{\tau}$ such that $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\Xi,\Delta) \tilde{\subseteq} cl_{FH}(\Xi,\Delta) \tilde{\subseteq} (\aleph,)$.*

Proof. Let $(\Pi,\tilde{\tau},\Delta)$ be a fuzzy parameterized fuzzy hypersoft T_3- space and $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\aleph,) \tilde{\in} \tilde{\tau}$. Since $(\Pi,\tilde{\tau},\Delta)$ is a fuzzy parameterized fuzzy hypersoft T_3 $-$space for the FPFHP $P_{\mu(\alpha)}^{(\alpha,\pi)}$ and FHCS $(\aleph,)^c$, there exist $(\Xi_1,\Delta_1),(\Xi_2,\Delta_2) \tilde{\in} \tilde{\tau}$ such that $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\Xi_1,\Delta_1), (\aleph,)^c \tilde{\subseteq} (\Xi_2,\Delta_2)$ and $(\Xi_1,\Delta_1) \tilde{\cap} (\Xi_2,\Delta_2) = 0_{(\Pi_{FH},\Delta)}$. Thus, we have $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\Xi_1,\Delta_1) \tilde{\subseteq} (\Xi_2,\Delta_2)^c \tilde{\subseteq} (\aleph,)$. Since $(\Xi_2,\Delta_2)^c$ is a FHCS, so $cl_{FH}(\Xi_1,\Delta_1) \tilde{\subseteq} (\Xi_2,\Delta_2)^c$.

Conversely, suppose that $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\cap} (\Upsilon,\Omega) = 0_{(\Pi_{FH},\Delta)}$ and (Υ,Ω) is a FHCS on $(\Pi,\tilde{\tau},\Delta)$. Thus $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\Upsilon,\Omega)^c$ and from the condition of the theorem, we have $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\Xi,\Delta) \tilde{\subseteq} cl_{FH}(\Xi,\Delta) \tilde{\subseteq} (\Upsilon,\Omega)^c$. Then $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\Xi,\Delta)$, $(\Upsilon,\Omega) \tilde{\subseteq} (cl_{FH}(\Xi,\Delta))^c$ and $(\Xi,\Delta) \cap (cl_{FH}(\Xi,\Delta))^c = 0_{(\Pi_{FH},\Delta)}$ are satisfied, i.e., $(\Pi,\tilde{\tau},\Delta)$ is a fuzzy parameterized fuzzy hypersoft T_3 $-$space.

Definition 4.4 *A FPFHTS $(\Pi,\tilde{\tau},\Delta)$ over Π is called a fuzzy parameterized fuzzy hypersoft normal space if for every pair of disjoint FPFHCSs $(\aleph_{1,1})$ and $(\aleph_{2,2})$, there exists disjoint FPFHOSs (Ξ_1,Δ_1) and (Ξ_2,Δ_2) such that $(\aleph_{1,1}) \tilde{\subseteq} (\Xi_1,\Delta_1)$ and $(\aleph_{2,2}) \tilde{\subseteq} (\Xi_2,\Delta_2)$. $(\Pi,\tilde{\tau},\Delta)$ is said to be a fuzzy parameterized fuzzy hypersoft T_4- space if it is both a fuzzy parameterized fuzzy hypersoft normal and fuzzy parameterized fuzzy hypersoft T_1- space.*

Theorem 4.5 *Let $(\Pi, \tilde{\tau}, \Delta)$ be a FPFPFHTS over Π. Then $(\Pi, \tilde{\tau}, \Delta)$ is a fuzzy parameterized fuzzy hypersoft T_4 −space iff for each FHCS $(\aleph,)$ and FHOS (Ξ, Δ) with $(\aleph,) \tilde{\subseteq} (\Xi, \Delta)$, there exists a FHOS (Υ, Ω) such that*

$$(\aleph,) \tilde{\subseteq} (\Upsilon, \Omega) \tilde{\subseteq} cl_{FH}(\Upsilon, \Omega) \tilde{\subseteq} (\Xi, \Delta).$$

Proof. Let $(\Pi, \tilde{\tau}, \Delta)$ be a fuzzy parameterized fuzzy hypersoft $T_4 -$ space, $(\aleph,)$ be a FHCS on $(\Pi, \tilde{\tau}, \Delta)$ and $(\aleph,) \tilde{\subseteq} (\Xi, \Delta) \tilde{\in} \tilde{\tau}$. Then $(\Xi, \Delta)^c$ is a FHCS and $(\aleph,) \tilde{\cap} (\Xi, \Delta)^c = 0_{(\Pi_{FH}, \Delta)}$. Since $(\Pi, \tilde{\tau}, \Delta)$ is a fuzzy parameterized fuzzy hypersoft T_4 −space, there exists FPFHOSs (Υ_1, Ω_1) and (Υ_2, Ω_2) such that $(\aleph,) \tilde{\subseteq} (\Upsilon_1, \Omega_1)$, $(\Xi, \Delta)^c \tilde{\subseteq} (\Upsilon_2, \Omega_2)$ and $(\Upsilon_1, \Omega_1) \tilde{\cap} (\Upsilon_2, \Omega_2) = 0_{(\Pi_{FH}, \Delta)}$. This implies that

$$(\aleph,) \tilde{\subseteq} (\Upsilon_1, \Omega_1) \tilde{\subseteq} (\Upsilon_2, \Omega_2)^c \tilde{\subseteq} (\Xi, \Delta).$$

$(\Upsilon_2, \Omega_2)^c$ is a FHCS and $cl_{FH}(\Upsilon_1, \Omega_1) \tilde{\subseteq} (\Upsilon_2, \Omega_2)^c$ is satisfied. Thus,

$$(\aleph,) \tilde{\subseteq} (\Upsilon, \Omega) \tilde{\subseteq} cl_{FH}(\Upsilon, \Omega) \tilde{\subseteq} (\Xi, \Delta)$$

is obtained.

Conversely, suppose that $(\aleph_{1,1})$ and $(\aleph_{2,2})$ be two disjoint FPFHCSs on $(\Pi, \tilde{\tau}, \Delta)$. Then $(\aleph_{1,1}) \tilde{\subseteq} (\aleph_{2,2})$. From the condition of theorem, the exists a FHOS (Υ, Ω) such that

$$(\aleph_{1,1}) \tilde{\subseteq} (\Upsilon, \Omega) \tilde{\subseteq} cl_{FH}(\Upsilon, \Omega) \tilde{\subseteq} (\aleph_{2,2})^c.$$

Therefore, $(\Upsilon, \Omega), (cl_{FH}(\Upsilon, \Omega))^c$ are FPFHOSs and $(\aleph_{1,1}) \tilde{\subseteq} (\Upsilon, \Omega)$, $(\aleph_{2,2}) \tilde{\subseteq} (cl_{FH}(\Upsilon, \Omega))^c$ and $(\Upsilon, \Omega) \tilde{\cap} (cl_{FH}(\Upsilon, \Omega))^c = 0_{(\Pi_{FH}, \Delta)}$ are obtained. So, $(\Pi, \tilde{\tau}, \Delta)$ is a fuzzy parameterized fuzzy hypersoft T_4 −space.

In the following theorems, the hereditary properties of separation axioms on FPFHTS are investigated.

Theorem 4.6 *Let $(\Pi, \tilde{\tau}, \Delta)$ be a FPFHTS and $(\aleph,)$ be a FPFHS over Π. If $(\Pi, \tilde{\tau}, \Delta)$ is a fuzzy parameterized fuzzy hypersoft T_i −spaces, then the fuzzy parameterized fuzzy hypersoft topological subspace $\left((\aleph,), \tilde{\tau}_{(\aleph,)},\right)$ is a fuzzy parameterized fuzzy hypersoft T_i −space for $i = 0,1,2,3$.*

Proof. Let $P_{\mu(\alpha)}^{(\alpha,\pi)}, P_{\mu(\beta)}^{(\beta,y)} \tilde{\in} \left((\aleph,), \tilde{\tau}_{(\aleph,)},\right)$ such that $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\cap} P_{\mu(\beta)}^{(\beta,y)} = 0_{(\Pi_{FH},\Delta)}$. Hence, there exists FPFHOSs $(\aleph_{1},_{1})$ and $(\aleph_{2},_{2})$ satisfying the conditions of fuzzy parameterized fuzzy hypersoft T_i −spaces such that $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\aleph_{1},_{1})$ and $P_{\mu(\beta)}^{(\beta,y)} \tilde{\in} (\aleph_{2},_{2})$. Then $P_{\mu(\alpha)}^{(\alpha,\pi)} \tilde{\in} (\aleph_{1},_{1}) \tilde{\cap} (\aleph,)$ and $P_{\mu(\beta)}^{(\beta,y)} \tilde{\in} (\aleph_{2},_{2}) \tilde{\cap} (\aleph,)$. Also, FPFHOSs $(\aleph_{1},_{1}) \tilde{\cap} (\aleph,)$ and $(\aleph_{2},_{2}) \tilde{\cap} (\aleph,)$ in $\tilde{\tau}_{(\aleph,)}$ satisfying the conditions of fuzzy parameterized fuzzy hypersoft T_i −space for $i = 0,1,2,3$.

Theorem 4.7 *Let $(\Pi, \tilde{\tau}, \Delta)$ be a FPFHTS over Π. If $(\Pi, \tilde{\tau}, \Delta)$ is a fuzzy parameterized fuzzy hypersoft T_4 −space and $(\aleph,)$ is a FHCS on $(\Pi, \tilde{\tau}, \Delta)$, then $\left((\aleph,), \tilde{\tau}_{(\aleph,)},\right)$ is a fuzzy parameterized fuzzy hypersoft T_4 −space.*

Proof. Let $(\Pi, \tilde{\tau}, \Delta)$ be a fuzzy parameterized fuzzy hypersoft T_4 −space and $(\aleph,)$ be a FHCS on $(\Pi, \tilde{\tau}, \Delta)$. Let $(\aleph_{1},_{1})$ and $(\aleph_{2},_{2})$ be two FPFHCSs on $\left((\aleph,), \tilde{\tau}_{(\aleph,)},\right)$ such that $(\aleph_{1},_{1}) \tilde{\cap} (\aleph_{2},_{2}) = 0_{(\Pi_{FH},\Delta)}$. When $(\aleph,)$ is a FHCS on $(\Pi, \tilde{\tau}, \Delta)$, $(\aleph_{1},_{1})$ and $(\aleph_{2},_{2})$ are FPFHCSs on $(\Pi, \tilde{\tau}, \Delta)$. Since $(\Pi, \tilde{\tau}, \Delta)$ is a fuzzy parameterized fuzzy hypersoft $T_4 -$ space, there exist FPFHOSs (Ξ_1, Δ_1) and (Ξ_2, Δ_2) such that $(\aleph_{1},_{1}) \tilde{\subseteq} (\Xi_1, \Delta_1)$, $(\aleph_{2},_{2}) \tilde{\subseteq} (\Xi_2, \Delta_2)$ and $(\Xi_1, \Delta_1) \tilde{\cap} (\Xi_2, \Delta_2) = 0_{(\Pi_{FH},\Delta)}$. Then $(\aleph_{1},_{1}) = (\Xi_1, \Delta_1) \tilde{\cap} (\aleph,)$, $(\aleph_{2},_{2}) = (\Xi_2, \Delta_2) \tilde{\cap} (\aleph,)$ and $[(\Xi_1, \Delta_1) \tilde{\cap} (\aleph,)] \tilde{\cap} [(\Xi_2, \Delta_2) \tilde{\cap} (\aleph,)] = 0_{(\Pi_{FH},\Delta)}$. This implies that $\left((\aleph,), \tilde{\tau}_{(\aleph,)},\right)$ is fuzzy parameterized fuzzy hypersoft T_4 −space.

5. Conclusion

Separation axioms are fundamental, widely used, and intriguing concepts in topology, essential for constructing more specialized topological spaces. Motivated by this, the current paper investigates the separation axioms within fuzzy parameterized fuzzy hypersoft topologies. Initially, the study presents some basic properties of the fuzzy parameterized fuzzy hypersoft point concept. Next, we define the notion of fuzzy parameterized fuzzy hypersoft Ti-spaces (i=0,1,2,3,4). We then explore key properties of the newly introduced fuzzy parameterized fuzzy hypersoft separation axioms. In future research, concepts like fuzzy parameterized fuzzy hypersoft compactness and connectedness will be further examined. Ultimately, we anticipate that the concepts introduced in this paper will have broad applications in various fields.

References

Ahmad, B., & Kharal, A. (2009). On fuzzy soft sets. *Advances in Fuzzy Systems*, 2009.

Aktaş, H., & Çagman, N. (2007). Soft sets and soft groups. *Information Sciences, 177*(2726–2735).

Ali, M. I., Feng, F., Liu, X. Y., Min, W. K., & Shabir, M. (2009). On some new operations in soft set theory. *Computers and Mathematics with Applications, 57*(1547–1553).

Çagman, N., Çitak, F., & Enginoglu, S. (2010). Fuzzy parameterized fuzzy soft set theory and its applications. *Turkish Journal of Fuzzy Systems, 1*(21–35).

Chen, D., Tsang, E. C. C., Yeung, D. S., Wong, X. (2005). The parametrization reduction of soft sets and its applications. *Computers and Mathematics with Applications, 49*(757–763).

Feng, F., Jun, Y. B., & Zhao, X. (2008). Soft semirings. *Computers and Mathematics with Applications, 56*(2621–2628).

Gayen, S., Smarandache, F., Jha, S., Singh, M. K., Broumi, S., & Kumar, R. (2020). Introduction to plithogenic hypersoft subgroup. *Neutrosophic Sets and Systems, 33*(1), 14.

Kharal, A., & Ahmad, B. (2009). Mappings on fuzzy soft classes. *Advances in Fuzzy Systems, 2009*.

Martin, N., & Smarandache, F. (2020). Concentric plithogenic hypergraph based on plithogenic hypersoft sets – A novel outlook. *Neutrosophic Sets and Systems, 33*(1), 5.

Maji, P. K., Biswas, R., & Roy, A. R. (2003). Soft set theory. *Computers and Mathematics with Applications, 45*(555–562).

Maji, P. K., Biswas, R., & Roy, A. R. (2001). Fuzzy soft sets. *Journal of Fuzzy Mathematics, 9*(589–602).

Molodtsov, D. (1999). Soft set theory—first results. *Computers and Mathematics with Applications, 37*(4-5), 19–31.

Ozturk, T. Y., & Bayramov, S. (2014). Soft mappings space. *The Scientific World Journal*. https://doi.org/10.1155/2014/307292

Pei, D., & Miao, D. (2005). From soft sets to information systems. *Proceedings of IEEE International Conference on Granular Computing, 2*(617–621).

Pawlak, Z. (1982). Rough sets. *International Journal of Computer and Information Sciences, 11*(5), 341–356.

Rana, S., Qayyum, M., Saeed, M., Smarandache, F., & Khan, B. (2019). Plithogenic fuzzy whole hypersoft set, construction of operators, and their application in frequency matrix multi-attribute decision-making technique. *Neutrosophic Sets and Systems, 28*(1).

Rahman, A. U., Saeed, M., Mohammed, M. A., Jaber, M. M., & Garcia-Zapirain, B. (2022). A novel fuzzy parameterized fuzzy hypersoft set and Riesz summability approach based decision support system for diagnosis of heart diseases. *Diagnostics, 12*(7), 1546.

Roy, A. R., & Maji, P. K. (2007). A fuzzy soft set theoretic approach to decision making problems. *Journal of Computational and Applied Mathematics, 203*(2), 412–418.

Roy, S., & Samanta, T. K. (2012). A note on fuzzy soft topological spaces. *Annals of Fuzzy Mathematics and Informatics, 3*(305–311).

Saeed, M., Ahsan, M., Siddique, M. K., & Ahmad, M. R. (2020). A study of the fundamentals of hypersoft set theory. *International Journal of Scientific & Engineering Research, 11*(1).

Saqlain, M., Jafar, N., Moin, S., Saeed, M., & Broumi, S. (2020). Single and multi-valued neutrosophic hypersoft set and tangent similarity measure of single-valued neutrosophic hypersoft sets. *Neutrosophic Sets and Systems, 32*(1), 20.

Saqlain, M., Moin, S., Jafar, M. N., Saeed, M., & Smarandache, F. (2020). Aggregate operators of neutrosophic hypersoft set. *Neutrosophic Sets and Systems, 32*(1), 18.

Smarandache, F. (2018). Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic Sets and Systems, 22*, 168–170.

Tanay, B., & Kandemir, M. B. (2011). Topological structure of fuzzy soft sets. *Computers and Mathematics with Applications, 61*(2952–2957).

Yang, X., Lin, T. Y., Yang, J., Li, Y., & Yu, D. (2009). Combination of interval-valued fuzzy set and soft set. *Computers and Mathematics with Applications, 58*(3), 521–527.

Yolcu, A., & Ozturk, T. Y. (2021). Fuzzy hypersoft sets and its application to decision-making. In *Theory and application of hypersoft set* (Pons Publishing House, Brussels, pp. 50).

Yolcu, A., & Ozturk, T. Y. (2024a). Fuzzy parameterized fuzzy hypersoft topological spaces. *Matematik, Yaz Yayınları*, 46–66.

Yolcu, A., & Ozturk, T. Y. (2024b). Some properties of fuzzy parameterized fuzzy hypersoft topological spaces. *Proceedings of the I. International Ayasofya Scientific Researches and Innovation Congress*, Istanbul, Turkey.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control, 8*(3), 338–353.

CHAPTER V

Some Notes On Locally Symmetric Almost α-Kenmotsu Pseudo-Metric Manifolds

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Introduction

Local symmetry refers to a property of a mathematical object, such as a manifold or a space, where symmetry exists at each point locally. Namely, a transformation or symmetry operation presents for every point in the object that leaves the object invariant and acts transitively on a small neighborhood around that point. Local symmetry can be declared in different ways depending on the type of object under consideration. For instance, in a locally symmetric space, such as a locally symmetric Riemannian or a

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pseudo-Riemannian manifold, the isometries move transitively on the entire space, not just locally around each point. Local symmetry has essential applications in various fields of mathematics and physics. It provides insights into the geometric properties of manifolds, helps classify and understand different types of spaces, and plays a crucial role in formulating physical theories (Duggal, 1990), (O'Neill, 1983).

Almost Kenmotsu structures are a special class of almost contact metric structures, recently studied in (Kim & Pak, 2005), (Öztürk et al., 2017-2021), (Dileo & Pastore, 2009), (Naik et al., 2020), (Venkatesha et al., 2021), (Jun et al., 2005). An almost contact metric manifold (M, ϕ, ξ, η, g) is said to be an almost Kenmotsu manifold if $d\eta = 0$ and $d\Phi = 2(\eta \wedge \Phi)$, where Φ is the fundamental 2-form endowed with the structure. A normal almost Kenmotsu manifold is known as Kenmotsu manifold defined by $\nabla_X \xi = X - \eta(X)\xi$ and $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$ (Kenmotsu, 1972). Also, the Kenmotsu manifold is not compact since $div\xi = 2n$ and has a structure closely related to the warped product. A $(2n + 1)$-dimensional Kenmotsu manifold M is locally a warped product $M = (-\varepsilon, +\varepsilon) \times_f N^{2n}$, where $(-\varepsilon, +\varepsilon)$ is an open interval, N^{2n} is a Kaehler manifold and $f(t) = ce^t$ such that c is a positive constant. Furthermore, a locally symmetric Kenmotsu manifold is a manifold of constant sectional curvature -1. In other words, local symmetry is a substantial restriction for Kenmotsu manifolds. Dileo and Pastore studied locally symmetric almost Kenmotsu manifolds (Dileo & Pastore, 2007). Let M be a locally symmetric almost Kenmotsu manifold with dimension $(2n + 1)$. Then, the Lie derivative holds $R(X, Y)\xi = 0$ for any $X, Y \in D$, and the Lie derivative does not vanish. So, such a manifold is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant curvature -4 and a flat n-dimensional manifold.

A Riemannian manifold M is said to be locally symmetric if its curvature tensor R is parallel, i.e., $\nabla R = 0$. Boeckx and Cho obtained that a locally symmetric contact metric space is either Sasakian with constant curvature 1 or locally isometric to the unit tangent bundle of a Euclidean space (Boeckx & Cho, 2006). For a $(2n+1)$-dimensional Kenmotsu manifold M, if M is locally symmetric, then M is semi-symmetric, i.e., $R.R = 0$. Thus, locally symmetric spaces are semi-symmetric, but the converse is not necessarily true. Besides, Wang and Liu studied a locally symmetric almost Kenmotsu manifold of dimension $(2n+1), (n>1)$ with a CR-integrable structure (Wang & Liu, 2015). They proved that such a manifold is locally isometric to either the hyperbolic space of constant sectional curvature -1 or the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.

Calvaruso and Perrone introduced a systematic study of contact structures with associated pseudo-Riemannian metrics (Calvaruso & Perrone, 2010). Then, some authors studied contact pseudo-metric manifolds (Calvaruso, 2011), (Perrone, 2014). Following these studies, many authors focused on almost contact pseudo-metric manifolds (Venkatesha et al., 2019), (Naik et al., 2019). The relevance of the physics of contact pseudo-metric structures is indicated in (Bejancu et al., 1993), (Duggal, 1986, 1989). The help of the contact pseudo-metric structure gives more insight into the geometry of space-time, which is necessary for physical problems in relativity (Duggal, 1990).

Wang and Liu introduced the geometry of almost Kenmotsu pseudo metric manifolds and investigated the analogies and differences connected with the Riemannian metric tensor (Wang & Liu, 2016). Also, they showed some results related to local symmetry and nullity conditions. Then Öztürk considered almost α-Kenmotsu pseudo-metric structures and their basic properties (Öztürk, 2020-2023). Mainly, some results related to the η-parallelity of h, ϕh, and τ were obtained. Furthermore, some classification results on such manifolds with CR-integrable structures were given.

In this study, we consider almost α-Kenmotsu pseudo-metric manifolds. First, we recall the concept of almost α-Kenmotsu pseudo-metric structure and its basic curvature properties. Then, we investigate locally symmetric almost α-Kenmotsu pseudo-metric manifolds. We obtained some results related to the local symmetry. Finally, two illustrative examples of locally symmetric almost α-Kenmotsu pseudo-metric manifolds are constructed.

Preliminaries

An almost contact structure on a $(2n+1)$-dimensional smooth manifold M endowed with a triple (ϕ, ξ, η), where ϕ is a (1,1)-type tensor field, ξ is an characteristic vector field, and η is a 1-form which defines

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi \tag{1}$$

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad rank\phi = 2n \tag{2}$$

(Yano & Kon, 1984). A pseudo-Riemannian metric g on M is said to be compatible with the almost contact structure (ϕ, ξ, η) if

$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y) \tag{3}$$

for any vector fields $X, Y \in \Gamma(TM)$. Throughout the study, we shall denote by $\Gamma(TM), \nabla$, and D the Lie algebra of all tangent vector fields on M, the Levi Civita connection of pseudo-Riemannian metric g, and the distribution orthogonal to ξ called the contact distribution, i.e.,

$$D = Ker(\eta) = \{X: \quad \eta(X) = 0\} \tag{4}$$

(Blair, 2002). Also, we notice that

$$\eta(X) = \varepsilon g(X, \xi), \; g(\xi, \xi) = \varepsilon, \; \varepsilon = \pm 1. \tag{5}$$

An almost contact manifold M endowed with a compatible pseudo-metric (pseudo-Riemannian metric) is called an almost contact pseudo-metric manifold and the fundamental 2-form Φ of M is defined as $\Phi(X, Y) = g(X, \phi Y)$ for any $X, Y \in \Gamma(TM)$ on M (O'Neill, 1983). Let M be an almost contact pseudo-metric manifold with structure (ϕ, ξ, η, g). If the following conditions are held, then M is said to be an almost α-Kenmotsu pseudo-metric manifold

$$d\eta = 0, \;\; d\Phi = 2\alpha(\eta \wedge \Phi) \tag{6}$$

for $\alpha \neq 0, \alpha \in R$ (Öztürk, 2021). The identically vanishing of the following tensor defined by

$$N_\phi = [\phi, \phi] + 2d\eta \otimes \xi \tag{7}$$

which expresses the normality of almost contact pseudo-metric structure, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. The normality of an almost α-Kenmotsu pseudo-metric manifold can be given by

$$(\nabla_X \phi)Y = -\alpha[\varepsilon g(X, \phi Y)\xi + \eta(Y)\phi X] \tag{8}$$

for any $X, Y \in \Gamma(TM)$. A normal almost α-Kenmotsu pseudo-metric manifold is said to be an α-Kenmotsu pseudo-metric manifold (Öztürk, 2021).

Basic Curvature Properties

This section will recall the basic formula and some curvature properties of almost α-Kenmotsu pseudo-metric manifolds for later usage. The framework that supports the proof of the following propositions on almost Kenmotsu pseudo-metric manifolds is investigated in detail by Öztürk (2021-2023):

Proposition 1. Let M be an almost contact metric manifold and ∇ be the Riemannian connection. Then, we have

$$(\nabla_X \Phi)(Y,Z) = g(Y,(\nabla_X \phi)Z) \tag{9}$$

$$(\nabla_X \eta)Y = g(Y,\nabla_X \xi) = (\nabla_X \Phi)(\xi,\phi Y) \tag{10}$$

$$(\nabla_X \Phi)(Y,Z) + (\nabla_X \Phi)(\phi Y,\phi Z) = \eta(Z)(\nabla_X \eta)\phi Y - \eta(Y)(\nabla_X \eta)\phi Z \tag{11}$$

$$2d\eta(X,Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X \tag{12}$$

$$3d\Phi(X,Y,Z) = \bigoplus_{X,Y,Z} (\nabla_X \Phi)(Y,Z) \tag{13}$$

Here, $\bigoplus_{X,Y,Z}$ denotes the cyclic sum over the vector fields X, Y and Z (Chinea & Gonzalez, 1990).

Proposition 2. Let M be an almost contact pseudo-metric manifold. Then, the following equation can be written as

$$\begin{aligned} 2g((\nabla_X \phi)Y,Z) &= 3d\Phi(X,\phi Y,\phi Z) - 3d\Phi(X,Y,Z) \\ &\quad + g(N^{(0)}(Y,Z),\phi X) + \varepsilon N^{(1)}(Y,Z)\eta(X) \\ &\quad + 2\varepsilon d\eta(\phi Y,X)\eta(Z) - 2\varepsilon d\eta(\phi Z,X)\eta(Y) \end{aligned} \tag{14}$$

for any $X,Y,Z \in \Gamma(TM)$, where $N^{(0)}, N^{(1)}$ is defined by:

$$N^{(0)}(X,Y) = N_\phi(X,Y) + 2d\eta(X,Y)\xi \tag{15}$$

and

$$N^{(1)}(X,Y) = (L_{\phi X}\eta)Y - (L_{\phi Y}\eta)X \tag{16}$$

respectively. Here, L_X denotes the Lie derivative in the direction of X (Wang & Liu, 2016).

Proposition 3. Let M be an almost α-Kenmotsu pseudo-metric manifold. Then, we have

$$h(X) = \tfrac{1}{2}(L_\xi \phi)X, \quad h(\xi) = 0 \tag{17}$$

$$\nabla_X \xi = -\alpha\phi^2 X - \phi h X \tag{18}$$

$$\nabla_\xi \xi = 0, \ \ \nabla_\xi \phi = 0 \tag{19}$$

$$(\phi \circ h)X = -(h \circ \phi)X \tag{20}$$

$$(\nabla_X \eta)Y = \alpha[\varepsilon g(X,Y) - \eta(X)\eta(Y)] + \varepsilon g(\phi Y, hX) \tag{21}$$

for any $X, Y, Z \in \Gamma(TM)$ (Öztürk, 2021).

Proposition 4. Let M be an almost α-Kenmotsu pseudo-metric manifold. For any $X, Y, Z \in \Gamma(TM)$, we have

$$\begin{aligned} 2g\big((\nabla_X \phi)Y, Z\big) \ &= \ -2g\alpha(\varepsilon g(X,\phi Y)\xi + \eta(Y)\phi X, Z) \\ &\quad + g\big(N^{(0)}(Y,Z), \phi X\big) \end{aligned} \tag{22}$$

(Öztürk, 2021).

Proposition 5. Let M be an almost α-Kenmotsu pseudo-metric manifold. Then, the following curvature conditions are satisfied:

$$\begin{aligned} R(X,Y)\xi \ &= \ \alpha^2[\eta(X)Y - \eta(Y)X] - \alpha[\eta(X)\phi hY - \eta(Y)\phi hX] \\ &\quad + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y \end{aligned} \tag{23}$$

$$R(X,\xi)\xi = \alpha^2\phi^2 X + 2\alpha\phi hX - h^2 X + \phi(\nabla_\xi h)X \tag{24}$$

$$(\nabla_\xi h)X = -\phi R(X,\xi)\xi - \alpha^2\phi X - 2\alpha hX - \phi h^2 X \tag{25}$$

$$R(X,\xi)\xi - \phi R(\phi X,\xi)\xi = 2[\alpha^2\phi^2 X - h^2 X] \tag{26}$$

$$S(X,\xi) = -2n\alpha^2\eta(X) - (div(\phi h))X \tag{27}$$

$$S(\xi,\xi) = -[2n\alpha^2 + tr(h^2)] \tag{28}$$

$$div\xi = 2\alpha n, \ div\eta = -2\alpha n\varepsilon \tag{29}$$

for any $X, Y \in \Gamma(TM)$ (Öztürk, 2021).

Main Results

In this section, we consider an almost α-Kenmotsu pseudo-metric manifold (M, ϕ, ξ, η, g) of dimension $(2n+1)$ which is locally symmetric, that is, $\nabla R = 0$. Thus we state the following results:

Theorem 1. Let M be a $(2n+1)$-dimensional almost α-Kenmotsu pseudo-metric manifold. Then M is an α-Kenmotsu pseudo-metric manifold if and only if Eq. (8) holds for any $X, Y \in \Gamma(TM)$.

Proof. According to the hypothesis, we assume M is an α-Kenmotsu pseudo-metric manifold. In this case, the proof is clear from Eq. (22) and $N^{(0)}(Y, Z) = 0$. Contrarily, suppose that M is an almost α-Kenmotsu pseudo-metric manifold. In view of Eq. (8) for $Y = \xi$, it follows that

$$\nabla_X \xi = -\alpha\phi^2 X.$$

A straightforward computation gives

$$d\eta(X, Y) = \varepsilon\alpha g(X, \phi^2 Y) - \varepsilon\alpha g(Y, \phi^2 X) = 0 \tag{30}$$

for any $X, Y \in \Gamma(TM)$. On the other hand, taking into account of Eqs. (8) and (13), we observe that

$$d\Phi(X, Y, Z) = 2\alpha\eta(Z)g(X, \phi Y) + 2\alpha\eta(X)g(Y, \phi Z) - 2\alpha\eta(Y)g(X, \phi Z). \tag{31}$$

Then, Eq. (31) gives

$$g\big((\nabla_X h)Y, -\phi^2 Z\big) - \eta(X)g\left(\left(\nabla_\xi h\right)Y, Z\right) - \eta(Y)g\big((\nabla_X h)\xi, Z\big) = 0 \tag{32}$$

such that

$$d\Phi(X,Y,Z) = 2\alpha(\eta \wedge \Phi)(X,Y,Z).$$

Eventually, putting $h' = -\phi h$ in Eq. (22) and considering $\nabla\xi = -\alpha\phi^2$, we deduce

$$h' = h = 0. \tag{33}$$

Finally, in view of Eqs. (7) and (8), we have

$$N_\phi(X,Y) = -\phi(\nabla_X \phi Y) + \phi^2(\nabla_Y X) + \phi(\nabla_Y \phi X) - \phi^2(\nabla_Y X) \tag{34}$$

$$+\nabla_{\phi X}\phi Y - \phi(\nabla_{\phi X} Y) + \phi(\nabla_{\phi Y} X) - \nabla_{\phi Y}\phi X$$

$$= -\alpha\phi(\varepsilon g(\phi X, Y)\xi - \eta(Y)\phi X) + \alpha\phi(\varepsilon g(\phi Y, X)\xi - \eta(X)\phi Y)$$

$$-\alpha(\varepsilon g(\phi^2 Y, X)\xi - \eta(X)\phi^2 Y) + \alpha(\varepsilon g(\phi^2 X, Y)\xi - \eta(Y)\phi^2 X) = 0$$

for any $X, Y \in \Gamma(TM)$. Thus the proof ends by using Eqs. (33) and (34).

Theorem 2. Let M be a $(2n+1)$-dimensional locally symmetric almost α-Kenmotsu pseudo-metric manifold. Then we have $\nabla_\xi h = 0$.

Proof. The proof is similar to the case, which can be seen in (Wang & Liu, 2016). The proof of this proposition is inspired by Dileo and Pastore using $h' = -\phi h$, Eqs. (25) and (26) (Dileo & Pastore, 2007). The result does not change whether the ξ characteristic vector field is time-like or space-like for almost α-Kenmotsu pseudo-metric case. Because Eqs. (25) and (26) are independent of ε.

Theorem 3. Let M be a $(2n+1)$-dimensional almost α-Kenmotsu pseudo-metric manifold. Then the following conditions are held:

(i) The integral submanifold of D is almost Kaehler manifold,

(ii) The integral submanifold of D is totally umbilical if and only if $h = 0$.

Proof. Analogously, by using Proposition 3.1 in (Kim & Pak, 2005), we complete the proof for the pseudo-metric case.

Theorem 4. Let M be a $(2n+1)$-dimensional locally symmetric almost α-Kenmotsu pseudo-metric manifold. Then the following conditions are equivalent:

(i) $h = 0$,

(ii) M is an α-Kenmotsu pseudo-metric manifold.

Also, when the above conditions are provided, M has constant sectional curvature with $K = -\varepsilon\alpha^2$.

Proof. $(i) \Rightarrow (ii)$ Suppose thay M is an almost α-Kenmotsu pseudo-metric manifold and $h = 0$. Then follows from Eq. (23) we have

$$R(X,Y)\xi = \alpha^2[\eta(X)Y - \eta(Y)X]. \tag{35}$$

For any $X, Y, W \in \Gamma(TM)$, Eq. (35) takes the form

$$\begin{aligned}(\nabla_W R)(X,Y)\xi &= (\nabla_W R)(X,Y)\xi - R(\nabla_W X, Y)\xi \\ &\quad -R(X,\nabla_W Y)\xi - R(X,Y)\nabla_W \xi \\ &= -\alpha R(X,Y)W + \varepsilon\alpha^3[g(X,W)Y - g(Y,W)X].\end{aligned} \tag{36}$$

Since M is locally symmetric, Eq. (36) becomes

$$R(X,Y)W = -\alpha^2\varepsilon[g(Y,W)X - g(X,W)Y]. \tag{37}$$

By the help of Eq. (37), the sectional curvature of M defined by $K = -\alpha^2\varepsilon$. Here, it is noted that $\alpha \neq 0$. Moreover, denoting by $\widetilde{M}$ and $\tilde{\nabla}$ the integral manifold of the contact distribution D and the Levi-Civita connection of $\widetilde{M}$, let us consider the pseudo-metric (pseudo Riemannian) immersion such that $\widetilde{M} \to M$. We remark that the second fundamental form of a pseudo-metric immersion defined by

$$B(X,Y) = -\varepsilon\alpha g(Y,X)\xi. \tag{38}$$

In other words, the totally umbilical submanifold of $\widetilde{M}$ holds Eq. (38) on M. Let $\tilde{R}$ be a Riemannian curvature tensor of $\widetilde{M}$. Then we have

$$R(X,Y)W = \tilde{R}\,(X,Y)W - \alpha^2\varepsilon[g(Y,W)X - g(X,W)Y] \tag{39}$$

for $X, Y, W \in D$. Taking into account of Eqs. (37) and (38) we have easily seen that $\tilde{R}$ vanishes. Hence, M is flat and Kaehler. Therefore, using Theorem 3, the first side of the proof is completed.

$(ii) \Rightarrow (i)$ Suppose that M is an α-Kenmotsu pseudo-metric manifold. With the help of Eq. (8), we get $\nabla\xi = -\alpha\phi^2$. So it is obvious that

$$h' = -\phi h = 0 \Leftrightarrow h = 0. \tag{40}$$

Thus the above relation makes the proof complete.

Theorem 5. Let M be a $(2n+1)$-dimensional almost α-Kenmotsu pseudo-metric manifold of constant curvature K. Then M is an α-Kenmotsu pseudo-metric manifold such that $K = -\varepsilon\alpha^2$.

Proof. By the hypothesis, we assume M is an almost α-Kenmotsu pseudo-metric manifold with constant curvature K. It follows that M is locally symmetric. According to Theorem 2, we can write $\nabla_\xi h = 0$. Moreover, we have

$$R(X,Y)\xi = \varepsilon K[\eta(Y)X - \eta(X)Y]. \tag{41}$$

Follows from Eqs. (23) and (41), we obtain

$$\begin{aligned} 0 = (\varepsilon K + \alpha^2)(\eta(X)Y - \eta(Y)X) + \alpha\eta(X)h'Y \\ -\alpha\eta(Y)h'X - (\nabla_Y h')X + (\nabla_X h')Y. \end{aligned} \tag{42}$$

Then putting $Y = \xi$ in Eq. (42), we get

$$0 = (\varepsilon K + \alpha^2)(\eta(X)\xi - X) - h^2X - 2\alpha h'X \tag{43}$$

where $(\nabla_\xi h')X = 0$. For $X \in D$, if X is an eigenvector of h with eigenvalue of ζ, then we deduce

$$0 = (\varepsilon K + \alpha^2 + \zeta^2)X - 2\alpha\zeta\phi X. \tag{44}$$

Since X and ϕX are linearly independent, Eq. (44) reduces to

$$\varepsilon K + \alpha^2 + \zeta^2 = 2\alpha\zeta = 0. \qquad (45)$$

Hence, Eq. (45) implies that $\zeta = 0$ and $K = -\varepsilon\alpha^2$. Thus, the proof is then completed by the application of Theorem 4.

Corollary 1. Let M be a $(2n+1)$-dimensional almost α-Kenmotsu pseudo-metric manifold. Suppose the rank of the locally symmetric almost α-Kenmotsu pseudo-metric manifold M equals 1. In that case, M has constant curvature K. It is α-Kenmotsu pseudo-metric with $K = -\varepsilon\alpha^2$ and $h = 0$. If there exists no constant curvature of M, then the rank of M must be greater than 1 and $h \neq 0$.

Examples

Example 1. Let us denote $IR^3(x, y, z)$ the standart coordinates and consider the manifold

$$M = \{(x, y, z) \in IR^3, \qquad z \neq 0\}.$$

Then the vector fields with respect to the local pseudo ϕ-basis are as follows:

$$e_1 = e^{z^5}\left(\frac{\partial}{\partial x}\right),\ e_2 = e^{z^5}\left(\frac{\partial}{\partial y}\right),\ e_3 = \left(\frac{\partial}{\partial z}\right)$$

where the pseudo-Riemannian metric tensor product is defined as:

$$g = \left(e^{-2z^5}\right)(dx^2 + dy^2) + \varepsilon dz^2.$$

Moreover, ϕ is a (1,1)-type tensor defined by:

$$\phi(e_1) = e_2, \qquad \phi(e_2) = -e_1, \qquad \phi(e_3) = 0$$

and η is an 1-form given by $\eta(X) = \varepsilon g(X, e_3)$. Using the linearity of g and ϕ, we have

$$\phi^2 X = -X + \eta(X)e_3, \ \ g(e_3, e_3) = \varepsilon.$$

Then the Levi-Civita connection ∇ gives

$$[e_1, e_3] = -5\varepsilon z^3 e_1, \ \ [e_2, e_3] = -5\varepsilon z^3 e_2, \ \ [e_1, e_2] = 0.$$

So, the almost contact metric structure yields

$$\Phi\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = -e^{-2z^5}.$$

Since $\eta = dz$, we deduce

$$d\Phi = -10z^3(\eta \wedge \Phi),$$

with $\alpha(z) = -5z^3$. We note that N_ϕ identically vanishes, and then the manifold is an α-Kenmotsu pseudo-metric. Follows from the non-zero components of the curvature tensor R, we discover that M is a manifold of constant sectional curvature $K = -\varepsilon\alpha^2$. Thus, Theorem 4 and Theorem 5 are valid.

Example 2. We provide an example of almost α-Kenmotsu pseudo-metric manifold which is locally symmetric. Let us consider the $M \subset IR^3$ manifold such that

$$M = \{(x, y, z) \in IR^3 \ \}.$$

Here, (x, y, z) are the standart coordinates in IR^3. The vector fields are as follows:

$$e_1 = \gamma_2 e^{-\alpha z}\left(\frac{\partial}{\partial x}\right) + \gamma_1 e^{-\alpha z}\left(\frac{\partial}{\partial y}\right), e_2 = -\gamma_1 e^{-\alpha z}\left(\frac{\partial}{\partial x}\right) + \gamma_2 e^{-\alpha z}\left(\frac{\partial}{\partial y}\right), e_3 = \left(\frac{\partial}{\partial z}\right).$$

Let g be the pseudo-metric tensor product given by:

$$g = ({k_1}^2 + {k_2}^2)^{-1}(dx^2 + dy^2) + \varepsilon dz^2$$

where k_1, k_2 are defined by $k_1(z) = \gamma_2 e^{-\alpha z}, k_2(z) = \gamma_1 e^{-\alpha z}$ with ${\gamma_1}^2 + {\gamma_2}^2 \neq 0, \alpha \neq 0$ for constants γ_1, γ_2 and α. It is clear that $\{e_1, e_2, e_3\}$ are linearly independent at each point of M. Moreover, we can write the following equations:

$$\phi(e_3) = 0, \phi(e_1) = e_2, \phi(e_2) = -e_1,$$
$$\phi^2 X = -X + \eta(X)e_3, \eta(X) = \varepsilon g(e_3, X), \eta(e_3) = g(e_3, e_3) = \varepsilon,$$
$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y)$$

for any $X, Y \in \Gamma(TM)$.

According to the above equations, we can say that there exists an almost contact pseudo-metric structure (ϕ, ξ, η, g) on M. In order to check, whether it is almost α-Kenmotsu pseudo metric or not, we verify the condition $d\Phi = 2\alpha(\eta \wedge \Phi)$. Hence, we obtain

$$\Phi\left(\left(\frac{\partial}{\partial x}\right), \left(\frac{\partial}{\partial y}\right)\right) = -({k_1}^2 + {k_2}^2)^{-1} = -({\gamma_1}^2 + {\gamma_2}^2)^{-1} e^{2\alpha z}.$$

Since $\eta = dz$, we observe that $d\Phi = 2\alpha(\eta \wedge \Phi)$ on M. Here, we remark that $N_\phi = 0$. Thus M is an α-Kenmotsu pseudo-metric manifold and $h = 0$. Then using the non-zero components of the curvature tensor R, we obtain that M has constant sectional curvature $K = -\varepsilon\alpha^2$. As a result, Theorem 4 and Theorem 5 are verified.

Discussion and Conclusion

Almost Kenmotsu manifolds have been studied extensively in Riemannian geometry and have applications in various fields, including theoretical physics and mathematical biology, which provide a geometric framework for exploring the interplay between contact geometry, Riemannian geometry, and symmetries on manifolds. While almost Kenmotsu manifolds and local symmetry are essential in the theory of manifolds, there is no inherent connection between almost Kenmotsu manifolds and local symmetry. An almost Kenmotsu manifold may or may not possess local symmetry, depending on its specific geometric properties. Moreover, it is well known that the existence of the characteristic vector field in a Kenmotsu manifold establishes the connection between Kenmotsu manifolds and local symmetry.

In light of these explanations, this study deals with almost α-Kenmotsu pseudo-metric manifolds, which are locally symmetric. We present some results about locally symmetric almost α-Kenmotsu pseudo-metric manifolds. Our future works aim to investigate semi-symmetric and locally symmetric almost α-Kenmotsu or (pseudo-metric) manifolds on soliton theory.

REFERENCES

Bejancu, A. & Duggal, K. L. (1993). Real hypersurfaces of indefinite Kaehler manifolds. *Internat. J. Math. and Math. Sci., 16*, 545-556.

Blair, D. E. (2002). *Riemann Geometry of Contact and Symplectic Manifolds*. Springer: Birkhâuser Boston, MA, USA,

Boeckx, E. & Cho, J. T. (2006). Locally symmetric contact metric manifolds. *Monatsh. Math.*, *148,* 269-281.

Calvaruso, G. & Perrone, D. (2010). Contact pseudo-metric manifolds. *Diff. Geom. and its Appl., 28,* 615-634.

Calvaruso, G. (2011). Contact Lorentzian manifolds. *Diff. Geom. and its Appl., 29*, 541-551.

Chinea, D. & Gonzalez, C. (1990). A classification of almost contact metric manifolds. *Annali di Matematica Pura ed Applicata, 156*, 15-36.

Dileo, G. & Pastore, A. M. (2007). Almost Kenmotsu manifolds and local symmetry. *Bull. Belg. Math. Soc. Simon Stevin, 14*, 343-354.

Dileo, G. & Pastore, A. M. (2009). Almost Kenmotsu manifolds and nullity distributions. *Journal of Geom*etry, *93*, 46-61.

Duggal, K. L. (1986). CR-Structures and Lorentzian geometry. *Acta Appl. Math.*, *7*, 211-223.

Duggal, K. L. (1989). Lorentzian geometry of CR Submanifolds. *Acta Appl. Math.*, *7*, 171-193.

Duggal, K. L. (1990). Space time manifolds and contact structures. *Internat. J. Math. and Math. Science, 13*, 545-554.

Jun, J. B., De, U. C. & Pathak, G. (2005). On Kenmotsu manifolds. *J. Korean Math. Soc., 42,* 435-445. Doi: https://doi.org/10.4134/JKMS.2005.42.3.435

Kenmotsu, K. (1972). A class of contact Riemannian manifold. *Tohoku Math. J., 24,* 93-103. Doi: 10.2748/tmj/1178241594

Kim, T. W. & Pak, H. K. (2005). Canonical foliations of certain classes of almost contact metric structures. *Acta Math. Sinica, Eng. Ser. Aug., 21,* 841-846. Doi: https://doi.org/10.1007/s10114-004-0520-2

Naik, D. M., Venkatesha, V. & Kumara H. A. (2020). Some results on almost Kenmotsu manifolds. *Note Math., 40*, 87-100.

Naik, D. M., Venkatesha, V. & Prakasha, D. G. (2019). Certain results on Kenmotsu pseudo-metric manifolds. *Miskolc Math. Notes.*, *20*, 1083-1099.

O'Neill, B. (1983). *Semi-Riemannian Geometry*; Academic Press: New York, USA.

Öztürk, H., Mısırlı, İ. & Öztürk, S. (2017). Almost α-Cosymplectic Manifolds with η-Parallel Tensor Fields, *Academic Journal of Science, 7(3),* 605-612.

Öztürk, H., Öztürk, S. & Yadav, S. K. (2018). A Note on Almost Alpha Kenmotsu Manifolds, *Academic Journal of Science, 08(02),* 225-232.

Öztürk, H., Öztürk, S. & Taş, E. (2018). Some Remarks on Almost Alpha Cosymplectic Manifolds, *International Journal of Arts and Sciences, 11(01),* 397-404.

Öztürk, S. & Öztürk, H. (2020). On alpha-Kenmotsu pseudo-metric manifolds. *AKU J. Sci. Eng*., *20*, 975-982.

Öztürk, S. & Öztürk, H. (2021). Almost α-cosymplectic pseudo metric manifolds. *J. Math*., *2021*, 4106025.

Öztürk, S. & Öztürk, H. (2021). Certain class of almost α-cosymplectic manifolds. *J. Math*., *2021*, 9277175.

Öztürk, H. (2022). The investigation of some tensor conditions for α-Kenmotsu pseudo-metric structures. *AKU J. Sci. Eng*., *22*, 1314-1322.

Öztürk, H. & Öztürk, S. (2023). Almost α-Kenmotsu pseudo Riemannian manifolds with CR-Integrable structure. *Symmetry*., *15*, *353*, 1-13.

Perrone, D. (2014). Contact pseudo-metric manifolds of constant curvature and CR geometry. *Results in Mathematics., 66*, 213-225.

Venkatesha, V., Naik, D. M. & Vanlı, A. T. (2021). Second order parallel tensor on almost Kenmotsu manifolds. *Kyungpook Math. Journal, 61,* 191-203.

Venkatesha, V., Naik, D. M. & Tripathi, M. M. (2019). Certain results on almost contact pseudo-metric manifolds. *Journal of Geometry, 110(2)*, Publication no: 41.

Wang, Y. & Liu, X. (2015). Locally symmetric CR-integrable almost Kenmotsu manifolds. *Mediterr. J. Math., 12*, 159-171.

Wang, Y. & Liu, X. (2016). Almost Kenmotsu pseudo-metric manifolds. *Analele Stiin. ale Univer. Al I Cuza din Iasi-Matematica*., *62,* 241-256.

Yano, K. & Kon, M. (1984). *Structures on Manifolds*; World Scientific Publishing Cor.: Singapore.

CHAPTER VI

Some Results on α-Kenmotsu Manifolds Admitting Ricci Solitons

Hakan ÖZTÜRK[1]

Introduction

Almost contact structures were discussed by Gray on single dimensional spaces with the reduction of the structural group (Gray, 1959). Later, almost contact metric structures were constructed by using the metric tensor (Sasaki & Hatakeyama, 1962). Following this work, the same authors presented the normality condition on almost contact metric structures. This condition, which satisfies the $J^2 = -I$ equation, means that the J complex structure can be integrable. Cosymplectic manifolds, characterized as a subclass of almost contact metric structures, were first studied by Goldberg and Yano (Goldberg & Yano, 1969). After this pioneering study, Olszak

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carried out valuable works on cosymplectic manifolds (Olszak, 1981, 1989).

The history of Kenmotsu manifolds begins with their introduction by Katsuei Kenmotsu in the early 1970s (Kenmotsu, 1972). These manifolds were defined as a natural generalization of Sasakian manifolds, which are closely related to contact geometry and almost Hermitian geometry. The author aimed to describe a broader class of manifolds that extend the properties of Sasakian manifolds while maintaining certain structural conditions. Specifically, Kenmotsu manifolds arise as a generalization of almost contact metric structures, where the condition on the derivative of the 1-form η is modified. Kenmotsu manifolds often appear naturally as warped product manifolds. These constructions were important in understanding their geometric properties and curvature. Whether a Kenmotsu manifold is compact depends on the global topology of the manifold. The defining equations of the Kenmotsu structure do not guarantee compactness. Also, Kenmotsu warped products are a well-studied generalization within the broader family of warped product manifolds. They provide a rich field of exploration in differential geometry, especially in studying curvature, topology, and unique metrics. An almost Kenmotsu manifold is not guaranteed to be normal. However, a Kenmotsu manifold, a special case of an almost Kenmotsu manifold, is always normal. By requiring only the equation $d\eta = \eta \wedge \Phi$, almost Kenmotsu manifolds provide more flexibility in constructing examples. This equation makes them more adaptable for applications in theoretical physics, cosmology, and other fields where less restrictive geometric structures may better model certain phenomena.

Furthermore, by generalizing almost Kenmotsu structures, almost α-Kenmotsu manifolds were defined (Janssens & Vanhecke, 1981). Later, almost α-Kenmotsu and almost cosymplectic structures were combined to define almost α-cosymplectic manifolds, a subclass of almost contact metric manifolds (Kim & Pak, 2005).

Nowadays, the study of Kenmotsu manifolds under specific geometric flows (e.g., Ricci flow) has produced interesting results. For almost Kenmotsu structures, the relaxed conditions allow for a more diverse range of curvature properties to be studied, including Ricci solitons, Einstein metrics, and specific curvature constraints related to almost contact structures. An important tool in differential geometry and general relativity, the Ricci flow describes the time evolution of the metric tensor field on a manifold. It was first introduced by Hamilton (Hamilton, 1982). Hamilton introduced the most significant step in proving the Poincaré hypothesis. Subsequently, Hamilton carried out a study of Ricci flow on surfaces (Hamilton, 1988). However, the metric could generate 'singularities' during the Ricci flow. For example, the curvature of a region could approach infinity. Understanding and managing such situations is an important part of the mathematical theory of Ricci flow. Specialized methods called 'surgery' have been developed to resolve these singularities, which causes the flow to stop. To solve this problem, Perelman went to a classification of 3-dimensional manifolds. In other words, Perelman used a new method to free the flow from singularities (Perelman, 2002). As a result, the Poincaré hypothesis was eventually solved.

The Ricci flow is used to 'flatten' the curvature structure of a manifold. The time-dependent metric tensor modulates the curvature distribution and, in some cases, gives information about the topological structure of the manifold. This process can be analogized with the heat equation. Just as the heat equation smooths the temperature differences with time, the Ricci flow stabilizes the curvature differences of the manifold with time. Ricci solitons arise from the Ricci flow throughout the process. Ricci solitons show the formation of singularities in the Ricci flow and appear as self-similar solutions. When we look at the literature, we see that Riemannian manifolds with Einstein-like structures are investigated to find examples of Ricci solitons. Ricci solitons have an important place, especially in physics. They are usually expressed as quasi Einstein. Thus, the subject of Ricci solitons has become an important research area (Tripathi, 2008), (Yadav & Öztürk, 2019), (Öztürk & Yadav, 2023), (Öztürk & Bektaş, 2023).

In this study, Ricci solitons on α-Kenmotsu manifolds are investigated. In particular, some results are obtained for α-Kenmotsu manifolds admitting Ricci solitons with η-Einstein, Ricci recurrent, generalized Ricci recurrent, and generalized recurrent conditions.

Preliminaries

An almost contact manifold is an odd-dimensional manifold M which carries a field ϕ of endomorphisms of the tangent spaces, a vector field ξ, called characteristic or Reeb vector field, and a 1-form η

satisfying $\phi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$, where $I: TM \to TM$ is the identity mapping. From the definition it follows also that $\phi\xi = 0$, $\eta \circ \phi = 0$ and that the (1,1)-tensor field ϕ has constant rank $2n$ (Blair, 1976). An almost contact manifold (M, ϕ, ξ, η) is said to be normal when the tensor field N

$$N = [\phi, \phi] + 2d\eta \otimes \xi \tag{1}$$

vanishes identically, $[\phi, \phi]$ denoting the Nijenhuis tensor of ϕ. It is well known that any almost contact manifold (M, ϕ, ξ, η) admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for any vector fields X, Y on M. This metric g is called a compatible metric and the manifold M together with the structure (M, ϕ, ξ, η) is said to be an almost contact metric manifold. As an immediate consequence of Eq. (2), one has

$$\eta(X) = g(X, \xi). \tag{3}$$

The 2-form Φ of M defined by

$$\Phi(X, Y) = g(X, \phi Y) \tag{4}$$

is called the fundamental 2-form of the almost contact metric manifold M. Almost contact metric manifolds such that both η and Φ are closed are called almost cosymplectic manifolds. Also, an almost contact metric manifolds such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ are almost Kenmotsu manifolds (Kenmotsu, 1972). A normal almost cosymplectic manifold is a cosymplectic manifold, and a normal almost Kenmotsu manifold is a Kenmotsu manifold. Moreover, an almost contact metric manifold M is said to be an almost α-Kenmotsu if

$$d\eta = 0, d\Phi = 2\alpha(\eta \wedge \Phi). \tag{5}$$

Here, α is a non-zero real constant. In a special case, for $\alpha = 1$, M is an almost Kenmotsu manifold. The normal almost α-Kenmotsu manifold is also called an α-Kenmotsu manifold. Geometrical properties and examples of almost α-Kenmotsu manifolds are studied by many authors (Jensens & Vanhecke, 1981), (Olszak, 1981), (Kim & Pak, 2005).

Proposition 1. Let (M, ϕ, ξ, η, g) a $(2n+1)$-dimensional almost α-Kenmotsu manifold. M is α-Kenmotsu manifold if and only if

$$(\nabla_X \phi)Y = \alpha[g(\phi X, Y)\xi - \eta(Y)\phi X] \tag{6}$$

for any $X, Y \in \chi(M)$ (Öztürk, 2021).

Proposition 2. Let (M, ϕ, ξ, η, g) a $(2n+1)$-dimensional α-Kenmotsu manifold. Then the following curvature properties are satisfied:

$$\nabla_X \xi = \alpha X - \alpha\eta(X)\xi \tag{7}$$

$$(\nabla_X \eta)Y = \alpha[g(X, Y) - \eta(X)\eta(Y)] \tag{8}$$

$$R(X, Y)\xi = -(\alpha^2 + \xi(\alpha))[\eta(Y)X - \eta(X)Y] \tag{9}$$

$$R(X, \xi)Y = (\alpha^2 + \xi(\alpha))[g(Y, X)\xi - \eta(Y)X] \tag{10}$$

$$R(X, \xi)\xi = (\alpha^2 + \xi(\alpha))\phi^2 X \tag{11}$$

$$\eta(R(X, Y)Z) = (\alpha^2 + \xi(\alpha))[-\eta(X)g(Y, Z) + \eta(Y)g(X, Z)] \tag{12}$$

$$S(X, \xi) = -2(\alpha^2 + \xi(\alpha))n\eta(X) \tag{13}$$

$$Q\xi = -2(\alpha^2 + \xi(\alpha))n\xi \tag{14}$$

$$S(\phi X, \phi Y) = \left(\alpha^2 + \xi(\alpha)\right)S(X, Y) + 2n\left(\alpha^2 + \xi(\alpha)\right)\eta(X)\eta(Y) \tag{15}$$

for any $X, Y \in \chi(M)$.

Here α is a differentiable function such that $d\alpha \wedge \eta = 0$ (Öztürk et al., 2017).

Definition 1. Let (M, ϕ, ξ, η, g) a $(2n+1)$-dimensional α-Kenmotsu manifold. For any $X, Y \in \chi(M)$, if the following condition holds

$$S(X,Y) = \mu_1 g(X,Y) + \mu_2 \eta(X)\eta(Y) \tag{16}$$

then M is an η-Einstein manifold, where μ_1 and μ_2 are functions on M. In special case, when $\mu_2 = 0$, M is an Einstein manifold (Blair, 1976).

Proposition 3. Let (M, ϕ, ξ, η, g) a $(2n+1)$-dimensional α-Kenmotsu manifold. Then we have,

$$\mu_1 + \mu_2 = -2n\alpha^2 \tag{17}$$

$$r = \mu_1(2n+1) + \mu_2 \tag{18}$$

$$\mu_1 = \frac{r + 2n\alpha^2}{(2n+1) - 1} \tag{19}$$

$$\mu_2 = \left[\frac{2n(\alpha^2 + r) + r}{(2n+1) - 1}\right] \tag{20}$$

for any $X, Y \in \chi(M)$. Here, r is scalar curvature on M and α is parallel along the characteristic vector field ξ (Öztürk & Bektaş, 2023).

α-Kenmotsu Structures Admitting Ricci Solitons

Definition 2. (M, g_0) be an n-dimensional Riemannian manifold. The following partial differential equation is said to be a Ricci flow which modifies the metric tensor g:

$$\frac{\partial}{\partial s}\big(g(s)\big) + 2S\big(g(s)\big) = 0, \ \ g(0) = g_0 \tag{21}$$

(Hamilton, 1982).

Definition 3. (M, g) be an n-dimensional Riemannian manifold. If the following equation

$$(L_V\, g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0 \tag{22}$$

holds, then (M,g) is said to be a Ricci soliton for arbitrary vector fields X, Y, V on M. Here, λ is a real scalar, the vector field V is the potential vector field of the Ricci soliton, and $L_V g$ is the Lie derivative of the g metric in the V direction. In this case, the Ricci soliton is denoted by (M,g,V,λ). The Ricci soliton (M,g,V,λ) is called the shrinking, steady and expanding Ricci soliton for the cases $\lambda < 0,\ \lambda = 0$ and $\lambda > 0,$ respectively (Hamilton, 1988).

Definition 4. Let (M,g) be an n-dimensional Riemannian manifold and $L_V g$ be the Lie derivative of the metric g in the direction V. Then, we have

$$(L_V g)(X,Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V) \tag{23}$$

(Yano & Kon, 1984).

Definition 5. Let (M,ϕ,ξ,η,g) be a $(2n+1)$-dimensional α-Kenmotsu manifold. If there exists a Ricci soliton (g,V,λ) on M, (M,g,V,λ) is called an α-Kenmotsu manifold admitting a Ricci soliton (Hamilton, 1988), (Kenmotsu, 1972).

Proposition 4. Let (M,g,V,λ) be a $(2n+1)$-dimensional α-Kenmotsu manifold admitting a Ricci soliton. If the potential vector field V is given as the characteristic vector field ξ, then the Ricci curvature tensor field holds

$$S(X,Y) = -(\alpha+\lambda)g(X,Y) + \alpha\eta(X)\eta(Y). \tag{24}$$

Here, α is parallel along the characteristic vector field ξ (Öztürk & Bektaş, 2023).

Proposition 5. Let (M,g,V,λ) be a $(2n+1)$-dimensional α-Kenmotsu manifold admitting a Ricci soliton. If the potential vector field V is given as the characteristic vector field ξ, then the curvature properties of M are held:

$$S(X,\xi) = -\lambda\eta(X) \tag{25}$$

$$QX = \alpha\eta(X)\xi - (\alpha+\lambda)X \tag{26}$$

$$Q\xi = -\lambda\xi, \tag{27}$$

$$S(\xi,\xi) = -\lambda \tag{28}$$

$$r = \alpha - (2n+1)(\alpha+\lambda). \tag{29}$$

Here, α is parallel along the characteristic vector field ξ (Öztürk & Bektaş, 2023).

Theorem 1. Let (M,ϕ,ξ,η,g) a $(2n+1)$-dimensional η-Einstein α-Kenmotsu manifold. Then, we have

$$S(X,Y) = (\alpha^2 + r/2n)g(X,Y) - [r/2n + \alpha^2(2n+1)]\eta(X)\eta(Y) \tag{30}$$

(Öztürk, 2022).

Main Results

In this chapter, Ricci solitons on α-Kenmotsu manifolds are studied. Some results are obtained for α-Kenmotsu manifolds admitting Ricci solitons using η-Einstein, Ricci recurrent, generalized Ricci recurrent, and generalized recurrent conditions.

Theorem 2. Let (M,ϕ,ξ,η,g) be a $(2n+1)$-dimensional α-Kenmotsu manifold. If, for $n \geq 1$, M has η-Einstein structure, then the Ricci soliton (g,ξ,λ), which is given by varying scalar curvature, is everywhere expanding. Here, α is parallel along the characteristic vector field ξ.

Proof. From Eq. (30), we note that M is an η-Einstein α-Kenmotsu manifold. Let us show that the Ricci soliton structure on M has a varying scalar curvature. A symmetric parallel covariant tensor field $h(X,Y)$ on M, for $V=\xi$, we have

$$h(X,Y) = \left(L_\xi g\right)(X,Y) + 2S(X,Y). \tag{31}$$

By using Eqs. (23) and (30), Eq. (31) takes the form

$$h(X,Y) = 2[r/2n + \alpha(\alpha+1)]g(X,Y) \quad (32)$$
$$-2[r/2n + \alpha(1+(2n+1)\alpha)]\eta(X)\eta(Y).$$

Then taking the covariant derivative of both sides of Eq. (32) with respect to the vector field W, for $W = \xi, X, Y \in \{Sp\}^{\perp}$, and $\nabla h = 0$, we get

$$\nabla_{\xi} r + 2n\nabla_{\xi}(\alpha^2 + \alpha) = 0 \quad (33)$$

where $\nabla_{\xi}\alpha = 0$. If we integrate both sides of Eq. (33) with respect to ξ, we deduce

$$r = c. \quad (34)$$

Here, c is an integral constant. Therefore, Ricci solitons exist with scalar curvature on M. We shall finally investigate the (g, ξ, λ) Ricci soliton. With the help of Eq. (22), it can be written as

$$h(X,Y) = -2\lambda g(X,Y). \quad (35)$$

If we replace X and Y by ξ in Eq. (35), it follows that

$$h(\xi,\xi) = -4n\alpha^2. \quad (36)$$

Then taking into account of Eqs. (35) and (36), we obtain

$$\lambda = 2n\alpha^2. \quad (37)$$

Since α is parallel along the characteristic vector field ξ, α is constant along ξ. Therefore, λ will be positive which completes the proof.

Corollary 1. Let (M, ϕ, ξ, η, g) be a 3-dimensional α-Kenmotsu manifold. If M has η-Einstein structure, then the Ricci soliton (g, ξ, λ), which is given by varying scalar curvature, is everywhere expanding. Here, α is parallel along the characteristic vector field ξ.

İspat. Let us assume that M is a 3-dimensional α-Kenmotsu manifold. Then, we have

$$R(X,Y)Z = -g(X,Z)QY + g(Y,Z)QX + S(Y,Z)X - S(X,Z)Y \quad (38)$$

$$+(r/2)[g(X,Z)Y - g(Y,Z)X]$$

(De & Pathak, 2004). If ξ is replaced by Z in Eq. (38) and then the required arrangements are carried out, it yields

$$\eta(Y)QX - \eta(X)QY + \left(2\alpha^2 + \frac{r}{2}\right)[\eta(X)Y - \eta(Y)X] \quad (39)$$

$$= \alpha^2[\eta(X)Y - \eta(Y)X].$$

Again, if ξ replaces Y in Eq. (39), we have

$$QX = \left(\frac{r}{2} + \alpha^2\right)X - \left(\frac{r}{2} + 3\alpha^2\right)\eta(X)\xi. \quad (40)$$

If the inner product of both sides of Eq. (40) with respect to the vector field W, Eq. (40) reduces to

$$S(X,W) = \left(\frac{r}{2} + \alpha^2\right)g(X,W) - \left(\frac{r}{2} + 3\alpha^2\right)\eta(X)\eta(W). \quad (41)$$

In this case, the 3-dimensional α-Kenmotsu manifold has an η-Einstein structure. Using the methodology in Theorem 2, we shall prove that the Ricci soliton has a varying scalar curvature. So then, we deduce

$$h(X,W) = 2\left[\frac{r}{2} + \alpha^2 + \alpha\right]g(X,W) - 2\left[\frac{r}{2} + 3\alpha^2 + \alpha\right]\eta(X)\eta(W). \quad (42)$$

Taking the covariant derivative of both sides of Eq. (42) with respect to the vector field $W = \xi$, for $W = \xi$ and $\nabla h = 0$, we find

$$\nabla_\xi r + 2\nabla_\xi(\alpha^2 + \alpha) = 0. \quad (43)$$

Here, due to the hypothesis, we note that $\nabla_\xi \alpha = 0$. If both sides of Eq. (43) can be integrated with respect to ξ, one can see that $r = c$. This means that the (g, ξ, λ) Ricci soliton on the 3-dimensional α-Kenmotsu manifold has a scalar varying curvature. This ends the proof.

Definition 6. Let (M, g) be an n-dimensional Riemannian manifold and ∇ be a Levi-Civita connection on M. Then, we have

$$(\nabla_Z S)(Y, \xi) = \nabla_Z S(Y, \xi) - S(\nabla_Z Y, \xi) - S(Y, \nabla_Z \xi) \tag{44}$$

(Blair, 1976).

Lemma 1. Let (M, ϕ, ξ, η, g) be a $(2n + 1)$-dimensional recurrent or ϕ-recurrent Kenmotsu manifold. The characteristic vector field ξ and the B 1-form associated with an arbitrary vector field ρ are co-directional. Also, B is defined by

$$B(Z) = \eta(\rho)\eta(Z). \tag{45}$$

Here, when Z replaces ξ, we have

$$B(\xi) = \eta(\rho) \tag{46}$$

(De et al., 2009).

Definition 7. Let (M, g) be an n-dimensional Riemannian manifold. If there exists a non-zero 1-form B on M such that

$$(\nabla_Z S)(X, Y) = B(Z)S(X, Y) \tag{47}$$

then M is said to be a Ricci-recurrent manifold (De et al., 2009).

Theorem 3. Let (M, ϕ, ξ, η, g) be a $(2n + 1)$-dimensional α-Kenmotsu manifold with 1-form B. If M is a Ricci-recurrent α-Kenmotsu manifold for $n \geq 1$, then the (g, ξ, λ) Ricci soliton on M satisfies the following conditions:

(i) expanding if $\alpha > 0$ and $B(\xi) < \alpha$, or $\alpha < 0$ and $B(\xi) > \alpha$,

(ii) shrinking if $\alpha > 0$ and $B(\xi) > \alpha$, or $\alpha < 0$ and $B(\xi) < \alpha$,

(iii) steady if $B(\xi) = \alpha$.

Here, α is parallel along the characteristic vector field ξ.

Proof. Let M be a Ricci-recurrent α-Kenmotsu manifold. Then Eqs. (45) and (46) holds. If Y replaces ξ in Eq. (47) and makes use of Eq. (13), we have

$$(\nabla_Z S)(X,\xi) = B(Z)S(X,\xi) \tag{48}$$

and

$$(\nabla_Z S)(X,\xi) = -2n\alpha^2 B(Z)\eta(X) \tag{49}$$

such that $\nabla_\xi \alpha = 0$. With the help of Eqs. (7), (25), and (44), the left side of Eq. (49) takes the form

$$-\nabla_Z \lambda\eta(X) - S(\nabla_Z X,\xi) - \alpha S(X,Z) - \alpha\lambda\eta(X)\eta(Z).$$

By arranging the last formula, it follows that

$$\begin{aligned}(\nabla_Z S)(X,\xi) \; &= -\lambda g(\alpha Z - \alpha\eta(Z)\xi, X)\\ &\quad -\alpha S(X,Z) - \alpha\lambda\eta(X)\eta(Z)\end{aligned}$$

and $(\nabla_Z S)(X,\xi) = -\alpha[2n\alpha^2 g(X,Z) + S(X,Z)]$ (50)

If the Eqs. (49) and (50) are to be taken into account, we obtain

$$-2n\alpha^2 B(Z)\eta(X) + \alpha\lambda g(X,Z) + \alpha S(X,Z) = 0. \tag{51}$$

Since α is constant along ξ and $\alpha \neq 0$, Eq. (51) yields

$$S(X,Z) = -2n\alpha[\alpha g(X,Z) - B(Z)\eta(X)]. \tag{52}$$

Putting $X = \xi$ in Eq. (52), we have

$$S(Z,\xi) = 2n\alpha B(Z) - 2n\alpha^2\eta(Z). \tag{53}$$

Using Lemma 1 for $X = \xi$, Eq. (53) can be written as

$$S(\xi,\xi) = -2n\alpha\eta(\xi)[\alpha - \eta(\rho)]. \tag{54}$$

In view of Eq. (46), Eq. (54) becomes

$$S(\xi,\xi) = -2n\alpha[\alpha - B(\xi)]. \tag{55}$$

Taking into account of Eq. (28) in Eq. (55), we get

$$\lambda = 2n\alpha\big(\alpha - B(\xi)\big). \tag{56}$$

Thus, since $\alpha \neq 0$ and $n \geq 1$, the proof of all cases is clear using Eq. (56). It completes the proof.

Definition 8. Let (M, g) be an n-dimensional Riemannian manifold. If there exist non-zero 1-forms A and B on M such that

$$(\nabla_Z R)(X,Y)W = A(Z)R)(X,Y)W + B(Z)[g(Y,W)X - g(X,W)Y] \tag{57}$$

then M is said to be a generalized recurrent manifold. Here, there exist vector fields U and V such that

$$A(Z) = g(Z,U), B(Z) = g(Z,V) \tag{58}$$

(De & Guha, 1991).

Theorem 4. Let (M, ϕ, ξ, η, g) be a $(2n+1)$-dimensional α-Kenmotsu manifold with 1-forms A and B. If M is a generalized recurrent α-Kenmotsu manifold for $n \geq 1$, then the (g, ξ, λ) Ricci soliton on M satisfies the following conditions:

(i) expanding if $\alpha > 0$ and $B(\xi) < \left(\frac{2n-1}{2n+1}\right)\alpha A(\xi)$,

(ii) shrinking if $\alpha < 0$ and $B(\xi) > \left(\frac{2n-1}{2n+1}\right)\alpha A(\xi)$,

(iii) steady if $\alpha > 0$ and $B(\xi) = \left(\frac{2n-1}{2n+1}\right)\alpha A(\xi)$.

Here, α is parallel along the characteristic vector field ξ.

Proof. By the hypothesis, we assume that M is a generalized recurrent α-Kenmotsu manifold. Then, by the second Bianchi identity for arbitrary vector fields on M, we get

$$A(Z)R(X,Y)W + B(Z)[g(Y,W)X - g(X,W)Y] \tag{59}$$
$$A(X)R(Y,Z)W + B(X)[g(Z,W)Y - g(Y,W)Z]$$
$$+A(Y)R(Z,X)W + B(Y)[g(X,W)Z - g(Z,W)X] = 0.$$

Contracting Eq. (59) with X, we obtain

$$A(Z)S(Y,W) + (2n+1)B(Z)g(Y,W) + R(Y,Z,W,U) \tag{60}$$
$$+B(Y)g(Z,W) - B(Z)g(Y,W) - A(Y)S(Z,W)$$
$$-(2n+1)B(Y)g(Z,W) = 0.$$

Again, by contracting Eq. (60) with Y and W, we have

$$rA(Z) + 2n(2n+1)B(Z) - 2S(Z,U) = 0. \tag{61}$$

Replacing Z by ξ in (61), Eq. (61) becomes

$$r\eta(U) + 2n(2n+1)\eta(V) - 2S(U,\xi) = 0. \tag{62}$$

Make use of Eqs. (25) and (29) in Eq. (62), it follows that

$$[\alpha - (2n+1)(\alpha+\lambda)]\eta(U) - 2n(2n+1)\eta(V) + 2\lambda\eta(U) = 0. \tag{63}$$

Arranging the last equation, Eq. (63) turns into

$$\lambda = -\frac{2n(2n+1)\eta(V)}{(2n-1)\eta(U)} + 2n\alpha. \tag{64}$$

Applying Lemma 1 for $U = V = \xi$, we have

$$\lambda = -\frac{2n(2n+1)B(\xi)}{(2n-1)A(\xi)} + 2n\alpha. \tag{65}$$

Here, α is constant along ξ and $\alpha \neq 0$. When $\lambda < 0$, Eq. (65) holds

$$B(\xi) > \left(\frac{2n-1}{2n+1}\right)\alpha A(\xi) \tag{66}$$

for $\alpha < 0$, such that $A(\xi) \neq 0$. Also, when $\lambda > 0$, Eq. (65) holds

$$B(\xi) < \left(\frac{2n-1}{2n+1}\right) \alpha A(\xi) \tag{67}$$

where $\alpha > 0$. Finally, in the case $\lambda = 0$, we obtain

$$B(\xi) = \left(\frac{2n-1}{2n+1}\right) \alpha A(\xi) \tag{68}$$

where $\alpha > 0$. This conclusion completes the proof.

Definition 9. Let (M, g) be an n-dimensional Riemannian manifold. If there exist non-zero 1-forms A and B on M such that

$$(\nabla_Z S)(Y, W) = A(Z)S(Y, W) + 2nB(Z)g(Y, W) \tag{69}$$

then M is said to be a generalized recurrent manifold. Here, there exist vector fields Y and W such that

$$A(Z) = g(Z, Y), B(Z) = g(Z, W) \tag{70}$$

(De & Guha, 1991).

Theorem 5. Let (M, ϕ, ξ, η, g) be a $(2n + 1)$-dimensional α-Kenmotsu manifold with 1-forms A and B. If M is a generalized Ricci-recurrent α-Kenmotsu manifold for $n \geq 1$, then the (g, ξ, λ) Ricci soliton on M satisfies the following conditions:

(i) expanding if $\alpha > 0$ and $B(\xi) > \alpha^2(A(\xi) - \alpha)$, or $\alpha < 0$ and $B(\xi) < \alpha^2(A(\xi) - \alpha)$,

(ii) shrinking if $\alpha > 0$ and $B(\xi) < \alpha^2(A(\xi) - \alpha)$, or $\alpha < 0$ and $B(\xi) > \alpha^2(A(\xi) - \alpha)$,

(iii) steady if $B(\xi) = \alpha^2(A(\xi) - \alpha)$.

Here, α is parallel along the characteristic vector field ξ.

Proof. Let us assume that M is a generalized Ricci-recurrent α-Kenmotsu manifold and $\nabla_\xi \alpha = 0$. Replacing $W = \xi$ in Eq. (69), we have

$$(\nabla_Z S)(Y,\xi) = A(Z)S(Y,\xi) + 2nB(Z)\eta(Y). \tag{71}$$

Follows from Eq. (71), we get

$$(\nabla_Z S)(Y,\xi) = -2n\eta(Y)[\alpha^2 A(Z) - B(Z)]. \tag{72}$$

Moreover, using Eqs. (7), (13) and (44), we deduce

$$(\nabla_Z S)(Y,\xi) = -\alpha[S(Y,Z) + 2n\alpha^2 g(Y,Z)]. \tag{73}$$

According to Eqs. (72) and (73), we obtain

$$S(Y,Z) = -2n\alpha^2 g(Y,Z) + 2n\left[\alpha A(Z) - \left(\frac{1}{\alpha}\right)B(Z)\right]\eta(Y). \tag{74}$$

Putting $Y = Z = \xi$ in Eq. (74), Eq. (74) becomes

$$S(\xi,\xi) = -2n\alpha^2 + 2n\left[\alpha A(\xi) - \left(\frac{1}{\alpha}\right)B(\xi)\right]. \tag{75}$$

Using Eq. (28) in Eq. (75), we get

$$\lambda = 2n[\alpha^2 - \alpha A(\xi) + (1/\alpha)B(\xi)]. \tag{76}$$

Here, $n \geq 1$ and $\alpha \neq 0$. Considering Eq. (76), in order to λ vanishes, the following condition is held:

$$\alpha^2 - \alpha A(\xi) + (1/\alpha)B(\xi) = 0. \tag{77}$$

This means that

$$B(\xi) = -\alpha^2(\alpha - A(\xi)). \tag{78}$$

Therefore, the case *(iii)* is obvious from Eq. (78). Finally, for the other two cases, we have the following inequalites as follows:

$$\left(\frac{1}{\alpha}\right)B(\xi) > -\alpha^2 + \alpha A(\xi), \qquad \left(\frac{1}{\alpha}\right)B(\xi) < -\alpha^2 + \alpha A(\xi)$$

where $\alpha \neq 0$. This ends the proof.

Discussion and Conclusion

Ricci solitons and Einstein manifolds are two important concepts in Riemannian geometry defined based on the Ricci tensor. These two structures have an important relationship because Ricci solitons can be seen as a generalized form of Einstein manifolds. If $V = 0$, the Ricci soliton coincides with the Einstein metric. In this case, $S = \lambda g$, , where λ is an Einstein constant. That is, Einstein manifolds are a special subclass of Ricci solitons. Ricci solitons offer more general geometric structures by relaxing the constant curvature condition of the Einstein metric. Here, the vector field V can be seen as a 'deformation' of the metric. Ricci solitons are stable point solutions of the Ricci flow (solutions that change with measure or transformation). On the other hand, Einstein manifolds represent a more remarkable case of the Ricci flow since these solutions have fixed points that do not change with time. Ricci solitons have a wider range of behavior due to the vector field V. For example; a manifold can shrink, expand, or be steady under Ricci flow differently than Einstein manifolds. Hence, Ricci solitons are a fundamental construction stone of geometric analysis and offer a broader perspective on the way to Einstein manifolds.

In Kenmotsu manifolds, the Reeb vector field ξ may have a special relationship with the potential vector field V of the Ricci soliton. For example, when $V = \xi$, the Ricci soliton equation is compatible with the curvature conditions on the Kenmotsu manifold. The existence of Ricci solitons can topologically classify some special subclasses of Kenmotsu manifolds. Moreover, Ricci solitons provide a more flexible framework for studying the relation of Kenmotsu manifolds with Einstein structures.

Recurrent manifolds have a special recurrence structure of the Riemannian tensor. Since the Ricci tensor is obtained from a trace derivative of the Riemannian tensor, a direct relation between the Ricci soliton equation and the properties of recurrent manifolds can be established. If the Riemannian tensor on a recurrent manifold is repeated with a constant coefficient ($c(x)$ is constant), this can lead to the Ricci tensor having a similar recurrent structure. This case may produce a structure compatible with the Ricci soliton on the manifold. If a recurrent manifold is compatible with the Ricci soliton equation with constants $c(x)$ and λ, then this manifold displays a measured deformation under the Ricci flow. In particular, it is noticed that recurrent manifolds on constant Ricci solitons ($\lambda = 0$) offer a more natural structure. Symmetric and semi-symmetric manifolds, a special subclass of recurrent manifolds, may offer a more appropriate framework for Ricci solitons. Here, the Lie derivative becomes compatible with the symmetry conditions of the manifold. Recurrent manifolds refer to a special behavior of the curvature tensor and the Ricci tensor. At the same time, Ricci solitons are used to investigate these tensors' dynamical and time-varying structures. Together, these two concepts provide a broader understanding of the curvature properties of manifolds.

In light of this knowledge, this study is dedicated to obtaining results on α-Kenmotsu manifolds admitting Ricci solitons with some tensor conditions. Our further studies aim to obtain results on almost α-Kenmotsu manifolds under semi-symmetric spaces, local symmetry, and D -homothetic deformation on different Ricci solitons.

REFERENCES

Blair, D. E., (1976). *Contact manifolds in Riemannian geometry*. Lecture Notes in Math., Vol. 509, New York: Springer-Verlag, Berlin-Heidelberg.

De, U. C. & Guha, N. (1991). On Generalized Recurrent Manifolds, *Proc. Math. Soc., 7*, 7-11.

De, U. C. & Pathak, G. (2004). On 3-dimensional Kenmotsu manifolds. *Ind. J. Pure Applied Math., 35,* 159-165.

De, U. C, Yıldız, A. & Yalınız, A. F. (2009). On ϕ-Recurrent Kenmotsu Manifolds, *Turkish Journal of Mathematics*, *33*, 17-25. Doi: https://doi.org/10.3906/mat-0711-10

Goldberg, S. I. & Yano, K. (1969). Integrability of Almost Cosymplectic Structure, *Pacific J. Math., 31,* 373-382.

Gray, J. W. (1959), Some global properties of contact structures, *Ann. of Math., 69,* 421-450.

Hamilton, R. S. (1982). Three-manifolds with positive Ricci curvature. *Journal of Differential Geometry, 17(2)*, 255-306. Doi: 10.4310/jdg/1214436922

Hamilton, R. S. (1988). The Ricci flow on surfaces, Mathematics and General Relativity (Santa Cruz, CA, 1986), *Contemp. Math*, *71*, 301-307.

Janssens, D. & Vanhecke, L. (1981). Almost contact structures and curvature tensors. *Kodai Mathematical Journal, 4,* 1-27.

Kenmotsu, K., (1972). A class of contact Riemannian manifold. *Tohoku Math. J., 24,* 93-103. Doi: 10.2748/tmj/1178241594

Kim, T. W. & Pak, H. K. (2005). Canonical foliations of certain classes of almost contact metric structures. *Acta Math. Sinica, Eng. Ser. Aug., 21,* 841-846. Doi: https://doi.org/10.1007/s10114-004-0520-2

Olszak, Z., (1981). On Almost Cosymplectic Manifolds, *Kodai Mathematical Journal*, 4, 239-250.

Olszak, Z, (1989). Locally Conformal Almost Cosymplectic Manifolds, *College Math. J., 57,* 73-87.

Öztürk, S. & Öztürk, H. (2021). Certain class of almost α-cosymplectic manifolds, *Journal of Mathematics, 2021,* Article ID 9277175, 9 pages. Doi: https://doi.org/10.1155/2021/9277175

Öztürk, H., Mısırlı, İ. & Öztürk, S. (2017). Almost α-Cosymplectic Manifolds with η-Parallel Tensor Fields, *Academic Journal of Science, 7(3),* 605-612.

Öztürk, H., Öztürk, S. & Yadav, S. K. (2018). A Note on Almost Alpha Kenmotsu Manifolds, *Academic Journal of Science, 08(02),* 225-232.

Öztürk, H., Öztürk, S. & Taş, E. (2018). Some Remarks on Almost Alpha Cosymplectic Manifolds, *International Journal of Arts and Sciences, 11(01),* 397-404.

Öztürk, H. & Öztürk, S. (2018). On 3-Dimensional Almost Alpha Kenmotsu Manifolds, *International Journal of Engineering, Science and Mathematics, 7(12),* 130-137.

Öztürk, S. (2022). A Study on α-Kenmotsu Manifolds, Science and Mathematics Sciences, *Theory, Current Researches and New Trends 5, Chapter V*, 45-58, Ivpe, Montenegro.

Öztürk, H. & Yadav S. K. (2023). A Note on Ricci and Yamabe Solitons on Almost Kenmotsu Manifolds, *Novi Sad J. of Maths., 53(2),* 223-239. Doi: https://doi.org/10.30755/NSJOM.13375

Öztürk, H. & Bektaş, E. (2023). *α-Kenmotsu Manifoldlar Üzerinde Ricci Solitonlar,* Modern Matematikte Yeni Yaklaşımlar, Chapter III, 22-43, BİDGE Yayınları, ISBN: 978-625-372-065-0, Ankara.

Perelman, G. (2002). The Entropy Formula for the Ricci Flow and its Geometric Applications, *arXiv preprint math/0211159.*

Sasaki, S. & Hatakeyama, Y. (1962). On differentiable manifolds with contact metric structures. *Journal of the Mathematical Society of Japan, 14,* 249-271.

Tripathi, M. M. (2008). Ricci solitons in contact metric manifolds, *arXiv:0801.4222v1 [math.DG]*.

Yadav, S. K. & Öztürk, H. (2019). On (ε) -Almost Paracontact Metric Manifolds with Conformal η-Ricci Solitons, *Diff. Geo. Dynamical Systems, 21,* 202-215.

Yano, K. & Kon, M. (1984). *Structures on manifolds*. Series in Pure Math., Vol. 3, Singapore: World Scientific Publishing Co.

CHAPTER VII

Bijective Soft Rough Sets

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1 Introduction

More sophisticated and effective mathematical methods are needed to find solutions to some uncertain situations in real life. A variety of imperfect knowledge types are evident in the data pertaining to complex problems in engineering, marketing, and other fields. Therefore, various mathematical tools have emerged to explain uncertain situations and to determine useful information, such as fuzzy set theory (Zadeh, 1965) and rough set theory (Pawlak, 1982). While all of these theories are useful ways of describing imprecision, each of them has its own complexities, as Molodtsov (1999) points out.

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Soft set theory, given by Molodtsov (1999), is a rising tool to deal with vague events. Soft set theory in recent years has improved a lot both theoretically and practically. After that, Maji et al (2003) presented various algebraic manipulations in soft set theory and conducted a comprehensive theoretical work on soft sets. Aktas and Çağman (2007) give the concept of soft group. In (Feng et al., 2008), introduction of the notions of soft semirings is given. Acar et al. (2010) introduced initial concepts of soft rings. Maji et al. (2002) examined the applications of soft set theory to a decision making problem. Zou and Xiao (2008) studied the soft set data analysis approach under imperfect information. Gong et al. (2010) put forth the concept of bijective soft sets and delineated some of its fundamental operations.

The rough sets, initiated in (Pawlak, 1982), are effective mathematical tools to deal with impreciseness and granularity in information systems. The fundamental tenet of rough set theory pertain to the approximation of an arbitrary subset within a given universe, which is described by two definable subsets: an upper approximation and a lower approximation. The approach of rough set seems to be of essential importance in many fields for example in data analysis, image processing, intelligent systems knowledge discovery in database (Peters et al. 2010; Pawlak and Skowron, 2007; Pawlak, 2002). Lately, a great deal of research has been done by many scientists analysing the algebraic structure of rough sets (Bonikowaski, 1995; Bağırmaz and Özcan, 2015; Kuroki and Wang, 1996).

No theory mentioned above may give the best result alone. So some researchers have combined these theories. In fact, a soft set in place of an equivalence relation in Pawlak's rough sets is used to decompose the universe of discourse. Feng et al. (2011) explored the concept of soft rough set, which is a combination of rough and soft sets. Shabir et al. (2013) proposed an alternative approach to soft rough sets known as MSR sets. The aforementioned sets address the shortcomings identified in (Feng et al., 2011), specifically the potential for upper approximations of a subset of the discourse universe to lack a set that does not appear in Pawlak's rough sets.

Let us now introduce the basic terms such as rough sets, soft sets ve bijective soft sets that we will refer to in the following sections.

1.1 Rough sets

Definition 1 *Let's take a non-empty set U and an equivalence relation σ on the set U. In this case, the pair (U,σ) is called the approximation space.The equivalence class of $a \in U$ is denoted by $\sigma(a)$. For a subset $X \subseteq U$,*

$$\underline{X} = \bigcup_{a\in U} \{\sigma(a) : \sigma(a) \subseteq X\},$$

$$\overline{X} = \bigcup_{a\in U} \{\sigma(a) : \sigma(a) \cap X \neq \emptyset\}.$$

The sets $\underline{X}, \overline{X}$ are referred to as the lower and upper approximations of X with respect to (U, σ) respectively (Pawlak, 1982).

Example 1 *Let $U = \{m,n,r,s\}$ and the equivalence relation σ over U, defined by $\sigma(\mathrm{m}) = \{\mathrm{m}\}, \sigma(\mathrm{n}) = \{\mathrm{n}\}$ and $\sigma(r) = \{\mathrm{r},\mathrm{s}\}$. Let $X = \{m,n,s\}$. Then $\underline{X} = \{m,n\}$ and $\overline{X} = \{m,n,r,s\}$.*

1.2 Soft sets

Definition 2 (Molodtsov, 1999) *Let U is a certain set, called the universe, and E is a set of parameters representing the properties of the elements in U. If $A \subseteq E$ and $f: A \rightarrow P(U)$ is a set-valued mapping, a pair $S = (f, A)$ is called a soft set on U.*

An example illustrating the above definition is included below.

Example 2 *Let $U = \{k_1, k_2, k_3, k_4, k_5\}$ and $A = \{e_1, e_2\} \subseteq E = \{e_1, e_2, e_3\}$ denote a universe set and a set of parameters, respectively. Let $A = e_1, e_2$. Consider the set-valued mapping $f: A \rightarrow P(U)$, where $f(e_1) = \{k_1\}$ and $f(e_2) = \{k_3, k_4\}$.Thus, we have $(f, A) = \{(e_1, \{k_1\}), (e_2, \{k_3, k_4\})\}$ as a soft set.*

Definition 3 (Maji et al., 2002) *Let (f, A) and (g, B) be two soft sets over U. Then (f, A) is considered a soft subset of (g, B), denoted by $(f, A) \tilde{\subset} (g, B)$, if $A \subseteq B$ and $\forall\ e \in A$, $f(e)$ and $g(e)$ have the same approximations.*

Definition 4 (Maji et al., 2002) *Let (f, A) and (g, B) be two soft sets over U. The union of (f, A) and (g, B) is defined as (h, C), where $C = A \cup B$ and for all $e \in C$,*

$$h(e) = \begin{cases} f(e), if e \in A \backslash B, \\ g(e), if e \in B \backslash A, \\ f(e) \cup g(e), if \ \ e \in A \cap B. \end{cases}$$

This is denoted by $(f, A) \tilde{\cup} (g, B) = (h, C)$.

Definition 5 (Maji et al., 2002) *(AND operation on two soft sets). If (f, A) and (g, B) are two soft sets then "(f, A) AND (g, B)" (also denoted as $(f, A) \wedge (f, B)$) is defined by $(f, A) \wedge (f, B) = (h, A \times B)$, where $h(\alpha, \beta) = f(\alpha) \cap g(\beta)$ for all $(\alpha, \beta) \in A \times B$.*

Definition 6 (Gong et al., 2010) *(Bijective soft sets) Let* (f, A) *be a soft set on* U*, where* f *is a set-valued mapping* $f: A \to P(U)$ *and* A *is a non-empty set of parameters.* (f, A) *is called a bijective soft set if it satisfies the following conditions:*

1. $\cup_{e \in A} f(e) = U$,

2. For two arbitrary parameters $e_i, e_j \in A$, $e_i \neq e_j$, $f(e_i) \cap f(e_j) = \emptyset$.

In simpler terms, let $\mathfrak{C}$ be a subset of $P(U)$ such that $\mathfrak{C} = \{f(e_1), f(e_2), \dots, f(e_n)\}$, where $e_1, e_2, \dots, e_n \in A$. According to Definition 6, the set-valued mapping $f: A \to P(U)$ can be converted to a bijective function $f: A \to \mathfrak{C}$. This means that for every k in $\mathfrak{C}$ there is precisely one parameter e in A such that $f(e) = k$ and there are no unmapped elements in either A or $\mathfrak{C}$. In summary, each element of a bijective soft set over U can only be mapped to one parameter.

The following gives an example of Definition 6.

Example 3 *Let* (f, E) *be a soft set over the set* $U = \{l_1, l_2, l_3, l_4, l_5, l_6\}$ *and set of parameters is* $E = \{e_1, e_2, e_3, e_4\}$*. Let* $A = \{e_1, e_2, e_3\}$, $B = \{e_2, e_4\}$ *and* $C = \{e_2, e_3\}$*. Let the mapping* (f, E) *be given by*

$$f(e_1) = \{l_1\}, f(e_2) = \{l_2, l_3\}, f(e_3) = \{l_4, l_5, l_6\}, f(e_4) = \{l_1, l_4, l_5, l_6\}.$$

From Definition 6, (f, A) and (f, B) are bijective soft sets. Whereas (f, C) is not bijective soft set.

Proposition 1 (Gong et al., 2010) *If* (f, A) *and* (g, B) *are two bijective soft sets over* U*, then* $(h, C) = (f, A) \wedge (g, B)$ *is a bijective soft set.*

Definition 7 (Feng et al.,2011) *(Soft rough sets) Let* $S = (f, A)$ *represent a soft set on* U. *In this case, the pair* (U, S) *is called a soft approximation space. Depending on* (U, S), *for any subset X of* U, *following two operators are defined*

$$\underline{X} = \{x \in U: \exists e \in A[x \in f(e) \subseteq X]\},$$
$$\overline{X} = \{x \in U: \exists e \in A[x \in f(e), f(e) \cap X \neq \emptyset]\}.$$

The subsets $\underline{X}$ and $\overline{X}$ are called the lower and upper soft rough approximations of X on (U, S), respectively.

2 Bijective soft rough sets

This section describes rough approximations of bijective soft sets and introduces a hybrid notion called bijective soft rough sets.

In the classic Pawlak approach, the rough sets are defined as an approximation space consisting of a universe U and an equivalence relation σ ⊆ U × U. In the context of the concept of a bijective soft set, each element is mapped to a single parameter. The union of the discrete partitions formed according to the parameter set constitutes the universe of discourse. By Definition 6, given that $\mathfrak{C} = \{f(e_1), f(e_2), \dots, f(e_n)\}$, $with\ e_1, e_2, \dots, e_n \in A$, constitutes a partition and a covering of the universe of discourse, it is possible to consider *f(e)* as a basic granule of the universe of discourse, whereby the granular structure of the universe can be represented using a bijective soft set.

Let $S = (f, A)$ be a bijective soft set on U and the the corresponding pair $\mathfrak{B} = (U, S)$ is called a bijective soft approximation space. Then, the set $\mathfrak{C} = \{f(e_1), f(e_2), \dots, f(e_n)\}$, where $e_1, e_2, \dots, e_n$ are elements of A, will be called a class of values and can be defined by the equivalence relation $\sigma \subseteq U \times U$, defined as follows

$$(x, y) \in \sigma \Leftrightarrow x, y \in f(e)$$

for all $x, y \in U$ and only one $e \in A$. Thus $\sigma(x) = f(e) \Leftrightarrow x \in f(e)$. It can thus be demonstrated that the class of values $\mathfrak{C}_A$ of the bijective soft set (f, A) and the quotient set U / σ in $\mathfrak{B} = (U, S)$ can be identified.

Definition 8 *Let $S = (f, A)$ be a bijective soft set over U and $\mathfrak{B} = (U, S)$ be a bijective soft approximation space. Let X be a nonempty subset of U. Then the sets*
$\underline{A}_{\mathfrak{B}}(X) = \bigcup_{e\in A} \{f(e): f(e) \subseteq X\}$and $\overline{A}_{\mathfrak{B}}(X) = \bigcup_{e\in A} \{f(e): f(e) \cap X \neq \emptyset\}$
are called, respectively, lower and upper bijective soft rough approximations of X based on $\mathfrak{B} = (U, S)$. Moreover, $Bnd_{\mathfrak{B}}(X) = \underline{A}_{\mathfrak{B}}(X) - \overline{A}_{\mathfrak{B}}(X)$ is called bijective soft rough boundary regions of X. If $Bnd_{\mathfrak{B}}(X) = \emptyset$; X is said to be bijective soft rough definable; otherwise X is called a bijective soft rough set.

From this point on, throughout this paper, we will accept each bijective soft rough set S over the set U with the bijective soft rough approximation space $\mathfrak{B} = (U, S)$ and all bijective soft rough sets on $\mathfrak{B} = (U, S)$ will be denoted $\mathfrak{B}\mathcal{S}(U)$.
For the purpose of exemplifying this definition an example is given below.

Example 4 *Let the soft set (f, E) be defined on the set $U = \{l_1, l_2, l_3, l_4, l_5, l_6\}$ with parameters $E = \{e_1, e_2, e_3, e_4, e_5\}$. The mapping of (f, E) is as follows: $f(e_1) = \{l_1, l_2\}, f(e_2) = \{l_3, l_4, l_5, l_6\}, f(e_3) = \{l_1, l_2, l_3\}, f(e_4) = \{l_4, l_5\}$ and $f(e_5) = \{l_6\}$. We can tabulate this soft set as shown in Table 1. If $l_i \in f(e)$ then*

$l_{ij} = 1$, *otherwise* $l_{ij} = 0$, *where* l_{ij} *are the entries Table 1.*

	e_1	e_2	e_3	e_4	e_5
l_1	1	0	1	0	0
l_2	1	0	1	0	0
l_3	0	1	1	0	0
l_4	0	1	0	1	0
l_5	0	1	0	1	0
l_6	0	1	0	0	1

Table 1: Soft set (f, E).

Let $A = \{e_1, e_2\} \subseteq E$, $B = \{e_3, e_4, e_5\} \subseteq E$. From Definition 6, (f, A) and (f, B) are bijective soft sets.
Then

$$\mathfrak{C}_A = \{f(e_1), f(e_2)\} = \{\{l_1, l_2\}, \{l_3, l_4, l_5, l_6\}\},$$

$$\mathfrak{C}_B = \{f(e_3), f(e_4), f(e_5)\} = \{\{l_1, l_2, l_3\}, \{l_4, l_5\}, \{l_6\}\}.$$

For $X = \{l_1, l_2, l_3\} \subseteq U$ and $Y = \{l_4, l_6\} \subseteq U$ we can write

$$\underline{A}_{\mathfrak{B}}(X) = \{l_1, l_2\}, \overline{A}_{\mathfrak{B}}(X) = U$$

and

$$\underline{B}_{\mathfrak{B}}(Y) = \{l_6\}, \overline{B}_{\mathfrak{B}}(Y) = \{l_4, l_5, l_6\}.$$

Thus, by Definition 8, X and Y are bijective soft rough sets.

The following proposition is due to subsection 2.3 of [30], we will briefly show that these features are provided in the bijective soft rough sets.

Proposition 2 *Let* $S = (f, A) \in \mathfrak{B}\mathcal{S}(U)$. *Then, for every* $X, Y \subseteq U$ *following properties hold:*

1. $\underline{A}_{\mathfrak{B}}(X) \subset X \subset \overline{A}_{\mathfrak{B}}(X)$,

2. $\underline{A}_{\mathfrak{B}}(\emptyset) = \overline{A}_{\mathfrak{B}}(\emptyset) = \emptyset$,

3. $\underline{A}_{\mathfrak{B}}(U) = \overline{A}_{\mathfrak{B}}(U) = U$,

4. $\underline{A}_{\mathfrak{B}}(X \cap Y) = \underline{A}_{\mathfrak{B}}(X) \cap \underline{A}_{\mathfrak{B}}(Y)$,

5. $\overline{A}_{\mathfrak{B}}(X \cup Y) = \overline{A}_{\mathfrak{B}}(X) \cup \overline{A}_{\mathfrak{B}}(Y)$,

6. $X \subset Y \Rightarrow \underline{A}_{\mathfrak{B}}(X) \subset \underline{A}_{\mathfrak{B}}(Y)$,

7. $X \subset Y \Rightarrow \overline{A}_{\mathfrak{B}}(X) \subset \overline{A}_{\mathfrak{B}}(Y)$,

8. $\overline{A}_{\mathfrak{B}}(X \cap Y) \subseteq \overline{A}_{\mathfrak{B}}(X) \cap \overline{A}_{\mathfrak{B}}(Y)$,

9. $\underline{A}_{\mathfrak{B}}(X \cup Y) \supseteq \underline{A}_{\mathfrak{B}}(X) \cup \underline{A}_{\mathfrak{B}}(Y)$,

10. $\underline{A}_{\mathfrak{B}}\left(\underline{A}_{\mathfrak{B}}(X)\right) = \underline{A}_{\mathfrak{B}}(X)$,

11. $\overline{A}_{\mathfrak{B}}\left(\overline{A}_{\mathfrak{B}}(X)\right) = \overline{A}_{\mathfrak{B}}(X)$.

Proof. (1) If $x \in \underline{A}_{\mathfrak{B}}(X)$, then $x \in f(e) \subseteq X$ for only one $e \in A$. Hence $\underline{A}_{\mathfrak{B}}(X) \subset X$. Next, if $x \in X$, then, $x \in f(e)$ for only one $e \in A$, we have $f(e) \cap X \neq \emptyset$, and so $x \in \overline{A}_{\mathfrak{B}}(X)$. Thus $X \subseteq \overline{A}_{\mathfrak{B}}(X)$.
(2) and (3) directly follows from Definition 6 and Definition 8.
(4) Let $x \in \underline{A}_{\mathfrak{B}}(X) \cap \underline{A}_{\mathfrak{B}}(Y)$. Then

$$\Leftrightarrow x \in \underline{A}_{\mathfrak{B}}(X) \text{ and } x \in \underline{A}_{\mathfrak{B}}(Y)$$
$$\Leftrightarrow x \in f(e) \subseteq X \text{ and } x \in f(e) \subseteq Y \text{ for only one } e \in A$$
$$\Leftrightarrow x \in f(e) \subseteq X \cap Y \text{ for only one } e \in A$$
$$\Leftrightarrow x \in \underline{A}_{\mathfrak{B}}(X \cap Y).$$

Hence $\underline{A}_{\mathfrak{B}}(X \cap Y) = \underline{A}_{\mathfrak{B}}(X) \cap \underline{A}_{\mathfrak{B}}(Y)$.

(5) Let $x \in \overline{A}_{\mathfrak{B}}(X \cup Y)$. Then

$\Leftrightarrow x \in f(e) \cap (X \cup Y) \neq \emptyset \; for \; only \; one \; e \in A$

$\Leftrightarrow x \in (f(e) \cap X) \cup (f(e) \cap Y) \neq \emptyset \; for \; only \; one \; e \in A$

$\Leftrightarrow x \in (f(e) \cap X) \neq \emptyset \; or \; x \in (f(e) \cap Y) \neq \emptyset \; for \; only \; one \; e \in A$

$\Leftrightarrow x \in \overline{A}_{\mathfrak{B}}(X) \; or \; x \in \overline{A}_{\mathfrak{B}}(Y)$

$\Leftrightarrow x \in \overline{A}_{\mathfrak{B}}(X) \cup \overline{A}_{\mathfrak{B}}(Y)$.

(6) Let $X \subset Y$, then $X \cap Y = X$. By (4) we have

$$\underline{A}_{\mathfrak{B}}(X) = \underline{A}_{\mathfrak{B}}(X \cap Y) = \underline{A}_{\mathfrak{B}}(X) \cap \underline{A}_{\mathfrak{B}}(Y).$$

This implies that $\underline{A}_{\mathfrak{B}}(X) \subseteq \underline{A}_{\mathfrak{B}}(Y)$.

(7) Let $X \subset Y$, then $X \cup Y = Y$. By (5) we have

$$\overline{A}_{\mathfrak{B}}(Y) = \overline{A}_{\mathfrak{B}}(X \cup Y) = \overline{A}_{\mathfrak{B}}(X) \cup \overline{A}_{\mathfrak{B}}(Y).$$

This implies that $\overline{A}_{\mathfrak{B}}(X) \subset \overline{A}_{\mathfrak{B}}(Y)$.

(8) Let $x \in \overline{A}_{\mathfrak{B}}(X \cap Y)$. Then

$\Rightarrow x \in f(e) \cap (X \cap Y) \neq \emptyset \; for \; only \; one \; e \in A$

$\Rightarrow x \in (f(e) \cap X) \cap (f(e) \cap Y) \neq \emptyset \; for \; only \; one \; e \in A$

$\Rightarrow x \in (f(e) \cap X) \neq \emptyset \; and \; x \in (f(e) \cap Y) \neq \emptyset \; for \; only \; one \; e \in A$

$\Rightarrow x \in \overline{A}_{\mathfrak{B}}(X) \; and \; x \in \overline{A}_{\mathfrak{B}}(Y)$

$\Rightarrow x \in \overline{A}_{\mathfrak{B}}(X) \cap \overline{A}_{\mathfrak{B}}(Y)$.

Hence $\overline{A}_{\mathfrak{B}}(X \cap Y) \subseteq \overline{A}_{\mathfrak{B}}(X) \cap \overline{A}_{\mathfrak{B}}(Y)$.

(9) Let $x \in \underline{A}_{\mathfrak{B}}(X) \cup \underline{A}_{\mathfrak{B}}(Y)$. Then

$\Rightarrow x \in \underline{A}_{\mathfrak{B}}(X) \; or \; x \in \underline{A}_{\mathfrak{B}}(Y)$

$\Rightarrow x \in f(e) \subseteq X or x \in f(e) \subseteq Y \; for \; only \; one \; e \in A$

$\Rightarrow x \in f(e) \subseteq X \cup Y \; for \; only \; one \; e \in A$

$\Rightarrow x \in \underline{A}_{\mathfrak{B}}(X \cup Y)$.

Hence $\underline{A}_{\mathfrak{B}}(X \cup Y) \supseteq \underline{A}_{\mathfrak{B}}(X) \cup \underline{A}_{\mathfrak{B}}(Y)$.

(10) Let $Y = \underline{A}_{\mathfrak{B}}(X)$. By (1) we have $\underline{A}_{\mathfrak{B}}(Y) \subset Y$. Thus $\underline{A}_{\mathfrak{B}}\left(\underline{A}_{\mathfrak{B}}(X)\right) = \underline{A}_{\mathfrak{B}}(X)$.

Conversely, Let $Y = \underline{A}_{\mathfrak{B}}(X)$ and $x \in Y$. Then $x \in f(e) \subseteq X$ for only one $e \in A$. By Definition 8 we have $x \in f(e) \subseteq Y$ for only one $e \in A$, and so $x \in \underline{A}_{\mathfrak{B}}(Y)$. Thus $Y \subseteq \underline{A}_{\mathfrak{B}}(Y)$, and so $\underline{A}_{\mathfrak{B}}(X) \subseteq \underline{A}_{\mathfrak{B}}\left(\underline{A}_{\mathfrak{B}}(X)\right)$.

Hence $\underline{A}_{\mathfrak{B}}\left(\underline{A}_{\mathfrak{B}}(X)\right) = \underline{A}_{\mathfrak{B}}(X)$.

(11) Let $Y = \overline{A}_{\mathfrak{B}}(X)$. By (1) we have $\overline{A}_{\mathfrak{B}}(Y) \subset Y$. Thus $\overline{A}_{\mathfrak{B}}\left(\overline{A}_{\mathfrak{B}}(X)\right) \subseteq \overline{A}_{\mathfrak{B}}(X)$.

Conversely, Let $Y = \overline{A}_{\mathfrak{B}}(X)$ and $x \in Y$. Then $x \in f(e) \cap X \neq \emptyset$ for only one $e \in A$. By Definition 8 we have $x \in f(e) \cap Y \neq \emptyset$ for only one $e \in A$, and so $x \in \overline{A}_{\mathfrak{B}}(Y)$. Thus $Y \subseteq \overline{A}_{\mathfrak{B}}(Y)$, and so $\overline{A}_{\mathfrak{B}}\left(\overline{A}_{\mathfrak{B}}(X)\right) = \overline{A}_{\mathfrak{B}}(X)$.

Hence $\overline{A}_{\mathfrak{B}}\left(\overline{A}_{\mathfrak{B}}(X)\right) = \overline{A}_{\mathfrak{B}}(X)$.

Remark 1 *We should note that in [10], some properties are not valid as in rough sets and some properties are also valid under strong conditions but for example in the Proposition 2 of the current paper items (1) and (4) are satisfied without extra conditions. In the context of a bijective soft rough set, the aforementioned restrictions are lifted. Consequently, the bijective soft rough set concept offers a superior integration of roughness and parameterisation.*

Definition 9 *(Approximations for AND Operation of two Bijective Soft Rough Sets) Let $(f, A), (g, B) \in \mathfrak{B}\mathcal{S}(U)$. Then we can define the following two operations on bijective soft set $(h, C) = (f, A) \wedge (g, B)$; for every subset $X \subseteq U$*

$$\underline{C}_{\mathfrak{B}}(X) = \bigcup_{e \in C} \{h(e) : h(e) \subseteq X\},$$

$$\overline{C}_{\mathfrak{B}}(X) = \bigcup_{e \in C} \{h(e): h(e) \cap X \neq \emptyset\}$$

where $h(e) = f(\alpha) \cap g(\beta), \forall e = (\alpha, \beta) \in A \times B$.

Proposition 3 *Let* $(f, A), (g, B) \in \mathfrak{B}\mathcal{S}(U)$. *Then approximation on* $(h, C) = (f, A) \wedge (g, B)$, *for every* $X \subseteq U$ *following properties hold:*

1. $\underline{C}_{\mathfrak{B}}(X) \supseteq \underline{A}_{\mathfrak{B}}(X) \cap \underline{B}_{\mathfrak{B}}(X)$,

2. $\overline{C}_{\mathfrak{B}}(X) \subseteq \overline{A}_{\mathfrak{B}}(X) \cap \overline{B}_{\mathfrak{B}}(X)$.

Proof. (1) Let $x \in \underline{A}_{\mathfrak{B}}(X) \cap \underline{B}_{\mathfrak{B}}(X)$. Then $x \in \underline{A}_{\mathfrak{B}}(X)$ and $x \in \underline{B}_{\mathfrak{B}}(X)$. Thus, since (f, A) and (g, B) are two bijective soft sets, there are only one $(\alpha, \beta) \in A \times B$ that is $x \in f(\alpha) \subseteq X$ and $x \in g(\beta) \subseteq X$. Thus there is only one $e \in C$ that is $x \in h(e) = f(\alpha) \cap g(\beta) \subseteq X$. Hence $x \in \underline{C}_{\mathfrak{B}}(X)$.
(2) Let $x \in \overline{C}_{\mathfrak{B}}(X)$. Then $x \in h(e) \cap X \neq \emptyset$, for only one $e \in C$, where $h(e) = f(\alpha) \cap g(\beta)$, $e = (\alpha, \beta) \in A \times B$. Thus $x \in (f(\alpha) \cap X) \cap (g(\beta) \cap X) \neq \emptyset$, and so $x \in f(\alpha) \cap X \neq \emptyset$ and $x \in g(\beta) \cap X \neq \emptyset$, for only one $(\alpha, \beta) \in A \times B$. Therefore $x \in \overline{A}_{\mathfrak{B}}(X)$ and $x \in \overline{B}_{\mathfrak{B}}(X)$. Hence $\overline{C}_{\mathfrak{B}}(X) \subseteq \overline{A}_{\mathfrak{B}}(X) \cap \overline{B}_{\mathfrak{B}}(Y)$.

We can exemplify Proposition 3 as follows:

Example 5 *We reconsider the bijective soft sets* (f, A) *and* (f, B) *given in Example 4. For* $X = \{x_1, x_2, x_4\} \subseteq U$, *we obtain*

$$\underline{A}_{\mathfrak{B}}(X) = \{x_1, x_2\}, \overline{A}_{\mathfrak{B}}(X) = U$$

and

$$\underline{B}_{\mathfrak{B}}(X) = \emptyset, \overline{B}_{\mathfrak{B}}(X) = \{x_1, x_2, x_3, x_4, x_5\}.$$

Also

$$\underline{A}_{\mathfrak{B}}(X) \cap \underline{B}_{\mathfrak{B}}(X) = \emptyset$$

and

$$\overline{A}_{\mathfrak{B}}(X) \cap \overline{B}_{\mathfrak{B}}(X) = \{x_1, x_2, x_3, x_4, x_5\}.$$

Now taking $(h, C) = (f, A) \wedge (f, B), h(e) = f(\alpha) \cap f(\beta), \forall e = (\alpha, \beta) \in A \times B$. Then we obtain

$$\mathfrak{C}_C = \{f(e_1), f(e_2), f(e_3), f(e_4)\} = \{\{x_1, x_2\}, \{x_3\}, \{x_4, x_5\}, \{x_6\}\}$$

and

$$\underline{C}_{\mathfrak{B}}(X) = \{x_1, x_2\}, \overline{C}_{\mathfrak{B}}(X) = \{x_1, x_2, x_4, x_5\}.$$

Thus

$$\underline{C}_{\mathfrak{B}}(X) \nsubseteq \underline{A}_{\mathfrak{B}}(X) \cap \underline{B}_{\mathfrak{B}}(X) \text{and} \overline{A}_{\mathfrak{B}}(X) \cap \overline{B}_{\mathfrak{B}}(X) \nsubseteq \overline{C}_{\mathfrak{B}}(X)$$

Proposition 3 directly implies Corollary 1.

Corollary 1 *Let* $(f_i, A_i) \in \mathfrak{B}\mathcal{S}(U)$, *where* $(i = 1,2,3, \ldots, n)$. *Then approximations on* $(h_n, C_n) = \wedge_{i=1}^{n} (f_i, A_i)$, *for every* $X \subseteq U$ *following properties hold:*

1. $\left(\underline{C_n}\right)_{\mathfrak{B}}(X) \supseteq \cap_{i=1}^{n} \left(\underline{A_i}\right)_{\mathfrak{B}}(X)$,

2. $\left(\overline{C_n}\right)_{\mathfrak{B}}(X) \subseteq \cap_{i=1}^{n} \left(\overline{A_i}\right)_{\mathfrak{B}}(X)$.

Proposition 4 *Let* $(f_i, A_i) \in \mathfrak{B}\mathcal{S}(U)$, *where* $(i = 1,2,3, \ldots, n)$. *Then approximations on* $(h_n, C_n) = \wedge_{i=1}^{n} (f_i, A_i)$, *for every* $X \subseteq U$ *following properties hold:*

1. $\left(\underline{C_n}\right)_{\mathfrak{B}}(X) \supseteq \left(\underline{C_m}\right)_{\mathfrak{B}}(X)$,

2. $\left(\overline{C_n}\right)_{\mathfrak{B}}(X) \subseteq \left(\overline{C_m}\right)_{\mathfrak{B}}(X)$,

where $m \leq n$.

Proof. (1) Let $m = 2, n = 3$ and $x \in \left(\underline{C_2}\right)_{\mathfrak{B}}(X)$. Then $x \in h_2(e) \subseteq X$, for only one $e \in C_2$, where $h_2(e) = f_1(a_1) \cap f_2(a_2)$, $e = (a_1, a_2) \in A_1 \times A_2$. On the other hand, since (f_3, A_3) is a bijective soft sets over U there is only one $a_3 \in A_3$ that is $x \in f_3(a_3)$. Thus $x \in f_1(a_1) \cap f_2(a_2) \cap f_3(a_3) \subseteq f_1(a_1) \cap f_2(a_2) \subseteq X$, and so $x \in h_3(e) \subseteq h_2(e) \subseteq X$, for only one $e \in C_3$, where $h_3(e) = f_1(a_1) \cap f_2(a_2) \cap f_3(a_3)$, $e = (a_1, a_2, a_3) \in A_1 \times A_2 \times A_3$. Therefore $x \in \left(\underline{C_3}\right)_{\mathfrak{B}}(X)$. Hence $\left(\underline{C_2}\right)_{\mathfrak{B}}(X) \subseteq \left(\underline{C_3}\right)_{\mathfrak{B}}(X)$.
(2) Let $x \in \left(\overline{C_3}\right)_{\mathfrak{B}}(X)$. Then $x \in h_3(e) \cap X \neq \emptyset$, for only one $e \in C_3$, where $h_3(e) = f_1(a_1) \cap f_2(a_2) \cap f_3(a_3)$, $e = (a_1, a_2, a_3) \in A_1 \times A_2 \times A_3$. Thus $x \in \left(f_1(a_1) \cap f_2(a_2)\right) \cap X \neq \emptyset$, and so $x \in h_2(e) \cap X \neq \emptyset$ for only one $e \in C_2$, where $h_2(e) = f_1(a_1) \cap f_2(a_2)$, $e = (a_1, a_2) \in A_1 \times A_2$. Therefore $x \in \left(\overline{C_2}\right)_{\mathfrak{B}}(X)$. Hence $\left(\overline{C_3}\right)_{\mathfrak{B}}(X) \subseteq \left(\overline{C_2}\right)_{\mathfrak{B}}(X)$.

3 Bijective soft rough equality of sets

In [30], the concept of rough equality of sets is examined. This section demonstrates that bijective soft rough sets also manifest similar properties.

Definition 10 *Let $(f, A) \in \mathfrak{B}\mathcal{S}(U)$. Then, for every $X, Y \subseteq U$ we can define:*

1. $X \simeq_{\mathfrak{B}} Y$ if and only if $\underline{A}_{\mathfrak{B}}(X) = \underline{A}_{\mathfrak{B}}(Y)$,

2. $X \eqsim_{\mathfrak{B}} Y$ *if and only if* $\overline{A}_{\mathfrak{B}}(X) = \overline{A}_{\mathfrak{B}}(Y)$,

3. $X \approx_{\mathfrak{B}} Y$ *if and only if* $\underline{A}_{\mathfrak{B}}(X) = \underline{A}_{\mathfrak{B}}(Y), \overline{A}_{\mathfrak{B}}(X) = \overline{A}_{\mathfrak{B}}(Y)$.

These relations can be referred to as lower bijective soft rough relations, upper bijective soft rough relations, and bijective soft rough relations, respectively.

It is simple to check that $\simeq_{\mathfrak{B}}, \eqsim_{\mathfrak{B}}$ *and* $\approx_{\mathfrak{B}}$ are equivalence relations on $P(U)$.

Proposition 5 follows directly from Definition 10 and Proposition 2.

Proposition 5 *Let* $(f, A) \in \mathfrak{B}\mathcal{S}(U)$. *Then, for every* $X, Y, X_1, Y_1 \subseteq U$ *following properties hold:*

1. If $X \simeq_{\mathfrak{B}} Y$, then $X \simeq_{\mathfrak{B}} (X \cap Y) \simeq_{\mathfrak{B}} Y$,
2. If $X \eqsim_{\mathfrak{B}} Y$, then $X \simeq_{\mathfrak{B}} (X \cup Y) \simeq_{\mathfrak{B}} Y$,
3. If $X \simeq_{\mathfrak{B}} X_1$ and $Y \simeq_{\mathfrak{B}} Y_1$, then $X \cap Y \simeq_{\mathfrak{B}} X_1 \cap Y_1$
4. If $X \eqsim_{\mathfrak{B}} X_1$ and $Y \eqsim_{\mathfrak{B}} Y_1$, then $X \cup Y \eqsim_{\mathfrak{B}} X_1 \cup Y_1$,
5. If $X \simeq_{\mathfrak{B}} Y$, then $X \cap (-Y) \simeq_{\mathfrak{B}} \emptyset$,
6. If $X \eqsim_{\mathfrak{B}} Y$, then $X \cup (-Y) \simeq_{\mathfrak{B}} U$,
7. If $X \subseteq Y$ and $Y \eqsim_{\mathfrak{B}} \emptyset$, then $X \eqsim_{\mathfrak{B}} \emptyset$,
8. If $X \subseteq Y$ and $X \eqsim_{\mathfrak{B}} U$, then $Y \eqsim_{\mathfrak{B}} U$,
9. If $X \subseteq Y$ and $Y \simeq_{\mathfrak{B}} \emptyset$, then $X \simeq_{\mathfrak{B}} \emptyset$,
10. If $X \subseteq Y$ and $X \simeq_{\mathfrak{B}} U$, then $Y \simeq_{\mathfrak{B}} U$,

where $-X$ is an abbreviation for $U - X$.

4 Accuracy of bijective soft rough approximations

In this section, an accuracy measure for bijective soft rough sets is introduced.

Definition 11 *Let* $(f, A) \in \mathfrak{B}\mathcal{S}(U)$. *The accuracy measure of any subset* $X \subseteq U$ *with respect to* A *is defined as*

$$\beta_{\mathfrak{B}}^{A}(X) = \frac{|\underline{A}_{\mathfrak{B}}(X)|}{|\overline{A}_{\mathfrak{B}}(X)|}$$

Obviously $0 \leq \beta_{\mathfrak{B}}^{A}(X) \leq 1$. If $\beta_{\mathfrak{B}}^{A}(X) = 1$, X is crisp with respect to A, and otherwise, if $\beta_{\mathfrak{B}}^{A}(X) < 1$, X is bijective soft rough with respect to A.

Let us depict above definition by examples referring to Example 5.
For $X = \{x_1, x_2, x_4\} \subseteq U$ and $A \subseteq E$ we have $\underline{A}_{\mathfrak{B}}(X) = \{x_1, x_2\}, \overline{A}_{\mathfrak{B}}(X) = U$. For this case $\beta_{\mathfrak{B}}^{A}(X) = \frac{|\underline{A}_{\mathfrak{B}}(X)|}{|\overline{A}_{\mathfrak{B}}(X)|} = \frac{2}{6}$. It means that the parameter set A is less characteristic for X.
For $X = \{x_1, x_2, x_4\} \subseteq U$ and $B \subseteq E$ we have $\underline{B}_{\mathfrak{B}}(X) = \emptyset, \overline{B}_{\mathfrak{B}}(X) = \{x_1, x_2, x_3, x_4, x_5\}$. For this case $\beta_{\mathfrak{B}}^{B}(X) = \frac{|\underline{B}_{\mathfrak{B}}(X)|}{|\overline{B}_{\mathfrak{B}}(X)|} = \frac{0}{5}$. It means that this parameter set B is not characteristic for X.
For $X = \{x_1, x_2, x_4\} \subseteq U$ and $C \subseteq E$ we have $\underline{C}_{\mathfrak{B}}(X) = \{x_1, x_2\}, \overline{C}_{\mathfrak{B}}(X) = \{x_1, x_2, x_4, x_5\}$. For this case $\beta_{\mathfrak{B}}^{C}(X) = \frac{|\underline{C}_{\mathfrak{B}}(X)|}{|\overline{C}_{\mathfrak{B}}(X)|} = \frac{2}{4}$.
It means that the set X can be characterized partially by parameter sets A and B.
From our observations above we can give Proposition 6.

Proposition 6 *Let* $(f_i, A_i) \in \mathfrak{B}\mathcal{S}(U)$, *where* $(i = 1,2,3, \dots, n)$. *Let* $(h_n, C_n) = \wedge_{i=1}^{n} (f_i, A_i)$. *Then, for every* $X \subseteq U$ *and* $m \leq n$,

$$\beta_{\mathfrak{B}}^{C_m}(X) \leq \beta_{\mathfrak{B}}^{C_n}(X).$$

Proof. Follows directly from Definition 11 and Proposition 4.

5 Conclusion

Bu çalışma hem teorik hem de pratik özelliklere sahiptir. Bu çalışmada bijective yumuşak kaba kümeler tanımlanmış ve önemli özellikleri verilmiştir.

References

Molodtsov, D. (1999). Soft set theory – First results, Computers and Mathematics with Applications, vol. 37, no. 4-5, pp. 19–31.

Maji, P.K.,Biswas, R. & Roy, A.R. (2003). Soft set theory, Computers and Mathematics with Applications, vol. 45, no. 4-5, pp. 555–562.

Maji, P.K , Roy, A.R. & Biswas, R. (2002). An application of soft sets in a decision making problem, Computers and Mathematics with Applications, vol. 44, no.8-9, pp. 1077–1083.

Zadeh, L.A. (1965). Fuzzy sets, Infor. and Control 8, pp. 338-353.

Zou, Y. & Xiao, Z. (2008). Data analysis approaches of soft sets under incomplete information, Knowledge-Based Systems, vol. 21, no.8, pp. 941-945.

Aktas, H. & Çagman, N. (2007). Soft sets and soft groups, Inform. Sci. 177, pp. 2726–2735.

Feng, F., Jun, Y.B. & Zhao, X. (2008). Soft semirings, Computers and Mathematics with Applications. 56 (10), pp. 2621-2628.

Feng, F., Liu, X., Leoreanu-Fotea, V. & Young, J.B. (2011). "Soft sets and soft rough sets," Information Sciences, vol. 181, no. 6, pp. 125–1137.

Acar, U., Koyuncu, F. & Tanay, B. (2010). Soft sets and soft rings, Computers and Mathematics with Applications, vol.59, no.11, pp. 3458–3463.

Peters, J.F. & Pal, S.K. (2010). Cantor, Fuzzy, Near , and Rough Sets in Image Analysis, (In: Pal, S.K. and Peters, J. F.(eds.) Rough Fuzzy Image Analysis: Foundations and Methodologies, CRC Pres ,Taylor and Francis Group, Boca Raton, U.S.A.), 1.1-1.16.

Bonikowaski, Z. (1995). Algebraic structures of rough sets, in: W.P. Ziarko (Ed.), Rough Sets, Fuzzy Sets and Knowledge Discovery, Springer-Verlag, Berlin, pp. 242–247.

Biswas, R. & Nanda, S. (1994). Rough groups and rough subgroups, Bull. Polish Acad. Sci. Math., vol. 42, 251–254.

Bağırmaz, N. & Özcan, A.F. (2015). Rough semigroups on approximation spaces, International Journal of Algebra, vol. 9, pp. 339-350.

Kuroki, N. & Wang, P.P. (1996). The lower and upper approximations in a fuzzy group, Information Sciences, vol. 90, no.1-4, pp. 203–220.

Gong, K., Xiao, Z. & Zhang, X. (2010). The bijective soft set with its operations, Computers and Mathematics with Applications, vol. 60, no.8, pp. 2270–2278.

Pan, W. & Zhan, J. (2017). Soft Rough Groups and Corresponding Decision Making, Italian Journal of Pure and Applied Mathematics, vol.38, pp. 158-171.

Pawlak, Z. (1982). Rough sets, International Journal of Computer & Information Sciences vol. 11, no.5, pp. 341–356.

Pawlak, Z. & Skowron, A. (2007). Rough sets and Boolean reasoning, Information Sciences, vol.177, no.1, pp. 41–73.

Pawlak, Z. (2002). Rough sets and intelligent data analysis, Information Sciences, vol. 147, no.1-4, pp. 1-12.

Shabir, M., Ali M.I. & Shaheen T. (2013). Another approach to soft rough sets, Knowledge-Based Systems, vol. 40, 72–80.

CHAPTER VIII

The Narayana-Pell Sequence in Finite Groups

Özgür ERDAĞ [1]

1 Introduction and Preliminaries

If n is the year, then the Narayana problem can be modelled by the recurrence relation:

$$N_{n+3} = N_{n+2} + N_n$$

for $n \geq 0$ and with initial values $N_0 = N_1 = N_2 = 1$. This sequence is called the Narayana sequence (also called the Fibonacci-Narayana sequence or Narayana's cows sequence). (Allouche & Johnson, 1996)

The well-known Pell sequence $\{P_n\}$ is defined by the following recurrence relation:

$$P_{n+1} = 2P_n + P_{n-1}$$

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for $n \geq 1$, with initial conditions $P_0 = 0$ and $P_1 = 1$.

Deveci and Erdag (Deveci & Erdag, 2022) defined the Narayana-Pell sequence $\{n_k^P\}$ by the following homogeneous linear recurrence relation:

$$n_{k+5}^P = 3n_{k+4}^P - n_{k+3}^P - 2n_{k+1}^P - n_k^P \tag{1.1}$$

for $k \geq 0$, with initial conditions $n_0^P = \cdots = n_4^P = 0$ and $n_5^P = 1$.

By the recurrence relation (1.1), we have

$$\begin{bmatrix} n_{k+5}^P \\ n_{k+4}^P \\ n_{k+3}^P \\ n_{k+2}^P \\ n_{k+1}^P \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 & -2 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} n_{k+4}^P \\ n_{k+3}^P \\ n_{k+2}^P \\ n_{k+1}^P \\ n_k^P \end{bmatrix}$$

for the Narayana-Pell sequence $\{n_k^P\}$. Letting

$$N^P = \begin{bmatrix} 3 & -1 & 0 & -2 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The companion matrix $N^P = \left[n_{ij} \right]_{5\times 5}$ is called to be the Narayana-Pell matrix. It can be readily established by mathematical induction that for $\alpha \geq 4$

$$\left(N^P\right)^\alpha = \begin{bmatrix} n^P_{\alpha+4} & -n^P_{\alpha+3}-2n^P_{\alpha+1}-n^P_{\alpha} & -2n^P_{\alpha+2}-n^P_{\alpha+1} & -2n^P_{\alpha+3}-n^P_{\alpha+2} & -n^P_{\alpha+3} \\ n^P_{\alpha+3} & -n^P_{\alpha+2}-2n^P_{\alpha}-n^P_{\alpha-1} & -2n^P_{\alpha+1}-n^P_{\alpha} & -2n^P_{\alpha+2}-n^P_{\alpha+1} & -n^P_{\alpha+2} \\ n^P_{\alpha+2} & -n^P_{\alpha+1}-2n^P_{\alpha-1}-n^P_{\alpha-2} & -2n^P_{\alpha}-n^P_{\alpha-1} & -2n^P_{\alpha+1}-n^P_{\alpha} & -n^P_{\alpha+1} \\ n^P_{\alpha+1} & -n^P_{\alpha}-2n^P_{\alpha-2}-n^P_{\alpha-3} & -2n^P_{\alpha-1}-n^P_{\alpha-2} & -2n^P_{\alpha}-n^P_{\alpha-1} & -n^P_{\alpha} \\ n^P_{\alpha} & -n^P_{\alpha-1}-2n^P_{\alpha-3}-n^P_{\alpha-4} & -2n^P_{\alpha-2}-n^P_{\alpha-3} & -2n^P_{\alpha-1}-n^P_{\alpha-2} & -n^P_{\alpha-1} \end{bmatrix}.$$

We easily derive that $\det N^P = -1$.

Definition 1.1 A sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of terms in the shortest repeating subsequence is called the period of the sequence. In addition, if the first k terms in the sequence form a repeating subsequence then the sequence is simply periodic with period k. For example, the sequence $\{a,b,c,d,b,c,d,b,c,d,\ldots\}$ is periodic after the initial term a and has period 3 and also the sequence $\{a,b,c,d,a,b,c,d,a,b,c,d,\ldots\}$ is simply periodic with period 4.

The study of the linear recurrence sequences in groups began with the earlier work of Wall (Wall, 1960) where the ordinary Fibonacci sequences in cyclic groups were investigated. Other work on the Fibonacci sequences in cyclic groups is discussed in, see, for example, (Chang, 1986; Lü & Wang, 2006; Renault, 2013; Shah, 1968; Vinson, 1963). In the mid-eighties, Wilcox studied the Fibonacci sequences in abelian groups in (Wilcox, 1986). In (Campbell, Doostie, & Robertson, 1990), the theory was expanded to non-abelian groups by Campbell et al. There, they defined the Fibonacci orbit and the basic Fibonacci orbit of a 2-generator group. We also have the following definition of Fibonacci orbit for a finitely generated group $G=\langle A\rangle$, where $A=\{a_1,a_2,\ldots a_n\}$:

Definition 1.2 The Fibonacci orbit of with respect to the generating set A, written $F_A(G)$, is the sequence $x_i = a_{i+1}$, $0 \le i \le n-1$, $x_{i+n} = \prod_{j=1}^{n} x_{i+j-1}$ for $i \ge 0$.

If the sequence $F_A(G)$ is periodic, then the length of the period is called the Fibonacci lenght of G with respect to the generetaing set A, written $LEN_A(G)$ ((Campbell, 2003; Campbell & Campbell, 2009)

Let G be a 2-generator group and let

$$X = \{(x_1, x_2) \in G \times G \mid \langle\{x_1, x_2\}\rangle = G\}.$$

The notation (x_1, x_2) is said to be a generating pair for G. If G is a 2-generator group and (x_1, x_2) is a generating pair of G, then every element of G can be written as a word

$$(x_1)^{\alpha_1}(x_2)^{\alpha_2}(x_1)^{\alpha_3}\cdots(x_1)^{\alpha_{n-1}}(x_2)^{\alpha_n}$$

where $\alpha_i \in Z$, $1 \le i \le n$.

The concept of the Fibonacci length for two or more generators has also been considered; see, for example, (Aydin & Dikici, 1998; Campbell, Campbell, Doostie, & Robertson, 2004; Doostie & Golamie, 2000; Karaduman & Aydın, 2003; Knox, 1992). In next process, some special linear recurrence sequences defined by the aid of group elements have been studied by many authors; see, for example, (Akuzum, 2020; Akuzum & Deveci, 2020; Akuzum, Deveci & Rashedi, 2022; Campbell & Robertson, 1976; Deveci, Akdeniz & Akuzum, 2017; Deveci, Akuzum & Karaduman, 2015; Deveci, Artun & Akuzum, 2017; Deveci & Karaduman, 2015; Hulku, Erdag & Deveci, 2023; Kuloglu, Ozkan & Shannon, 2022; Mehraban & Hashemi, 2023).

Definition 1.3 The semidihedral group SD_{2^m} , $(m \geq 4)$ is defined by the presentation

$$SD_{2^m} = \left\langle x, y : x^{2^{m-1}} = y^2 = e,\ y^{-1}xy = x^{2^{m-1}-1} \right\rangle$$

Note that $\left|SD_{2^m}\right| = 2^m, |x| = 2^{m-1}$ and $|y| = 2$.

In this study, we consider the Narayana-Pell sequence in groups and then we define the Narayana-Pell orbit. Finally, we obtain the lengths of the periods of the Narayana-Pell orbit in the semidihedral group SD_{2^m} , $(m \geq 4)$ as applications of the results obtained.

2 Main Results

Let G be a finite j-generator group and let X be the subset of $\underbrace{G \times G \times G \cdots \times G}_{j}$ such that $(x_0, x_1, \ldots, x_{j-1}) \in X$ if and only if G is generated by $x_0, x_1, \ldots, x_{j-1}$. We call $(x_0, x_1, \ldots, x_{j-1})$ a generating j-tuple for G .

Definition 2.1. For a generating j-tuple $(x_0, x_1, \ldots, x_{j-1}) \in X$, we define the Narayana-Pell orbit as shown:

$$x_N^P(n+5) = \left(x_N^P(n)\right)^{-1} \left(x_N^P(n+1)\right)^{-2} \left(x_N^P(n+3)\right)^{-1} \left(x_N^P(n+4)\right)^{3}$$

for $n \geq 0$, with initial conditions

$$\begin{cases} x_N^P(0) = x_0, x_N^P(1) = x_1, \ldots, x_N^P(j-1) = x_{j-1}, x_N^P(j) = e, \ldots, x_N^P(4) = e & \text{if } j < 4, \\ x_N^P(0) = x_0, x_N^P(1) = x_1, x_N^P(2) = x_2, x_N^P(3) = x_3, x_N^P(4) = x_4, & \text{if } j = 4. \end{cases}$$

For a generating j-tuple $(x_0, x_1, \ldots, x_{j-1}) \in X$, the Narayana-Pell orbit is denoted by $x_N^P(G : x_0, x_1, \ldots, x_{j-1})$.

Theorem 2.1. If G is a finite group, then a Narayana-Pell orbit of G is simply periodic.

Proof. Suppose that ε is the order of the group G. Since there are ε^5 distinct 5-tuples of elements of G, at least one of the 5-tuples appears twice in the Narayana-Pell orbit. Thus, consider the subsequence following this 5-tuple. Because of the repeating, the Narayana-Pell orbit of the group G is periodic. Since the Narayana-Pell orbit is periodic, there exist natural number i and j with $i \equiv j \pmod 5$, such that

$$x_N^P(i) = x_N^P(j),\, x_N^P(i+1) = x_N^P(j+1), \ldots, x_N^P(i+5) = x_N^P(j+5).$$

By the definition relation of the Narayana-Pell orbit $x_N^P(G : x_0, x_1, \ldots, x_{j-1})$, we can easily derive

$$x_N^P(n) = \left(x_N^P(n+1)\right)^{-2}\left(x_N^P(n+3)\right)^{-1}\left(x_N^P(n+4)\right)^{3}\left(x_N^P(n+5)\right)^{-1}.$$

Therefore, we obtain $x_N^P(i) = x_N^P(j)$, and it then follows that

$$x_N^P(i-j) = x_N^P(0),\, x_N^P(i-j+1) = x_N^P(1), \ldots, x_N^P(i-j+5) = x_N^P(5).$$

which implies that the Narayana-Pell orbit $x_N^P(G : x_0, x_1, \ldots, x_{j-1})$ is simply periodic. □

We denote that the length of the period of the Narayana-Pell orbit $x_N^P(G : x_0, x_1, \ldots, x_{j-1})$ by $Lx_N^P(G : x_0, x_1, \ldots, x_{j-1})$.

In (Erdag & Deveci, 2022), Deveci and Erdağ denoted the period of the sequence $\{n_k^P(m)\}$, when read modulo m by $l^{n_k^P}(m)$.

Now we give the lenght of the periods of the Narayana-Pell orbit of the semidihedral group SD_{2^m} as applications of the results obtained.

Theorem 2.2. For generating pair (x, y), the length of the period of the Narayana-Pell orbit in the semidihedral group SD_{2^m} is $2^{m-2} \cdot l^{n_k^P}(2)$.

Proof. We consider the lenght of the period of the Narayana-Pell orbit in the semidihedral group by the aid of the period $l^{n_k^P}(m)$. We obtain Narayana-Pell orbit in the following form:

$$x_N^P(0) = x,\ x_N^P(1) = y,\ x_N^P(2) = e,\ x_N^P(3) = e,\ x_N^P(4) = e, \ldots,$$

$$x_N^P\left(2 \cdot l^{n_k^P}(2)\right) = x,\ x_N^P\left(2 \cdot l^{n_k^P}(2)+1\right) = x^{-4\alpha_1} y,\ x_N^P\left(2 \cdot l^{n_k^P}(2)+2\right) = e,$$

$$x_N^P\left(2 \cdot l^{n_k^P}(2)+3\right) = e,\ x_N^P\left(2 \cdot l^{n_k^P}(2)+4\right) = x^{-4\alpha_2}, \ldots,$$

where α_1 and α_2 are positive integers such that $\gcd(\alpha_1, \alpha_2) = 1$. Thus, for Using the above, the sequence becomes:

$$x_N^P(0) = x,\ x_N^P(1) = y,\ x_N^P(2) = e,\ x_N^P(3) = e,\ x_N^P(4) = e, \ldots,$$

$$x_N^P\left(2 \cdot l^{n_k^P}(2)\tau\right) = x,\ x_N^P\left(2 \cdot l^{n_k^P}(2)\tau+1\right) = x^{-4\tau\beta_1} y,\ x_N^P\left(2 \cdot l^{n_k^P}(2)\tau+2\right) = e,$$

$$x_N^P\left(2 \cdot l^{n_k^P}(2)\tau+3\right) = e,\ x_N^P\left(2 \cdot l^{n_k^P}(2)\tau+4\right) = x^{-4\tau\beta_2}, \ldots,$$

where $\beta_1, \beta_2 \in \Box$. So we need the smallest integer τ such that $2^{m-1} \cdot \eta = 4\tau$, $(\eta \in \Box)$ for $m \geq 4$. If we choose $\tau = 2^{m-3}$, we obtain

$$x_N^P\left(2^{m-2} \cdot l^{n_k^P}(2)\right) = x,\ x_N^P\left(2^{m-2} \cdot l^{n_k^P}(2)+1\right) = y,\ x_N^P\left(2^{m-2} \cdot l^{n_k^P}(2)+2\right) = e,$$

$$x_N^P\left(2^{m-2} \cdot l^{n_k^P}(2)+3\right) = e,\ x_N^P\left(2^{m-2} \cdot l^{n_k^P}(2)+4\right) = e, \ldots.$$

Since the elements succeeding $x_N^P\left(2^{m-2} \cdot l^{n_k^P}(2)\right)$, $x_N^P\left(2^{m-2} \cdot l^{n_k^P}(2)+1\right)$, $x_N^P\left(2^{m-2} \cdot l^{n_k^P}(2)+2\right)$, $x_N^P\left(2^{m-2} \cdot l^{n_k^P}(2)+3\right)$ and

$x_N^P\left(2^{m-2}\cdot l^{n_k^P}(2)+4\right)$ depend on x,y and e for their values, the cycle begins again with the $\left(2^{m-2}\cdot l^{n_k^P}(2)\right)$nd element. Thus it is verifed that the lenght of the period of the Narayana-Pell orbit in the semidihedral group SD_{2^m} is $2^{m-2}\cdot l^{n_k^P}(2)$. □

Example 2.1. For $m=5$, we consider the length of the period of the Narayana-Pell orbit in the semidihedral group SD_{32}. Using the relations of the semidihedral group SD_{32}, we have the sequence

$$
\begin{aligned}
&x, y, e, e, e, x^{-1}, x^{-5}y, x^{4}y, x^{-1}, xy, x^{3}y, x^{-3}y, x^{4}y, x^{-6},\\
&x^{-5}, x^{4}y, x^{4}, e, x^{-4}, x^{-7}, x^{5}y, y, x^{5}, x^{-5}y, x^{-3}y, x^{7}y, x^{-4}y, x^{6},\\
&x, x^{8}y, e, e, x^{8}, x^{7}, x^{3}y, x^{4}y, x^{7}, xy, x^{3}y, x^{5}y, x^{-4}y, x^{2},\\
&x^{-5}, x^{-4}y, x^{4}, e, x^{4}, x, x^{-3}y, y, x^{-3}, x^{-5}y, x^{-3}y, x^{-1}y, x^{-4}y, x^{-2},\\
&x, y, e, e, e, \ldots .
\end{aligned}
$$

Since $x_N^P(0)=x_N^P(56)=x$, $x_N^P(1)=x_N^P(57)=y$, $x_N^P(2)=x_N^P(58)=e$, $x_N^P(3)=x_N^P(59)=e$, $x_N^P(4)=x_N^P(60)=e$, the length of the period of the Narayana-Pell orbit is $Lx_N^P(SD_{32}:x,y)=56$.

3. CONCLUSION

In this study, we examined the Narayana-Pell sequence in groups. Firstly, we redefined the Narayana-Pell sequence by means of the elements of groups and call the Narayana-Pell orbit to this relation redefined. Then, we show that this orbit is simply periodic. In addition, we investigated the Narayana-Pell orbit in the semidihedral group SD_{2^m}. Finally, we obtained the lengths of the periods of the Narayana-Pell orbit in the semidihedral group SD_{2^m} as applications of the results obtained.

REFERENCES

Akuzum, Y. (2020). The Padovan-Padovan *p*-Sequences in Groups. *Turkish Journal of Science*, *5*(3), 285-289.

Akuzum, Y. & Deveci, Ö. (2020). The Hadamard-type *k*-step Fibonacci sequences in groups. *Communications in Algebra*, *48*(7), 2844-2856.

Akuzum, Y., Deveci, O. & Rashedi, M. E. (2022). The Hadamard-type *k*-step Pell sequences in Finite Groups. *Caspian Journal of Mathematical Sciences (CJMS)*, *11*(1), 304-312.

Allouche, J. P. & Johnson, T. (1996). Narayana's cows and delayed morphisms. (s. 2-7). Normandy, France: Proceedings of 3rd Computer Music Conference JIM96.

Aydın, H. & Dikici, R. (1998). General Fibonacci Sequences in Finite Groups. *Fibonacci Quarterly, 36*(3), 216-221.

Campbell, P.P. (2003). *Fibonacci length and efficieny in group presentations*. Ph.D. Thesis, University of St Andrews.

Campbell, C. M. & Campbell, P. P. (2009). The Fibonacci lengths of binary polyhedral groups and related groups. *Congressus Numerantium*, *194*, 95-102.

Campbell, C. M., Campbell, P. P., Doostie, H. & Robertson, E. F. (2004). On the Fibonacci length of powers of dihedral groups. In *Applications of Fibonacci Numbers: Volume 9: Proceedings of The Tenth International Research Conference on Fibonacci Numbers and Their Applications* (pp. 69-85). Springer Netherlands.

Campbell, C. M., Doostie, H. & Robertson, E. F. (1990). Fibonacci length of generating pairs in groups. In *Applications of Fibonacci Numbers: Volume 3 Proceedings of 'The Third International Conference on Fibonacci Numbers and Their Applications', Pisa, Italy, July 25–29, 1988* (pp. 27-35). Dordrecht: Springer Netherlands.

Campbell, C. M. & Robertson, E. F. (1976). On a group presentation due to Fox. *Canadian Mathematical Bulletin*, *19*(2), 247-248.

Chang, D. K. (1986). Higher-order Fibonacci sequences modulo *m*. *The Fibonacci Quarterly*, *24*(2), 138-139.

Deveci, O., Akdeniz, M. & Akuzum, Y. (2017). The Periods of The Pell *p*-Orbits of Polyhedral and Centro-Polyhedral Groups. *Jordan Journal of Mathematics and Statistics (JJMS)*, *10*(1), 1-9.

Deveci, O., Akuzum, Y. & Karaduman, E. (2015). The Pell-Padovan *p*-sequences and its applications. *Utilitas Mathematica*, *98*, 327-347.

Deveci, O., Artun, G. & Akuzum, Y. (2017). The Fibonacci *p*-Sequences and the Padovan *p*-Sequences in the Extended Triangle Groups. *Romai Journal*, *13*, 19-25.

Deveci, O. & Erdag, O. (2022). The Narayana-Pell Sequence And Its Binet Formulas. *6th International Congress on Life, Social, and Health Sciences in a Changing World*, 2-3 July 2022, Şişli / İstanbul, Turkey, (294-298).

Deveci, O. & Karaduman, E. (2015). The Pell sequences in finite groups. *Utilitas Mathematica*, *96*, 263-276.

Doostie, H. & Golamie, R. (2000). Computing on the Fibonacci lengths of finite groups. *International Journal of Applied Mathematics*, *4*(2), 149-156.

Erdag, O. & Deveci, O. (2022). The Narayana-Pell Sequence Modulo *m*. *6th International Congress on Life, Social, and Health Sciences in a Changing World*, 2-3 July 2022, Şişli / İstanbul, Turkey, (299-302).

Hulk, S., Erdag, O. & Deveci, O. (2023). Complex-type Narayana sequence and its application. *Maejo International Journal of Science & Technology*, *17*(2), 163-176.

Karaduman, E. & Aydın, H. (2003). General 2-step Fibonacci sequences in nilpotent groups of exponent p and nilpotency class 4. *Applied mathematics and computation*, *141*(2-3), 491-497.

Knox, S. W. (1992). Fibonacci sequences in finite groups. *The Fibonacci Quarterly*, *30*(2), 116-120.

Kuloğlu, B., Ozkan, E. & Shannon, A. G. (2022). The Narayana sequence in finite groups. *The Fibonacci Quarterly*, *60*(5), 212-221.

Lü, K. & Wang, J. (2006). *k*-step Fibonacci sequence modulo *m*. *Utilitas Mathematica*, *71*.

Mehraban, E. & Hashemi, M. (2023). Fibonacci length and the generalized order *k*-Pell sequences of the 2-generator p-groups of nilpotency class 2. *Journal of Algebra and Its Applications*, *22*(03), 2350061.

Renault, M. (2013). The period, rank, and order of the *(a,b)*-Fibonacci sequence mod *m*. *Mathematics Magazine*, *86*(5), 372-380.

Shah, A. P. (1968). Fibonacci sequence modulo *m*. *The Fibonacci Quarterly*, *6*(2), 139-141.

Vinson, J. (1963). The relation of the period modulo *m* to the rank of apparition of *m* in the Fibonacci sequence. *The Fibonacci Quarterly*, *1*(2), 37-46.

Wall, D. D. (1960). Fibonacci series modulo *m*. *The American Mathematical Monthly*, *67*(6), 525-532.

J. Wilcox, H. (1986). Fibonacci sequences of period *n* in groups. *The Fibonacci Quarterly*, *24*(4), 356-361.

CHAPTER IX

The Complex-Type Narayana-Jacobsthal Numbers

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1. Introduction

It is well known that reduced sequences (El Naschie, 2005; Fraenkel & Klein, 1996; Kirchoof & Rutishauser, 1990; Mandelbaum, 1972; Spinadel, 2002; Stein, 1993) are often discovered at the intersection of interdisciplinary relationships. Numerous features of algebraic reduction sequences, such as their generating function and exponential, permanental, and combinatorial representations, have been investigated by numerous researchers and continue to be investigated. (Akuzum & Deveci, 2021; Deveci, Akuzum, & Karaduman, 2015; Erdag & Deveci, 2019; Erdag & Deveci, 2022) (Erdag, Shannon, & Deveci, 2018; Gogin & Myllari, 2007; Horadam , 1961; Horadam , 1963; Ozkan, 2007; Stakhov & Rozin, 2006; Tasci & Firengiz, 2010). Many of

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these investigations have yielded different results by using matrices that correspond to reduced sequences. In (Deveci & Shannon, 2021; Deveci & Shannon, 2018), the authors defined the new sequences using quaternions and complex numbers, and then they gave various features. Some reduced sequences are redefined using complex numbers and adapted to a new approach to number theory (Deveci & Shannon, 2021; Deveci, Erdag, & Gungoz, 2023; Erdag, Halıcı, & Deveci , 2022; Horadam , 1963; Hulku, Erdag, & Deveci, 2023). In this study, a new reduction relation called the complex-type Narayana-Jacobsthal numbers was defined. By considering the generating matrices of these numbers in the form of companion matrices, super-diagonal matrices were defined, enabling the derivation of permanental representations of these numbers through the permanent values of these matrices. Generating functions were derived by examining the structural properties of the defined numbers, and their exponential and combinatorial representations were established using series and binomial expansions based on these generating functions.

2. Preliminaries

Deveci and Akuzum, (Deveci & Akuzum, 2022)defined the Narayana-Jacobsthal sequence as shown:

$$n_{k+5}^{J} = 2n_{k+4}^{J} + n_{k+3}^{J} - n_{k+2}^{J} - n_{k+1}^{J} - 2n_{k}^{J}$$

for the integers $k \geq 0$,with the initial conditions $n_0^J = n_1^J = n_2^J = n_3^J = 0$ and $n_4^J = 1$.

Suppose the $(n+k)$th term of a sequence is defined recursively as a linear combination of the preceding k terms. Then, it can be represented as:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

where $c_0, c_1, \ldots, c_{k-1}$ are constans.

Kalman (Kalman, 1982)established that number sequences can be formulated through matrix representations. By applying the companion matrix method, he derived explicit closed-form expressions for generalized sequences. The companion matrix A_k is constructed as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Also, he proved that

$$(A_k)^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

3. Main Results

We next define the complex-type Narayana-Jacobsthal numbers by integer constants $n_0^{c,J} = n_1^{c,J} = n_2^{c,J} = n_3^{c,J} = 0$ and $n_4^{c,J} = 1$ and the recurrence relation:

$$n_{k+5}^{c,J} = 2i.n_{k+4}^{c,J} - n_{k+3}^{c,J} + i.n_{k+2}^{c,J} - n_{k+1}^{c,J} - 2i.n_{k}^{c,J} \tag{3.1}$$

for $n \geq 0$.

By (3.1), we can construct a generating matrix G for the complex-type Narayana-Jacobsthal numbers as follows:

$$G = \begin{bmatrix} 2i & -1 & i & -1 & -2i \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}_{5\times 5}$$

The companion matrix is referred to as the complex-type Narayana-Jacobsthal matrix.

Using induction on α, we derive:

$$(G)^{\alpha} = \begin{bmatrix} n_{\alpha+4}^{c,J} & n_{\alpha+5}^{c,J} - 2i.n_{\alpha+4}^{c,J} & i.n_{\alpha+3}^{c,J} - n_{\alpha+2}^{c,J} - 2i.n_{\alpha+1}^{c,J} & -n_{\alpha+3}^{c,J} - 2i.n_{\alpha+2}^{c,J} & -2i.n_{\alpha+3}^{c,J} \\ n_{\alpha+3}^{c,J} & n_{\alpha+4}^{c,J} - 2i.n_{\alpha+3}^{c,J} & i.n_{\alpha+2}^{c,J} - n_{\alpha+1}^{c,J} - 2i.n_{\alpha}^{c,J} & -n_{\alpha+2}^{c,J} - 2i.n_{\alpha+1}^{c,J} & -2i.n_{\alpha+2}^{c,J} \\ n_{\alpha+2}^{c,J} & n_{\alpha+3}^{c,J} - 2i.n_{\alpha+2}^{c,J} & i.n_{\alpha+1}^{c,J} - n_{\alpha}^{J} - 2n_{\alpha-1}^{J} & -n_{\alpha+1}^{c,J} - 2i.n_{\alpha}^{c,J} & -2i.n_{\alpha+1}^{c,J} \\ n_{\alpha+1}^{c,J} & n_{\alpha+2}^{c,J} - 2i.n_{\alpha+1}^{c,J} & i.n_{\alpha}^{c,J} - n_{\alpha-1}^{c,J} - 2i.n_{\alpha-2}^{c,J} & -n_{\alpha}^{c,J} - 2i.n_{\alpha-1}^{c,J} & -2i.n_{\alpha}^{c,J} \\ n_{\alpha}^{c,J} & n_{\alpha+1}^{c,J} - 2i.n_{\alpha}^{c,J} & i.n_{\alpha-1}^{c,J} - n_{\alpha-2}^{c,J} - 2i.n_{\alpha-3}^{c,J} & -n_{\alpha-1}^{c,J} - 2i.n_{\alpha-2}^{c,J} & -2i.n_{\alpha-1}^{c,J} \end{bmatrix}$$

for $\alpha \geq 3$. Also, it is clear that $\det G = -2i$.

Definition 3.1. A $u \times v$ real matrix $M = \left[m_{i,j} \right]$ is referred to as a contractible matrix in the k^{th} column (or row) if the k^{th} column (or row) contains exactly two non-zero entries.

Suppose that $x_1, x_2, \ldots, x_u$ are row vectors of the matrix M. If M is contractible in the k^{th} column such that $m_{i,k} \neq 0, m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1)\times(v-1)$ matrix $M_{ij:k}$ obtained from M by replacing the i^{th} row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the j^{th} row.

The k^{th} column is called the contraction in the k^{th} column relative to the i^{th} row and the j^{th} row.

In (Brualdi & Gibson, 1977), Brualdi and Gibson showed that $per(M) = per(N)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Let $r \geq 5$ be a positive integer and let $S_r = \left[s_{k,j}^r \right]$ is the $r \times r$ super-diagonal matrix defined such that:

$$S_r = \begin{bmatrix} 2i & -1 & i & -1 & -2i & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2i & -1 & i & -1 & -2i & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2i & -1 & i & -1 & -2i & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 2i & -1 & i & -1 & -2i & 0 \\ 0 & 0 & \cdots & 0 & 1 & 2i & -1 & i & -1 & -2i \\ 0 & 0 & 0 & \cdots & 0 & 1 & 2i & -1 & i & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2i & -1 & i \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2i & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 2i \end{bmatrix}.$$

Theorem 3.1. For $r \geq 5$,

$$perS_r = n_{r+4}^{c,J}.$$

Proof. Assume that the equation holds for $r \geq 5$. We now prove it for $r+1$. By expanding the permanent $perS_r$ using the Laplace expansion with respect to the first row, we obtain:

$$perS_{r+1} = 2i.perS_r - perS_{r-1} + i.perS_{r-2} - perS_{r-3} - 2i.perS_{r-4}$$

Given that $perS_r = n_{r+4}^{c,J}$, $perS_{r-1} = n_{r-3}^{c,J}$, $perS_{r-2} = n_{r-2}^{c,J}$, $perS_{r-3} = n_{r+4}^{c,J}$ and $perS_{r-4} = n_r^{c,J}$, which follow from the definition of the complex-type Narayana-Jacobsthal number, we conclude:

$$perS_{r+1} = n_{r+5}^{c,J}.$$

Hence, the proof is concluded.

Let $D_r = \left[d_{k,j}^r \right]$ be the $r \times r$ matrix defined as:

$$d_{k,j}^r = \begin{cases} 2i & \text{if } k=t \text{ and } j=t \text{ for } 1 \le t \le r-2, \\ -1 & \text{if } k=t \text{ and } j=t+1 \text{ for } 1 \le t \le r-1, \\ & \text{and} \\ & k=t \text{ and } j=t+2 \text{ for } 1 \le t \le r-3, \\ i & \text{if } k=t \text{ and } j=t+2 \text{ for } 1 \le t \le r-2, \\ -2i & \text{if } k=t \text{ and } j=t+4 \text{ for } 1 \le t \le r-4, \\ 1 & \text{if } k=t \text{ and } j=t-1 \text{ for } 2 \le t \le r-2, \\ & \text{and} \\ & k=t \text{ and } j=t \text{ for } r-1 \le t \le r, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we introduce the $r \times r$ matrix $P_r = \left[p_{k,j}^r \right]$ in the following form:

$$\begin{array}{c} \qquad\qquad (r-2)\text{th} \\ \qquad\qquad \downarrow \\ P_r = \begin{bmatrix} 1 & \cdots & 1 & 0 \\ 1 & & & \\ 0 & & D_{r-1} & \\ \vdots & & & \\ 0 & & & \end{bmatrix} \end{array} \quad \text{for } r > 5.$$

Theorem 3.2. Let $n_r^{c,J}$ be the r th the complex-type Narayana-Jacobsthal number. Then

i. For $r \geq 5$,

$$perD_r = n_{r+2}^{c,J}.$$

ii. For $r > 5$,

$$perP_r = \sum_{y=0}^{r+1} n_y^{c,J}.$$

Proof. We apply the method of induction on r.

i. Assume that $perD_r = n_{r+2}^{c,J}$. for $r \geq 5$. Now consider the case $r+1$. By expanding the permanent $perD_r$ using the Laplace expansion along the first row and using the definition of the matrix D_r, we get:

$$perD_{r+1} = 2i.perD_r - perD_{r-1} + i.perD_{r-2} - perD_{r-3} - 2i.perD_{r-4}$$

Substituting the assumed values, this becomes:

$$perD_{r+1} = 2i.n_{r+2}^{c,J} - n_{r+1}^{c,J} + i.n_r^{c,J} - n_{r-1}^{c,J} - 2i.n_{r-2}^{c,J}$$

Thus, the result holds.

ii. Next, we expand $perP_r$ using the Laplace expansion along the first row. We obtain:

$$perP_r = perP_{r-1} + perD_{r-1}.$$

By applying the result from part (i) in Theorem 3.2 and the inductive argument, the proof follows directly.

A matrix M is called convertible if there is an $n \times n$ $(1,-1)$-matrix K such that $perM = \det(M \circ K)$, where $M \circ K$ denotes the Hadamard product of M and K.

Let $r>5$, and let H be the $r\times r$ matrix, defined by

$$H=\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

Corollary 3.1. For $r>5$,

$$\det\left(S_r \circ H\right)=n_{r+4}^{c,J},$$

$$\det\left(D_r \circ H\right)=n_{r+2}^{c,J},$$

and

$$\det\left(P_r \circ H\right)=\sum_{y=0}^{r+1} n_y^{c,J}.$$

Proof. Since

$$perS_r=\det\left(S_r \circ H\right)=n_{r+4}^{c,J},$$

$$perD_r=\det\left(D_r \circ H\right)=n_{r+2}^{c,J},$$

and

$$perP_r=\det\left(P_r \circ H\right)=\sum_{y=0}^{r+1} n_y^{c,J},$$

By Theorem 3.1 and Theorem 3.2, the results follow immediately.

Let $C\left(c_1,c_2,\ldots,c_v\right)$ be a $v\times v$ companion matrix as follows:

$$C(c_1,c_2,\ldots,c_v)=\begin{bmatrix} c_1 & c_2 & \cdots & c_v \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Theorem 3.3. (Chen & Louck, 1996).The (k,j) entry $c_{k,j}^{(n)}(c_1,c_2,\ldots,c_v)$ in the matrix $C^n(c_1,c_2,\ldots,c_v)$ is given by the following formula:

$$c_{k,j}^{(n)}(c_1,c_2,\ldots,c_v)=\sum_{(t_1,t_2,\ldots,t_v)}\frac{t_j+t_{j+1}+\cdots+t_v}{t_1+t_2+\cdots+t_v}\times\binom{t_1+\cdots+t_v}{t_1,\ldots,t_v}c_1^{t_1}\cdots c_v^{t_v} \qquad (3.2)$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+vt_v=n-k+j$, $\binom{t_1+\cdots+t_v}{t_1,\ldots,t_v}=\frac{(t_1+\cdots+t_v)!}{t_1!\cdots t_v!}$ is a multinomial coefficient, and the coefficients in (3.2) are defined to be 1 if $n=k-j$.

Here, we explore a combinatorial representation for the complex-type Narayana-Jacobsthal numbers.

Corollary 3.2. Suppose that $n_\alpha^{c,J}$ be the α the complex-type Narayana-Jacobsthal number. Then

i.

$$n_\alpha^{c,J}=\sum_{(t_1,t_2,\ldots,t_5)}\binom{t_1+t_2+\cdots+t_5}{t_1,t_2,\ldots,t_5}2i^{t_1}(-1)^{t_2+t_4}i^{t_3}(-2i)^{t_5}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+5t_5=\alpha-4$

ii.

$$n_{\alpha}^{c,J} = -\frac{1}{2i}\sum_{(t_1,t_2,\ldots,t_5)}\frac{t_5}{t_1+t_2+\cdots+t_5}\binom{t_1+t_2+\cdots+t_5}{t_1,t_2,\ldots,t_5}2i^{t_1}(-1)^{t_2+t_4}i^{t_3}(-2i)^{t_5}$$

where the summation is over nonnegative integers satisfying $t_1+2t_2+\cdots+5t_5=\alpha+1$

Proof. In Theorem 2.3, if we chose $k=5$, $j=1$, $c_1=2i, c_2=c_4=-1, c_3=i,$ and $c_5=-2i$ for the case i., $k=4$, $j=5$, $c_1=2i, c_2=c_4=-1, c_3=i,$ and $c_5=-2i$ for the case ii., then we can directly see the conclusions from $(G)^{\alpha}$.

It can be easily proven that the generating function for complex-type Narayana-Jacobsthal numbers is as follows:

$$g_j(x)=\frac{x^4}{1-2ix+x^2-ix^3+x^4+2ix^5}.$$

Theorem 3.4. The exponential representation of the complex-type Narayana-Jacobsthal numbers is given as follows:

$$g_j(x)=x^4\exp\left(\sum_{\alpha=1}^{\infty}\frac{1}{\alpha}x^{\alpha}\left(2i-x+ix^2-x^3-2ix^4\right)^{\alpha}\right).$$

Proof. It is evident that

$$\ln\frac{g_j(x)}{x^4}=-\ln\left(1-2ix+x^2-ix^3+x^4+2ix^5\right).$$

Using the function $\ln x$, we derive the following relation

$$-\ln\left(1-2ix+x^2-ix^3+x^4+2ix^5\right)=-\Big[-x\left(2i-x+ix^2-x^3-2ix^4\right)-\frac{1}{2}x^2\left(2i-x+ix^2-x^3-2ix^4\right)-\cdots-\frac{1}{\alpha}x^{\alpha}\left(2i-x+ix^2-x^3-2ix^4\right)-\cdots\Big].$$

As a result, we obtain

$$\ln \frac{g_j(x)}{x^4} = \exp\left(\sum_{\alpha=1}^{\infty} \frac{1}{\alpha} x^{\alpha} \left(2i - x + ix^2 - x^3 - 2ix^4\right)^{\alpha} \right).$$

Thus, we reach the conclusion.

We now focus on the sums of the complex-type Narayana-Jacobsthal numbers.

Let

$$W_{\alpha} = \sum_{t=1}^{\alpha} n_t^{c,J}$$

for $\alpha \geq 1$, let Q be the 6×6 matrix as follows:

$$Q = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ 0 & & G & \\ \vdots & & & \\ 0 & & & \end{bmatrix}.$$

Then, it follows by induction that

$$(Q)^{\alpha} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ W_{\alpha+3} & & & \\ W_{\alpha+2} & & (G)^{\alpha} & \\ \vdots & & & \\ W_{\alpha-1} & & & \end{bmatrix}.$$

4. Conclusion

In this study, we introduce the complex-type Narayana-Jacobsthal numbers. Subsequently, we establish the determinantal and permanental representations of these numbers using specific matrices obtained from the generating matrix of the complex-type Narayana-Jacobsthal numbers. Moreover, we derive their combinatorial and exponential representations, as well as their sums, with the help of the generating function and the generating matrix associated with these numbers.

References

Akuzum, Y., & Deveci, O. (2021). The arrowhead-Pell sequences. *Ars Combinatoria, 155*, 231-246.

Brualdi, R., & Gibson, P. M. (1977). Convex polyhedra of doubly stochastic matrices I: applications of permanent function. *Journal of Combinatorial Theory, 2*, 194-230.

Chen, W., & Louck, J. (1996). The combinatorial power of the companion matrix. *Linear Algebra and its Applications, 232*, 261-278.

Deveci, O., & Akuzum, Y. (2022). The Narayana-Jacobsthal Sequence and its Binet Formulas. *Al-Farabi 4th International Congress on Applied Sciences* (s. 480-484). Erzurum: Iksad Global Publishing House.

Deveci, O., & Shannon, A. (2018). The quaternion-Pell sequence. *Communication in Algebra, 46*(12), 5403–5409.

Deveci, O., & Shannon, A. (2021). The complex-type k-Fibonacci sequences and their applications. *Communication in Algebra, 3*, 1352-1367.

Deveci, O., Akuzum, Y., & Karaduman, E. (2015). The Pell-Padovan p-sequences and its applications. *Utilitas Mathematica, 98*, 327-347.

Deveci, O., Erdag, O., & Gungoz, U. (2023). The complex-type cyclic-Fibonacci sequence and its applications. *Journal of Mahani Mathematical Research, 12*(2), 235-246.

El Naschie, M. (2005). Deriving The Essential Featuares of Standard Model From The General Theory of Relativity. *Chaos, Solitons & Fractals, 26*, 1-6.

Erdag, O., & Deveci, O. (2019). The arrowhead-Fibonacci-Random-type sequences. *Utilitas Mathematica, 113*, 271-287.

Erdag, O., & Deveci, O. (2022). On the Connections between Padovan Numbers and Fibonacci p-Numbers. *Jordan Journal of Mathematics and Statistics, 15*(3), 507-521.

Erdag, O., Halıcı, S., & Deveci , O. (2022). The complex-type Padovan-p sequences. *Mathematica Moravica, 26*(1), 77-88.

Erdag, O., Shannon, A., & Deveci, O. (2018). The arrowhead-Pell-Random-type sequences. *Notes Number Theory Discrete Mathematics, 24*(1), 109-119.

Fraenkel, A., & Klein, S. (1996). Robutst Universal Complete Codes for Transmission and Compression. *Discrete Applied Mathematics, 64*, 31-55.

Gogin, N., & Myllari, A. (2007). The Fibonacci-Padovan sequence and MacWilliams transform matrices. *Programming and Computer Software, 33*, 74-79.

Horadam , A. (1961). A generalized Fibonacci sequence. *The American Mathematical Monthly, 68*(5), 455-459.

Horadam , A. (1963). Complex Fibonacci numbers and Fibonacci quaternions. *The American Mathematical Monthly, 70*(3), 289-291.

Hulku, S., Erdag, O., & Deveci, O. (2023). Complex-type Narayana sequence and its application. *Maejo International Journal of Science & Technology, 17*(2), 163-176.

Kalman, D. (1982). Generalized Fibonacci numbers by matrix methods. *The Fibonacci Quarterly, 20*, 73-76.

Kirchoof, B., & Rutishauser, R. (1990). The Phyllotavy of Costus (Costaceae). *Bot Gazette, 151*(1), 88-105.

Mandelbaum, D. (1972). Synchronization of Codes by means of Kautz's Fibonacci Encoding. *IEEE Transactions on Information Theory, 18*(2), 281-285.

Ozkan, E. (2007). On truncated Fibonacci sequences. *Indian Journal of Pure and Applied Mathematics, 38*(4), 241-251.

Spinadel, V. (2002). The Metallic Means Family and Forbidden Symmetries. *Int. Math. J., 2*(3), 279-288.

Stakhov, A., & Rozin, B. (2006). Theory of Binet formulas for Fibonacci and Lucas p-numbers. *Chaos, Solitons Fractals, 27*, 1162-1177.

Stein, W. (1993). Modelling The Evolution of Stelar Archictecture in Vascular Plants. *International Journal of Plant Sciences , 154*(2), 229-263.

Tasci, D., & Firengiz, M. (2010). Incomplete Fibonacci and Lucas p-numbers. *Mathematical and Computer Modelling, 52*, 1763-1770.

BÖLÜM X

Simulation in Statistics

Levent ÖZBEK[1]

1. Introduction

A certain expression of a real-world event, process or a system consisting of units and operating according to the internal relations between the units as well as the external relations with the environment is called a model. Although the expression can be done verbally, by drawing, by creating a physical similarity at a certain scale or in another way, the most valid expression is made with mathematics, the common language of science.

A model is the expression of the structure and operation of a phenomenon or system in the real world, depending on the concepts and laws of the scientific field it is related to (physics, chemistry, biology, geology, astronomy, economy, sociology, etc.). A model is an expression, a representation of a phenomenon in the real world.

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Since the real world is very complex, models simplify the phenomena and systems they want to explain and address them under certain assumptions. Models are not the reality themselves and no matter how complex they seem, they are an incomplete expression of the reality. In short, what is called a model is a product of the model builder's "understanding" of the reality.

Every model-building process is an abstraction process. The abstraction process is the transfer of images of the phenomena in the real world, free of details, to human thought. In order to establish and select a model, it is necessary to know the basic characteristics of the phenomenon or system in question, its internal relations between its units and its external relations with the environment. The success of the model, its practical and scientific usefulness, depends on the degree of accuracy in abstracting the essence of the phenomenon or system and how basic the characteristics taken into account are.

The observation of a feature or behavior related to an event, process or system on a model is called simulation. "Simulation" is a word that means imitation, resemblance. Let us remind you that in mathematical models, simulation is used when an analytical or numerical solution cannot be found, and instead of an optimal result, a set of "observation" results are obtained by experiments under different conditions.

Probability theory, when considered as an abstract mathematical discipline, is a part of Measure Theory, and when considered as an applied discipline in modeling the phenomenon of randomness, it is a part of Statistical Theory. Statistics is a branch of science that provides the necessary information and methods for establishing mathematical models about events, processes, and systems that contain "randomness", especially for testing the validity of these models and for drawing conclusions from these models. Randomness is a distinct characteristic of our environment. It would not be wrong to call Probability and Statistics Theory the science of randomness.

2. The Possibility of Meeting Layla and Majnun

Leyla and Mecnun decide to meet in front of a store in Kızılay between 17:00 and 18:00 on a weekend. The first one to arrive will wait for the other for 10 minutes and if they don't arrive, they will leave. What are the chances of Leyla and Mecnun meeting, assuming they act independently of each other?

Model :

If it is considered that Leyla and Mecnun will meet in a 60-minute period, we can express the experimental result with

$$\Omega = \left\{(x, y) \in R^2 \;\; : 0 \le x \le 60,\; 0 \le y \le 60\right\} \subset \Re^2$$

where x: Leyla's and y: Mecnun's arrival times are. If the Borel algebra in $\Re^2$ is restricted to Ω and B_Ω is taken as the probability

$$P(A) = \frac{\text{"Area measure of } A\text{"}}{\text{"Area measure of } \Omega\text{"}}$$

measure for $A \in B_\Omega$, the experiment is modeled (described) with the (Ω, B_Ω, P) probability space.

Let A: be the event of Leyla and Mecnun meeting.

$$A = \left\{(x, y) \in R^2 \;\; : \text{!x-y!} \le 10\right\}$$

is found as

$$P(A) = \frac{11}{36} = 0.305$$

When it is desired to simulate the experiment on this model, the algorithm steps related to this are as follows.

A1. The experiment is performed once by generating numbers from the distribution with RND and calculating the values

X=60*RND

Y=60*RND

A2. It is checked whether the event has occurred or not.

A3. The experiment is repeated n times.

A4. The number of occurrences of the event is counted. The number of occurrences is proportional to the number of experiments.

The program for this is written as follows.

```
INPUT " Enter the number of experiments =", n
FOR i = 1 TO n
   x = RND * 60
   y = RND * 60
   PSET (600 - (100 + x * 5), (100 + y * 5)), 5
   IF ABS(x - y) <= 10 THEN
      PSET (600 - (100 + x * 5), (100 + y * 5)), 10
      met = met + 1
   END IF
NEXT i
PRINT " Probability found by simulation ="; met / n; 11 / 36
```

When the program is run, an image like the one below will appear.

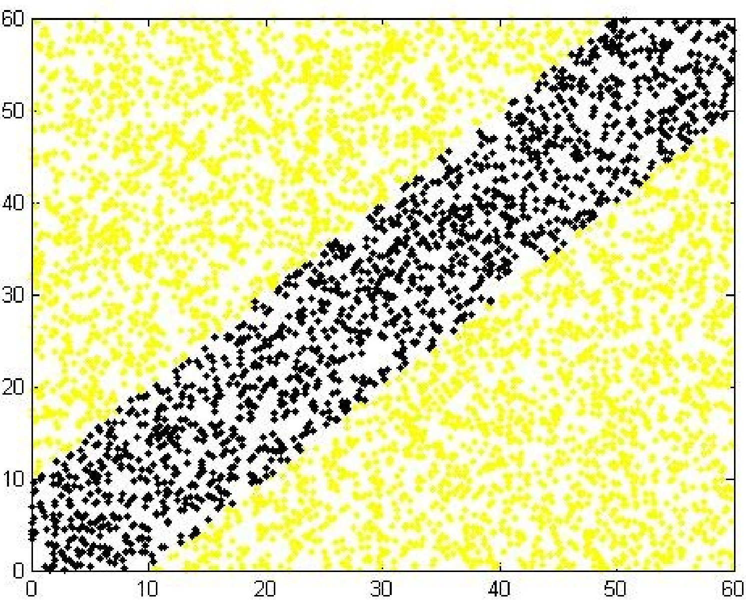

Figure 1. *Output image of the program.*

3. Calculating π Number Using Random Numbers

Let's consider the unit circle in the (x, y) coordinate system.

$$\Omega = \left\{(x, y) \in R^2 \ \ : \text{-}1 \le x \le 1,\ \text{-}1 \le y \le 1\right\} \subset \Re^2$$

Let's deal with the event

$$A = \left\{(x, y) \in R^2 \ \ : \mathrm{x}^2 + \mathrm{y}^2 \le 1\right\}$$

where the (x, y) point randomly selected from Ω falls inside the unit circle. If the Borel algebra in $\Re^2$ is restricted to Ω and B_Ω is taken as the probability measure for $A \in B_\Omega$, the experiment is modeled with the (Ω, B_Ω, P) probability space and the sought probability is found as

$$P(A) = \frac{\text{"Area measure of } A\text{"}}{\text{"Area measure of } \Omega\text{"}}$$

from here $P(A) = \frac{\pi}{4}$ can be written. The $P(A)$ probability can be found by simulation, if this probability value is multiplied by 4, this result will give us an estimate for $\pi = 4.P(A)$. The algorithm steps are as follows.

A1. The experiment is performed once by generating numbers from the RND and U(0,1) distribution and calculating the values

X=2*RND-1

Y=2*RND-1.

A2. If X^2+Y^2<=1, the event has occurred.

A3. The experiment is repeated n times.

A4. The number of occurrences of the event is counted.

The number of occurrences is proportional to the number of experiments.

The program for this is written as follows.

```
INPUT " Enter the number of experiments =", n
FOR i = 1 TO n
   x = 2 * RND - 1
```

```
    y = 2 * RND - 1
    PSET (600 - (300 + x * 150), (210 + y * 150)), 5
    IF x ^ 2 + y ^ 2 <= 1 THEN
        PSET (600 - (300 + x * 150), (210 + y * 150)), 10
        ok = ok + 1
    END IF
NEXT i
PRINT "Probability="; ok / n
PRINT "Eestimated pi value ="; 4 * ok / n
```

When the program is run, an image like the one below will appear.

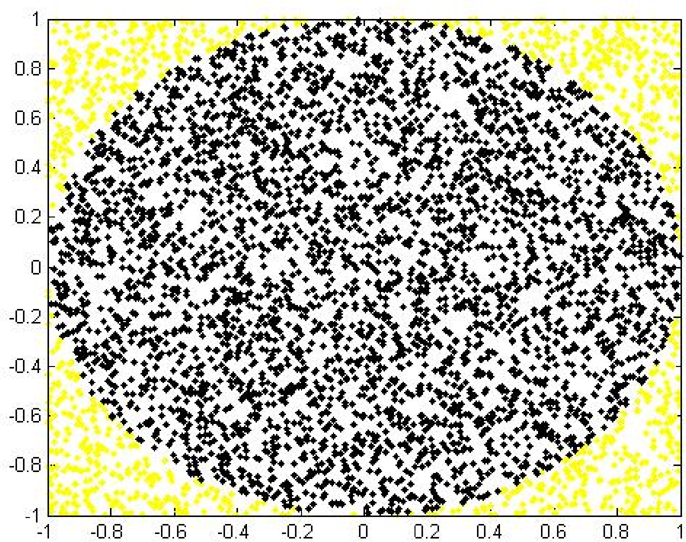

Figure 2. *Output image of the program.*

4. Marley Problem: What is the probability that a backgammon chip with a radius of 2 br. will not intersect the edges of the marley when it is randomly thrown onto the floor of a room paved with square marleys with side lengths of 20 br.?

Model: As a result of the experiment, let's observe the position of the center point of the coin on the marley where it falls (is located). The observation process can be done by shifting the marley where the center point of the coin is located to the starting point of a coordinate system as in Figure 3. The experiment can be thought of as if it were being done on a single marley. The set of possible outcomes for the coordinates of the center point of the coin to be (x, y) is

$$\Omega = \{(x,y) \in R^2 \ : 0 \le x \le 20,\ 0 \le y \le 20\} \subset \Re^2$$

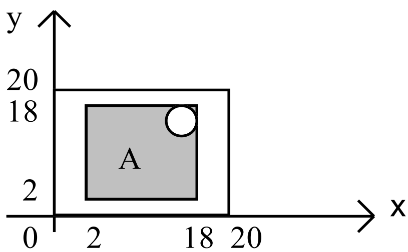

Figure 3. *Sample space*

If the restriction of the Borel algebra in $\Re^2$ to Ω is taken as the probability measure for $A \in B_\Omega$, with B_Ω as the B_Ω, we will have modeled (explained) the experiment with the (Ω, B_Ω, P) probability space.

$$P(A) = \frac{\text{"Area measure of } A\text{"}}{\text{"Area measure of } \Omega\text{"}}$$

A: Let the event be that the thrown chip does not intersect the edges of the marley.

$$A = \{(x, y) \in R^2 \ : \ 2 \leq x \leq 18, \ 2 \leq y \leq 18\}$$

is found as

$$P(A) = \frac{16^2}{20^2} = 0.64$$

The algorithm steps are as follows.

A1. The experiment is performed once by generating numbers from the RND and U(0,1) distribution and calculating the values

X=20*RND

Y=20*RND

A2. If 2<x<18 and 2<y<18, the event has occurred.

A3. The experiment is repeated n times.

A4. The number of occurrences of the event is counted.

The number of occurrences is proportional to the number of experiments.

References

Özbek, L., (2016). Reflections and Comments on Randomness. Econometrics Letters , vol.3, no.2, 28-33.

Özbek, L., (2009). Rasgelelik Üzerine Düşüncelerde Bir Gezinti. Koridor Dergisi , no.9, 4-7.

Özbek, L., (2002). Olasılık ve Olasılık Öğretimi Üzerine Bir Çalışma. Matematik Dünyası , vol.11, no.2, 19-23.

Özbek, L., (2000). Rasgele Dizi ve Pi. Matematik Dünyası , vol.9, no.1, 26-28.

Özbek, L., (2023). A Study On The Number pi And Geometric Distribution. Data Analysis and Advanced Modeling Techniques (pp.16-30), Ankara: BIDGE Publications.

Özbek, L., & Öztürk, F., (2004). Matematiksel Modelleme ve Simulasyon . Ankara: Gazi Yay..

Özbek, L., (2018). Olasılık ve Algoritma Tasarımına Giriş . Ankara: Akademisyen Yayınevi Kitabevi.

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